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Suzuki-type fixed point theorem for fuzzy mappings in ordered metric spaces

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Abstract
In this paper, a Suzuki-type fixed fuzzy point result for fuzzy mappings in complete ordered metric spaces is obtained. As an application, we establish the existence of coincidence fuzzy points and common fixed fuzzy points for a hybrid pair of a single-valued self-mapping and a fuzzy mapping. An example is also provided to support the main result presented herein.
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1 Introduction and preliminaries

Let X be a space of points with generic elements of X denoted by x and $I = [0,1]$. A fuzzy subset of X is characterized by a membership function such that each element in X is associated with a real number in the interval I. Let (X,d) be a metric space and a fuzzy set A in X is characterized by a membership function A. Then α-level set of A, denoted by A_α, is defined as

$$A_\alpha = \{x : A(x) \geq \alpha\}$$

for $\alpha \in (0,1]$ and for $\alpha = 0$, we have

$$A_0 = \overline{\{x : A(x) > 0\}},$$

where $\overline{B}$ denotes the closure of the non-fuzzy set B. A fuzzy set A in X is said to be an approximate quantity if and only if for $\alpha \in [0,1]$, A_α is a compact, convex subset of X and

$$\sup_{x\in X} A(x) = 1.$$

Let $W(X)$ be a family of all approximate quantities in X. A fuzzy set A is said to be more accurate than a fuzzy set B denoted by $A \subset B$ (that is, B includes A) if and only if $A(x) \leq B(x)$ for each x in X, where $A(x)$ and $B(x)$ denote the membership function of A and B, respectively. It is easy to see that if $0 < \alpha \leq \beta \leq 1$, then $A_\alpha \subseteq A_\beta$.

Corresponding to each $\alpha \in [0,1]$ and $x \in X$, the fuzzy point x_α of X is the fuzzy set $x_\alpha : X \to [0,1]$ given by

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

For $\alpha = 1$, we have

$$x_1(y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise} \end{cases} = \{x\}.$$

Let I^X be a collection of all fuzzy subsets of X and $W(X)$ be a subcollection of all approximate quantities. For $A, B \in W(X)$ and $\alpha \in [0,1]$, define

$$p_\alpha(A,B) = \inf\bigl\{d(x,y), x \in A_\alpha, y \in B_\alpha\bigr\},$$
$$D_\alpha(A,B) = \max\Bigl\{\sup_{x \in A_\alpha} d(x, B_\alpha), \sup_{y \in B_\alpha} d(y, A_\alpha)\Bigr\}$$

and

$$D(A,B) = \sup_\alpha D_\alpha(A,B).$$

Note that p_α is a nondecreasing function of α and D is a metric on $W(X)$. Let $\alpha \in [0,1]$. Define $W_\alpha(X) = \{A \in I^X : A_\alpha$ is nonempty, convex and compact$\}$. Let (X,d) be a metric space and Y be an arbitrary set. A mapping $F : Y \to W_\alpha(X)$ is called a fuzzy mapping, that is, $Fy \in W_\alpha(X)$ for each y in Y. Thus, if we characterize a fuzzy set Fy in a metric space X by a membership function Fy, then $Fy(x)$ is the grade of membership of x in Fy. Therefore, a fuzzy mapping F is a fuzzy subset of $Y \times X$ with a membership function $Fy(x)$.

In a more general sense than that given in [1], a mapping $F : X \to I^X$ is a fuzzy mapping over X [2] and $(F(x)x)$ is the fixed degree of x in $F(x)$.

Definition 1 ([3]) A fuzzy point x_α in X is called a fixed fuzzy point of the fuzzy mapping F if $x_\alpha \subset Fx$, that is, $(Fx)x \geq \alpha$ or $x \in (Fx)_\alpha$. That is, the fixed degree of x in Fx is at least α. If $\{x\} \subset Fx$, then x is a fixed point of a fuzzy mapping F.

Let $F : X \to W_\alpha(X)$ and $g : X \to X$.

A fuzzy point x_α in X is called a coincidence fuzzy point of the hybrid pair $\{F, g\}$ if $(gx)_\alpha \subset Fx$, that is, $(Fx)gx \geq \alpha$ or $gx \in (Fx)_\alpha$. That is, the fixed degree of gx in Fx is at least α. A fuzzy point x_α in X is called a common fixed fuzzy point of the hybrid pair $\{F, g\}$ if $x_\alpha = (gx)_\alpha \subset Fx$, that is, $x = gx \in (Fx)_\alpha$ (the fixed degree of x and gx in Fx is the same and is at least α).

We denote by $C_\alpha(F,g)$ and $F_\alpha(F,g)$ the set of all coincidence fuzzy points and the set of all common fixed fuzzy points of the hybrid pair $\{F, g\}$, respectively.

A hybrid pair $\{F, g\}$ is called *w-fuzzy compatible* if $g(Fx)_\alpha \subseteq (Fgx)_\alpha$ whenever $x \in C_\alpha(F,g)$.

A mapping g is called *F-fuzzy weakly commuting* at some point $x \in X$ if $g^2(x) \in (Fgx)_\alpha$.

Lemma 1 ([4]) *Let X be a nonempty set and $g : X \to X$. Then there exists a subset $E \subseteq X$ such that $g(E) = g(X)$ and $g : E \to X$ is one-to-one.*

Definition 2 Let X be a nonempty set. Then $(X, d, \preceq)$ is called an ordered metric space if (X, d) is a metric space and $(X, \preceq)$ is partially ordered.

Let $(X, \preceq)$ be a partially ordered set. Then $x, y \in X$ are said to be comparable if $x \preceq y$ or $y \preceq x$ holds.

Define

$$\nabla = \{(x, y) \in X \times X : x \preceq y \text{ or } y \preceq x\}.$$

An ordered metric space is said to satisfy the order sequential limit property if $(u_n, z) \in \nabla$ for all n, whenever a sequence $u_n \to z$ and $(u_n, u_{n+1}) \in \nabla$ for all n.

A mapping $F : X \to W_\alpha(X)$ is said to be an ordered fuzzy mapping if the following conditions are satisfied:

(a) $y \in F(x)_\alpha$ implies that $(y, x) \in \nabla$.

(b) $(x, y) \in \nabla$ implies that $(u, v) \in \nabla$ whenever $u \in (Fx)_\alpha$ and $v \in (Fy)_\alpha$.

The following lemmas are needed in the sequel.

Lemma 2 (Heilpern [1]) *Let (X, d) be a metric space, $x, y \in X$ and $A, B \in W(X)$:*

1. *if $p_\alpha(x, A) = 0$, then $x_\alpha \subset A$;*
2. *$p_\alpha(x, A) \leq d(x, y) + p_\alpha(y, A)$;*
3. *if $x_\alpha \subset A$, then $p_\alpha(x, B) \leq D_\alpha(A, B)$.*

Lemma 3 (Lee and Cho [5]) *Let (X, d) be a complete metric space and F be a fuzzy mapping from X into $W(X)$ and $x_0 \in X$. Then there exists an $x_1 \in X$ such that $\{x_1\} \subset Fx_0$.*

Zadeh [6] introduced the concept of a fuzzy set. Heilpern [1] introduced the concept of fuzzy mappings in a metric space and proved a fixed point theorem for fuzzy contraction mappings as a generalization of the fixed point theorem for multivalued mappings given by Nadler [7]. Estruch and Vidal [3] proved a fixed point theorem for fuzzy contraction mappings in complete metric spaces which in turn generalizes the Heilpern fixed point theorem. Further generalizations of the result given in [3] were proved in [8, 9]. Recently, Suzuki [10] generalized the Banach contraction principle and characterized the metric completeness property of an underlying space. Among many generalizations (see [11–13]) of the results given in [10], Dorić and Lazović [14] obtained Suzuki-type fixed point results for a generalized multivalued contraction in complete metric spaces.

On the other hand, the existence of fixed points in ordered metric spaces has been introduced and applied by Ran and Reurings [15]. Fixed point theorems in partially ordered metric spaces are hybrid of two fundamental principles: Banach contraction theorem with a contractive condition for comparable elements and a selection of an initial point to generate a monotone sequence. For results concerning fixed points and common fixed points in partially ordered metrics spaces, we refer to [16–22].

The aim of this paper is to investigate Suzuki-type fixed point results for fuzzy mappings in complete ordered metric spaces. As an application, a coincidence fuzzy point and a common fixed fuzzy point of the hybrid pair of a single-valued self-mapping and a fuzzy mapping are obtained. We provide an example to support the result.

Throughout this paper, let $\sigma : [0,1) \to (0,1]$ be the nonincreasing function defined by

$$\sigma(r) = \begin{cases} 1 & \text{if } 0 \le r < \frac{1}{2}, \\ 1 - r & \text{if } \frac{1}{2} \le r < 1. \end{cases} \tag{1}$$

2 Main results

The following theorem is the main result of the paper and is a generalization of [14, Theorem 2.1] for fuzzy mappings in ordered metric spaces.

Theorem 4 *Let $(X, d, \preceq)$ be a complete ordered metric space. If an ordered fuzzy mapping $F : X \to W_\alpha(X)$ satisfies*

$$\sigma(r)p_\alpha(x, Fx) \le d(x, y) \quad \textit{implies} \quad D_\alpha(Fx, Fy) \le rM_\alpha(F) \tag{2}$$

for all $(x, y) \in \nabla$, where

$$M_\alpha(F) = \max\left\{d(x, y), p_\alpha(x, Fx), p_\alpha(y, Fy), \frac{p_\alpha(x, Fy) + p_\alpha(y, Fx)}{2}\right\}.$$

Then there exists a point $x \in X$ such that $x_\alpha \subset Fx$ provided that X satisfies the order sequential limit property.

Proof Let r_1 be a real number such that $0 \le r < r_1 < 1$ and $u_1 \in X$. Since $(Fu_1)_\alpha$ is nonempty and compact, there exists $u_2 \in (Fu_1)_\alpha$ such that

$$d(u_1, u_2) = p_\alpha(u_1, Fu_1).$$

By the given assumption, we have $(u_1, u_2) \in \nabla$. Since $(Fu_2)_\alpha$ is nonempty and compact, there exists $u_3 \in (Fu_2)_\alpha$ such that

$$d(u_2, u_3) = p_\alpha(u_2, Fu_2) \le D_\alpha(Fu_1, Fu_2).$$

Also, $(u_2, u_3) \in \nabla$. Since $\sigma(r) < 1$, we obtain

$$\sigma(r)\rho_\alpha(u_1, Fu_1) \le p_\alpha(u_1, Fu_1) = d(u_1, u_2).$$

That is,

$$\sigma(r)p_\alpha(u_1, Fu_1) \le d(u_1, u_2).$$

So, we have

$$\begin{aligned} d(u_2, u_3) &\le D_\alpha(Fu_1, Fu_2) \\ &\le r \max\left\{d(u_1, u_2), p_\alpha(u_1, Fu_1), p_\alpha(u_2, Fu_2), \frac{p_\alpha(u_1, Fu_2) + p_\alpha(u_2, Fu_1)}{2}\right\} \end{aligned}$$

$$\begin{aligned}
&\le r_1 \max\left\{d(u_1,u_2), d(u_1,u_2), d(u_2,u_3), \frac{d(u_1,u_2)+p_\alpha(u_2,Fu_2)}{2}\right\}\\
&\le r_1 \max\left\{d(u_1,u_2), d(u_2,u_3), \frac{d(u_1,u_2)+d(u_2,u_3)}{2}\right\}.
\end{aligned}$$

Note that $d(u_2,u_3) \le d(u_1,u_2)$. If not, then the above inequality gives

$$\begin{aligned}
d(u_2,u_3) &\le r_1 \max\left\{d(u_2,u_3), d(u_2,u_3), \frac{d(u_2,u_3)+d(u_2,u_3)}{2}\right\}\\
&= r_1 d(u_2,u_3) < d(u_2,u_3) \quad \text{as } r_1 < 1,
\end{aligned}$$

a contradiction. Hence, $d(u_2,u_3) \le r_1 d(u_1,u_2)$. Continuing this process, we construct a sequence $\{u_n\}$ in X such that $u_{n+1} \in (Fu_n)_\alpha$ and $u_{n+2} \in (Fu_{n+1})_\alpha$ with

$$d(u_{n+1},u_{n+2}) = p_\alpha(u_{n+1},Fu_{n+1}) \le D_\alpha(Fu_n,Fu_{n+1}).$$

By the given assumption, we have $(u_n,u_{n+1}) \in \nabla$ and $(u_{n+1},u_{n+2}) \in \nabla$. As $\sigma(r) < 1$, so

$$\sigma(r)p_\alpha(u_n,Fu_n) \le p_\alpha(u_n,Fu_n) = d(u_n,u_{n+1}).$$

Therefore,

$$\begin{aligned}
d(u_{n+1},u_{n+2}) &\le D_\alpha(Fu_n,Fu_{n+1})\\
&\le r \max\left\{d(u_n,u_{n+1}), p_\alpha(u_n,Fu_n), p_\alpha(u_{n+1},Fu_{n+1}),\right.\\
&\qquad \left.\frac{p_\alpha(u_n,Fu_{n+1})+p_\alpha(u_{n+1},Fu_n)}{2}\right\}\\
&\le r_1 \max\left\{d(u_n,u_{n+1}), d(u_n,u_{n+1}), d(u_{n+1},u_{n+2}),\right.\\
&\qquad \left.\frac{d(u_n,u_{n+1})+p_\alpha(u_{n+1},Fu_{n+1})}{2}\right\}\\
&\le r_1 \max\left\{d(u_n,u_{n+1}), d(u_{n+1},u_{n+2}), \frac{d(u_n,u_{n+1})+d(u_{n+1},u_{n+2})}{2}\right\}.
\end{aligned}$$

We claim that $d(u_{n+1},u_{n+2}) \le d(u_n,u_{n+1})$. If not, then by the above inequality, we obtain

$$\begin{aligned}
d(u_{n+1},u_{n+2}) &\le r_1 \max\left\{d(u_{n+1},u_{n+2}), d(u_{n+1},u_{n+2}), \frac{d(u_{n+1},u_{n+2})+d(u_{n+1},u_{n+2})}{2}\right\}\\
&\le r_1 d(u_{n+1},u_{n+2}) < d(u_{n+1},u_{n+2}),
\end{aligned}$$

a contradiction as $r_1 < 1$. So, we have

$$d(u_{n+1},u_{n+2}) \le r_1 d(u_n,u_{n+1}) \le \cdots \le (r_1)^n d(u_1,u_2)$$

and

$$\sum_{n=1}^{\infty} d(u_{n+1},u_{n+2}) \le \sum_{n=1}^{\infty} (r_1)^n d(u_1,u_2) < \infty. \tag{3}$$

Hence, $\{u_n\}$ is a Cauchy sequence in X. Since X is complete, there is some point $z \in X$ such that $\lim_{n\to\infty} u_n = z$. As $(u_n, u_{n+1}) \in \nabla$ for all n, then by the assumption, $(u_n, z) \in \nabla$. Now, we show that for every pair $(x, z) \in \nabla$ with $x \neq z$, the following inequality holds:

$$p_\alpha(z, Tx) \leq r \max\{d(z,x), p_\alpha(x, Fx)\}.$$

As $\lim_{n\to\infty} u_n = z$, there exists a positive integer $n_0 \in N$ such that for all $n \geq n_0$, we have

$$d(z, u_n) \leq \frac{1}{3} d(z,x). \tag{4}$$

Now, for all $n \geq n_0$,

$$\begin{aligned}
\sigma(r) p_\alpha(u_n, Fu_n) &\leq p_\alpha(u_n, Fu_n) \\
&\leq d(u_n, z) + p_\alpha(z, Fu_n) \leq d(u_n, z) + d(z, u_{n+1}) \\
&\leq \frac{1}{3} d(z,x) + \frac{1}{3} d(z,x) \leq \frac{2}{3} d(z,x) \\
&= d(z,x) - \frac{1}{3} d(z,x) \\
&\leq d(z,x) - d(u_n, z) \leq d(u_n, x)
\end{aligned}$$

implies that

$$\begin{aligned}
p_\alpha(u_{n+1}, Fx) &\leq D_\alpha(Fu_n, Fx) \\
&\leq r \max\left\{d(u_n,x), p_\alpha(u_n, Fu_n), p_\alpha(x, Fx), \frac{p_\alpha(u_n, Fx) + p_\alpha(x, Fu_n)}{2}\right\} \\
&\leq r \max\left\{d(u_n,x), d(u_n, u_{n+1}), p_\alpha(x, Fx), \frac{p_\alpha(u_n, Fx) + d(x, u_{n+1})}{2}\right\},
\end{aligned}$$

which on taking limit as $n \to \infty$ gives

$$p_\alpha(z, Fx) \leq r \max\left\{d(z,x), p_\alpha(x, Fx), \frac{p_\alpha(z, Fx) + d(x,z)}{2}\right\}.$$

If

$$\max\left\{d(z,x), p_\alpha(x, Fx), \frac{p_\alpha(z, Fx) + d(x,z)}{2}\right\} = \frac{p_\alpha(z, Fx) + d(x,z)}{2},$$

then

$$\begin{aligned}
p_\alpha(z, Fx) &\leq r \frac{p_\alpha(z, Fx) + d(x,z)}{2} \leq \frac{r}{2} p_\alpha(z, Fx) + \frac{r}{2} d(x,z), \\
p_\alpha(z, Fx) &\leq \frac{r}{2-r} d(x,z) \leq \frac{r}{2-r} d(x,z) \leq r d(x,z).
\end{aligned}$$

Hence,

$$p_\alpha(z, Fx) \leq r \max\{d(z,x), p_\alpha(x, Fx)\}. \tag{5}$$

Now, we show that $z_\alpha \subset Fz$ for each $\alpha \in [0,1]$. First, consider the case $0 \leq r < 1/2$. Assume on the contrary that $z_\alpha \not\subseteq Fz$, that is, $z \notin (Fz)_\alpha$. Let $a \in (Fz)_\alpha$, as $(Fz)_\alpha$ is nonempty and compact, so for each $\alpha \in [0,1]$, we have

$$2rd(a,z) < p_\alpha(z,Fz). \tag{6}$$

Now, $a \in (Fz)_\alpha$ implies $(a,z) \in \nabla$ and $a \neq z$. From (5) we have

$$p_\alpha(z,Fa) \leq r \max\{d(z,a), p_\alpha(a,Fa)\}. \tag{7}$$

Now,

$$\sigma(r)p_\alpha(z,Fz) \leq p_\alpha(z,Fz) = d(z,a)$$

implies that

$$\begin{aligned}
D_\alpha(Fz,Fa) &\leq r \max\left\{d(z,a), p_\alpha(z,Fz), p_\alpha(a,Fa), \frac{p_\alpha(z,Fa) + p_\alpha(a,Fz)}{2}\right\} \\
&\leq r \max\left\{d(z,a), p_\alpha(z,Fz), p_\alpha(a,Fa), \frac{p_\alpha(z,Fa)}{2}\right\} \\
&\leq r \max\left\{d(z,a), p_\alpha(z,Fz), p_\alpha(a,Fa), \frac{d(z,a) + p_\alpha(a,Fa)}{2}\right\} \\
&\leq r \max\{d(z,a), p_\alpha(z,Fz), p_\alpha(a,Fa)\} \\
&\leq r \max\{d(z,a), p_\alpha(a,Fa)\}.
\end{aligned}$$

Hence,

$$D_\alpha(Fz,Fa) \leq r \max\{d(z,a), p_\alpha(a,Fa)\},$$

which further implies that

$$p_\alpha(a,Fa) \leq r \max\{d(z,a), p_\alpha(a,Fa)\} \quad \text{as } p_\alpha(a,Fa) \leq D_\alpha(Fz,Fa).$$

We claim that $p_\alpha(a,Fa) \leq d(z,a)$. If not, then the above inequality becomes

$$p_\alpha(a,Fa) \leq rp_\alpha(a,Fa) < p_\alpha(a,Fa) \quad \text{as } r < 1,$$

a contradiction, so we deduce that $p_\alpha(a,Fa) \leq rd(z,a)$. From inequality (7), we have

$$p_\alpha(z,Fa) \leq rd(z,a).$$

Therefore,

$$\begin{aligned}
p_\alpha(z,Fz) &\leq p_\alpha(z,Fa) + D_\alpha(Fz,Fa) \\
&\leq rd(z,a) + r \max\{d(z,a), p_\alpha(a,Fa)\} \\
&\leq rd(z,a) + rd(z,a) \leq 2rd(z,a) < p_\alpha(z,Fz),
\end{aligned}$$

a contradiction. Hence, $z_\alpha \subset Fz$.

Now, when $1/2 \leq r < 1$, we first prove that

$$D_\alpha(Fx, Fz) \leq r \max\left\{d(x,z), p_\alpha(x,Fx), p_\alpha(z,Fz), \frac{p_\alpha(x,Fz) + p_\alpha(z,Fx)}{2}\right\} \tag{8}$$

for all $(x,z) \in \nabla$. If $x = z$, then (8) holds trivially. So, assume that $x \neq z$. For every $n \in N$, one may find a sequence $y_n \in (Fx)_\alpha$ such that

$$d(z,y_n) \leq p_\alpha(z,Fx) + \frac{1}{n}d(x,z).$$

As $y_n \in (Fx)_\alpha$, this implies $(y_n, x) \in \nabla$. Using (7) we have

$$\begin{aligned}
p_\alpha(x,Fx) &\leq d(x,y_n) \\
&\leq d(x,z) + d(z,y_n) \\
&\leq d(x,z) + p_\alpha(z,Fx) + \frac{1}{n}d(x,z) \\
&\leq d(x,z) + r\max\{d(z,x), p_\alpha(x,Fx)\} + \frac{1}{n}d(x,z)
\end{aligned}$$

for all $n \in N$. If $d(x,z) \geq p_\alpha(x,Fx)$, then

$$\begin{aligned}
p_\alpha(x,Fx) &\leq d(x,z) + rd(z,x) + \frac{1}{n}d(x,z) \\
&\leq d(x,z) + rd(z,x) + \frac{1}{n}d(x,z) \leq \left(1 + r + \frac{1}{n}\right)d(x,z).
\end{aligned}$$

This implies that

$$\frac{1}{(1+r)}p_\alpha(x,Fx) \leq \left(1 + \frac{1}{(1+r)n}\right)d(x,z).$$

Hence, for $\frac{1}{2} \leq r < 1$, we obtain

$$\begin{aligned}
\sigma(r)p_\alpha(x,Fx) &= (1-r)p_\alpha(x,Fx) \\
&\leq \frac{1}{(1+r)}p_\alpha(x,Fx) \leq \left(1 + \frac{1}{(1+r)n}\right)d(x,z).
\end{aligned}$$

On taking the limit as $n \to \infty$, we have

$$\sigma(r)p_\alpha(x,Fx) \leq d(x,z).$$

If $d(x,z) \leq p_\alpha(x,Fx)$, then

$$\begin{aligned}
&p_\alpha(x,Fx) \leq d(x,z) + rp_\alpha(x,Fx) + \frac{1}{n}d(x,z), \\
&p_\alpha(x,Fx) - rp_\alpha(x,Fx) \leq d(x,z) + \frac{1}{n}d(x,z), \\
&(1-r)p_\alpha(x,Fx) \leq d(x,z) + \frac{1}{n}d(x,z).
\end{aligned}$$

On taking the limit as $n \to \infty$, we have

$$\sigma(r)p_\alpha(x, Fx) \le d(x, z).$$

By the given assumption, we have

$$D_\alpha(Fx, Fz) \le r \max\left\{d(x, z), p_\alpha(x, Fx), p_\alpha(z, Fz), \frac{p_\alpha(x, Fz) + p_\alpha(z, Fx)}{2}\right\}.$$

Thus, for any $x \neq z$, (8) holds true. Put $x = u_n$ in the above inequality to obtain

$$\begin{aligned} p_\alpha(z, Fz) &\le \lim_{n\to\infty} p_\alpha(u_{n+1}, Fz) \le \lim_{n\to\infty} D_\alpha(Fu_n, Fz) \\ &\le \lim_{n\to\infty} r \max\left\{d(u_n, z), d(u_n, u_{n+1}), p_\alpha(z, Fz), \frac{p_\alpha(u_n, Fz) + p_\alpha(z, Fu_n)}{2}\right\} \\ &= rp_\alpha(z, Fz) \end{aligned}$$

as $r < 1$, we get $p_\alpha(z, Fz) = 0$. Hence by Lemma 2, $z_\alpha \subset Fz$. □

Corollary 5 *Let $(X, d, \preceq)$ be a complete ordered metric space. If an ordered fuzzy mapping $F : X \to W_\alpha(X)$ satisfies*

$$\sigma(r)p_\alpha(x, Fx) \le d(x, y) \quad \textit{implies} \quad D_\alpha(Fx, Fy) \le rM_\alpha(F)$$

for all $(x, y) \in \nabla$, where

$$M_\alpha(F) = \max\{d(x, y), p_\alpha(x, Fx), p_\alpha(y, Fy)\}.$$

Then there exists a point $x \in X$ such that $x_\alpha \subset Fx$ provided that X satisfies the order sequential limit property.

Corollary 6 *Let $(X, d, \preceq)$ be a complete ordered metric space. If an ordered fuzzy mapping $F : X \to W_\alpha(X)$ satisfies*

$$\sigma(r)p_\alpha(x, Fx) \le d(x, y) \quad \textit{implies} \quad D_\alpha(Fx, Fy) \le \lambda M_\alpha(F)$$

for all $(x, y) \in \nabla$, where

$$M_\alpha(F) = d(x, y) + p_\alpha(x, Fx) + p_\alpha(y, Fy)\}$$

and $\lambda \in [0, \frac{1}{3})$, $r = 3\lambda$. Then there exists a point $x \in X$ such that $x_\alpha \subset Fx$ provided that X satisfies the order sequential limit property.

3 An application

Let $F : X \to W_\alpha(X)$ and $g : X \to X$. A pair $\{F, g\}$ is said to be an ordered fuzzy hybrid pair if the following conditions are satisfied:

(c) $gy \in F(x)_\alpha$ implies that $(y, x) \in \nabla$.
(d) $(x, y) \in \nabla$ gives $(u, v) \in \nabla$ whenever $gu \in (Fx)_\alpha$ and $gv \in (Fy)_\alpha$.
(e) $(gx, gy) \in \nabla$ whenever $(x, y) \in \nabla$ for all $x, y \in X$.

Theorem 7 *Let $(X,d,\preceq)$ be a complete ordered metric space. If an ordered fuzzy hybrid pair $\{F,g\}$ satisfies*

$$\sigma(r)p_\alpha(gx,Fx) \le d(gx,gy) \quad \textit{implies} \quad D_\alpha(Fx,Fy) \le rM_\alpha(F,g) \tag{9}$$

for all $(x,y) \in \nabla$, where

$$M_\alpha(F,g) = \max\left\{d(gx,gy), p_\alpha(gx,Fx), p_\alpha(gy,Fy), \frac{p_\alpha(gx,Fy)+p_\alpha(gy,Fx)}{2}\right\}.$$

Then $C_\alpha(F,g) \ne \phi$ provided that X satisfies the order sequential limit property and $(F(X))_\alpha \subseteq g(X)$ for each α. Moreover, F and g have a common fixed fuzzy point if any of the following conditions holds:

(f) *F and g are w-*fuzzy compatible, *$\lim_{n\to\infty} g^n x = u$ and $\lim_{n\to\infty} g^n y = v$ for some $x \in C_\alpha(F,g)$, $u \in X$ and g is continuous at u.*

(g) *g is F-*fuzzy weakly commuting *for some $x \in C_\alpha(g,F)$ and is a fixed point of g, that is, $g^2 x = gx$.*

(h) *g is continuous at x for some $x \in C_\alpha(g,F)$ and for some $u \in X$ such that* $\lim_{n\to\infty} g^n u = x$.

Proof By Lemma 1, there exists $E \subseteq X$ such that $g : E \to X$ is one-to-one and $g(E) = g(X)$. Define a mapping $\mathcal{A} : g(E) \to W_\alpha(X)$ by

$$\mathcal{A}gx = Fx \quad \text{for all } gx \in g(E). \tag{10}$$

As g is one-to-one on E, $\mathcal{A}$ is well defined. Also,

$$\sigma(r)p_\alpha(gx,Fx) \le d(gx,gy) \quad \text{implies} \quad D_\alpha(Fx,Fy) \le rM_\alpha(F,g) \tag{11}$$

for all $(x,y) \in \nabla$. Therefore,

$$\sigma(r)p_\alpha(gx,\mathcal{A}gx) \le d(gx,gy) \quad \text{implies} \quad D_\alpha(\mathcal{A}gx,\mathcal{A}gy) \le rM_\alpha(F,g)$$

for all $(gx,gy) \in \nabla$. Hence, $\mathcal{A}$ satisfies (2) and all the conditions of Theorem 4. Using Theorem 4 with a mapping $\mathcal{A}$, it follows that $\mathcal{A}$ has a fixed fuzzy point $u \in g(E)$. Now, it is left to prove that F and g have a coincidence fuzzy point. Since $\mathcal{A}$ has a fixed fuzzy point $u_\alpha \subset \mathcal{A}u$, we get $u \in (\mathcal{A}u)_\alpha$. As $(F(X))_\alpha \subseteq g(X)$, so there exists $u_1 \in X$ such that $gu_1 = u$, thus it follows that $gu_1 \in (\mathcal{A}gu_1)_\alpha = (Fu_1)_\alpha$. This implies that $u_1 \in X$ is a coincidence fuzzy point of F and g. Hence, $C_\alpha(F,g) \ne \phi$. Suppose now that (f) holds. Then for some $x_\alpha \in C_\alpha(F,g)$, we have $\lim_{n\to\infty} g^n x = u$, where $u \in X$. Thus $(g^{n-1}x,u) \in \nabla$. Since g is continuous at u, we have that u is a fixed point of g. As F and g are *w-fuzzy compatible*, and $(g^n x)_\alpha \in C_\alpha(F,g)$ for all $n \ge 1$. That is, $g^n x \in F(g^{n-1}x)_\alpha$ for all $n \ge 1$. Now,

$$\begin{aligned}\sigma(r)p_\alpha\left(g^n x, Fg^{n-1}x\right) &\le p_\alpha\left(g^n x, Fg^{n-1}x\right) = 0 \\ &\le d\left(gg^{n-1}x, gu\right)\end{aligned}$$

implies that

$$\begin{aligned}
p_\alpha(gu,Fu) &\le p_\alpha\bigl(gu,g^n x\bigr)+p_\alpha\bigl(g^n x,Fu\bigr)\\
&\le p_\alpha\bigl(gu,g^n x\bigr)+D_\alpha\bigl(F\bigl(g^{n-1}x\bigr),Fu\bigr)\\
&\le p_\alpha\bigl(gu,g^n x\bigr)+r\max\biggl\{d\bigl(gg^{n-1}x,gu\bigr),p_\alpha\bigl(g^n x,Fg^{n-1}x\bigr),p_\alpha(gu,Fu),\\
&\qquad \frac{p_\alpha(gu,Fg^{n-1}x)+p_\alpha(g^n x,Fu)}{2}\biggr\}.
\end{aligned}$$

On taking limit as $n\to\infty$, we get $p_\alpha(gu,Fu)\le rp_\alpha(gu,Fu)$ and therefore $p_\alpha(gu,Fu)=0$. By Lemma 2 we obtain $gu\in(Fu)_\alpha$. Consequently, $u=gu\in(Fu)_\alpha$. Hence, u_α is a common fixed fuzzy point of F and g. Suppose now that (g) holds. If for some $x_\alpha\in C_\alpha(F,g)$, g is *F-fuzzy weakly commuting* and $g^2x=gx$, then $gx=g^2x\in(Fgx)_\alpha$. Hence, $(gx)_\alpha$ is a common fixed fuzzy point of F and g. Suppose now that (h) holds and assume that for some $x_\alpha\in C_\alpha(F,g)$ and for some $u\in X$, $\lim_{n\to\infty}g^n u=x$ and $\lim_{n\to\infty}g^n v=y$. By the continuity of g at x and y, we get $x=gx\in(Fx)_\alpha$. The result follows. □

Example 1 Let $X=[0,1]$ be endowed with the usual metric. Let $\alpha\in(0,\frac{1}{3})$ and $r=\frac{1}{2}$, then $\sigma(r)=\frac{1}{2}$. Define a fuzzy mapping F from X into $W_\alpha(X)$ as

$$(F0)(x)=\begin{cases}1 & \text{if } x=0,\\ \alpha & \text{if } x\in(0,\frac{1}{3}],\\ \frac{\alpha}{3} & \text{if } x\in(\frac{1}{3},1]\end{cases}\quad \text{and}\quad (F1)(x)=\begin{cases}1 & \text{if } x=0,\\ 3\alpha & \text{if } x\in(0,\frac{1}{3}],\\ \frac{\alpha}{3} & \text{if } x\in(\frac{1}{3},1]\end{cases}$$

and for $z\in(0,1)$,

$$(Fz)(x)=\begin{cases}1 & \text{if } x=0,\\ \alpha & \text{if } x\in(0,\frac{1}{3}],\\ 0 & \text{if } x\in(\frac{1}{3},1].\end{cases}$$

Define a self-map $g:X\to X$ by $g(x)=x^2$. Then

$$\begin{aligned}
&(F0)_1=(F1)_1=(Fz)_1=\{0\},\qquad (F0)_\alpha=(F1)_\alpha=(Fz)_\alpha=\left[0,\frac{1}{3}\right],\\
&(F0)_{\frac{\alpha}{3}}=(F1)_{\frac{\alpha}{3}}=[0,1]\quad\text{and}\quad (Fz)_{\frac{\alpha}{3}}=\left[0,\frac{1}{3}\right].
\end{aligned}$$

Note that for all $x,y\in X$, we have

$$D_1(Fx,Fy)=H\bigl((Fx)_1,(Fy)_1\bigr)=D_\alpha(Fx,Fy)=H\bigl((Fx)_\alpha,(Fy)_\alpha\bigr)=0.$$

Also, for all $x,y\in\{0,1\}$, we have

$$D_{\frac{\alpha}{3}}(Fx,Fy)=H\bigl((Fx)_{\frac{\alpha}{3}},(Fy)_{\frac{\alpha}{3}}\bigr)=0.$$

And

$$D_{\frac{\alpha}{3}}(Fx, Fy) = H\big((Fx)_{\frac{\alpha}{3}}, (Fy)_{\frac{\alpha}{3}}\big) = 0 \quad \text{for all } x, y \in (0,1).$$

If $x \in \{0,1\}$ and $y \in (0,1)$, then $D_{\frac{\alpha}{3}}(Fx, Fy) = H((Fx)_{\frac{\alpha}{3}}, (Fy)_{\frac{\alpha}{3}}) = \frac{2}{3}$. So, for all $x, y \in X$, with $\sigma(r)p_\alpha(x, Fx) \le d(x, y)$, we have $D_\alpha(Fx, Fy) = 0$. Hence, for all $x, y \in X$,

$$D_\alpha(Fx, Fy) \le rM_\alpha(F) \quad \text{and} \quad D_\alpha(Fx, Fy) \le rM_\alpha(F, g)$$

hold true, where

$$M_\alpha(F) = \max\left\{d(x,y), p_\alpha(x, Fx), p_\alpha(y, Fy), \frac{p_\alpha(x, Fy) + p_\alpha(y, Fx)}{2}\right\}$$

and

$$M_\alpha(F, g) = \max\left\{d(gx, gy), p_\alpha(gx, Fx), p_\alpha(gy, Fy), \frac{p_\alpha(gx, Fy) + p_\alpha(gy, Fx)}{2}\right\}.$$

Hence, all the conditions of Theorem 7 are satisfied. Moreover, for each $x \in [0, \frac{1}{3}]$, we have $x_\alpha \subset F(x)$ and $(gx)_\alpha \subset F(x)$. For $\alpha = 1$, we have $\{0\} = \{g0\} \subset (F0)_1$.

4 Conclusion

The Banach contraction principle has become a classical tool to show the existence of solutions of functional equations in nonlinear analysis (see for details [23–26]). Suzuki-type fixed point theorems [10, 14] are the generalizations of the Banach contraction principle that characterize metric completeness of underlying spaces. Fuzzy sets and mappings play important roles in the process of fuzzification of systems. Suzuki-type fixed point theorems for fuzzy mappings obtained in this article can further be used in the process of finding the solutions of functional equations involving fuzzy mappings in fuzzy systems. In the main result, we not only extended the mapping to a fuzzy mapping, but also the underlying metric space has been replaced with ordered metric spaces. In this article, we defined coincidence fuzzy points and common fixed fuzzy points of the hybrid pair of a single-valued self-mapping and a fuzzy mapping and applied our main result to obtain the existence of coincidence fuzzy points and common fixed fuzzy points of the hybrid pair.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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