Demirci and Cangül *Fixed Point Theory and Applications* 2013, **2013**:77
http://www.fixedpointtheoryandapplications.com/content/2013/1/77

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

The constant term of the minimal polynomial of $\cos(2\pi/n)$ over $\mathbb{Q}$

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Abstract
Let $H(\lambda_q)$ be the Hecke group associated to $\lambda_q = 2\cos\frac{\pi}{q}$ for $q \geq 3$ integer. In this paper, we determine the constant term of the minimal polynomial of λ_q denoted by $P_q^*(x)$.

MSC: 12E05; 20H05

Keywords: Hecke groups; minimal polynomial; constant term

1 Introduction

The Hecke groups $H(\lambda)$ are defined to be the maximal discrete subgroups of $PSL(2,\mathbb{R})$ generated by two linear fractional transformations

$$T(z) = -\frac{1}{z} \quad \text{and} \quad S(z) = -\frac{1}{z+\lambda},$$

where λ is a fixed positive real number.

Hecke [1] showed that $H(\lambda)$ is Fuchsian if and only if $\lambda = \lambda_q = 2\cos\frac{\pi}{q}$ for $q \geq 3$ is an integer, or $\lambda \geq 2$. In this paper, we only consider the former case and denote the corresponding Hecke groups by $H(\lambda_q)$. It is well known that $H(\lambda_q)$ has a presentation as follows (see [2]):

$$H(\lambda_q) = \langle T, S \mid T^2 = S^q = I \rangle. \tag{1}$$

These groups are isomorphic to the free product of two finite cyclic groups of orders 2 and q.

The first few Hecke groups are $H(\lambda_3) = \Gamma = PSL(2,\mathbb{Z})$ (the modular group), $H(\lambda_4) = H(\sqrt{2})$, $H(\lambda_5) = H(\frac{1+\sqrt{5}}{2})$, and $H(\lambda_6) = H(\sqrt{3})$. It is clear from the above that $H(\lambda_q) \subset PSL(2,\mathbb{Z}[\lambda_q])$, but unlike in the modular group case (the case $q = 3$), the inclusion is strict and the index $[PSL(2,\mathbb{Z}[\lambda_q]) : H(\lambda_q)]$ is infinite as $H(\lambda_q)$ is discrete, whereas $PSL(2,\mathbb{Z}[\lambda_q])$ is not for $q \geq 4$.

On the other hand, it is well known that ζ, a primitive nth root of unity, satisfies the equation

$$x^n - 1 = 0. \tag{2}$$

In [3], Cangul studied the minimal polynomials of the real part of ζ, *i.e.*, of $\cos(2\pi/n)$ over the rationals. He used a paper of Watkins and Zeitlin [4] to produce further results.

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Also, he made use of two classes of polynomials called Chebycheff and Dickson polynomials. It is known that for $n \in \mathbb{N} \cup \{0\}$, the nth Chebycheff polynomial, denoted by $T_n(x)$, is defined by

$$T_n(x) = \cos(n \cdot \arccos x), \quad x \in \mathbb{R}, |x| \leq 1, \tag{3}$$

or

$$T_n(\cos\theta) = \cos n\theta, \quad \theta \in \mathbb{R}\ (\theta = \arccos x + 2k\pi, k \in \mathbb{Z}). \tag{4}$$

Here we use Chebycheff polynomials.

For $n \in \mathbb{N}$, Cangul denoted the minimal polynomial of $\cos(2\pi/n)$ over Q by $\Psi_n(x)$. Then he obtained the following formula for the minimal polynomial $\Psi_n(x)$.

Theorem 1 ([3, Theorem 1]) *Let $m \in \mathbb{N}$ and $n = [|m/2|]$. Then*

(a) *If $m = 1$, then $\Psi_1(x) = x - 1$, and if $m = 2$, then $\Psi_2(x) = x + 1$.*

(b) *If m is an odd prime, then*

$$\Psi_m(x) = \frac{T_{n+1}(x) - T_n(x)}{2^n(x-1)}. \tag{5}$$

(c) *If $4 \mid m$, then*

$$\Psi_m(x) = \frac{T_{n+1}(x) - T_{n-1}(x)}{2^{n/2}(T_{\frac{n}{2}+1}(x) - T_{\frac{n}{2}-1}(x)) \prod_{d|m,d\neq m,d|\frac{m}{2}}^{q-1} \Psi_d(x)}. \tag{6}$$

(d) *If m is even and $m/2$ is odd, then*

$$\Psi_m(x) = \frac{T_{n+1}(x) - T_{n-1}(x)}{2^{n-n'}(T_{n'+1}(x) - T_{n'}(x)) \prod_{d|m,d\neq m,d\ even}^{q-1} \Psi_d(x)}, \tag{7}$$

where $n' = \frac{\frac{m}{2}-1}{2}$.

(e) *Let m be odd and let p be a prime dividing m. If $p^2 \mid m$, then*

$$\Psi_m(x) = \frac{T_{n+1}(x) - T_n(x)}{2^{n-n'}(T_{n'+1}(x) - T_{n'}(x))}, \tag{8}$$

where $n' = \frac{\frac{m}{p}-1}{2}$. If $p^2 \mid m$, then

$$\Psi_m(x) = \frac{T_{n+1}(x) - T_n(x)}{2^{n-n'}(T_{n'+1}(x) - T_{n'}(x))\Psi_p(x)}, \tag{9}$$

where $n' = \frac{\frac{m}{p}-1}{2}$.

For the first four Hecke groups Γ, $H(\sqrt{2})$, $H(\lambda_5)$, and $H(\sqrt{3})$, we can find the minimal polynomial, denoted by $P_q^*(x)$, of λ_q over Qas $\lambda_3 - 1$, $\lambda_4^2 - 2$, $\lambda_5^2 - \lambda_5 - 1$, and $\lambda_6^2 - 3$, respectively. However, for $q \geq 7$, the algebraic number $\lambda_q = 2\cos\frac{\pi}{q}$ is a root of a minimal polynomial of degree ≥ 3. Therefore, it is not possible to determine λ_q for $q \geq 7$ as nicely as in the first four cases. Because of this, it is easy to find and study with the minimal polynomial of λ_q instead of λ_q itself. The minimal polynomial of λ_q has been used for many aspects in the literature (see [5–8] and [9]).

Notice that there is a relation

$$P_q^*(x) = 2^{\varphi(2q)/2} \cdot \Psi_{2q}\left(\frac{x}{2}\right)$$

between $P_q^*(x)$ and $\Psi_m(x)$.

In [10], when the principal congruence subgroups of $H(\lambda_q)$ for $q \geq 7$ prime were studied, we needed to know whether the minimal polynomial of λ_q is congruent to 0 *modulo p* for prime p and also the constant term of it *modulo p*.

In this paper, we determine the constant term of the minimal polynomial $P_q^*(x)$ of λ_q. We deal with odd and even q cases separately. Of course, this problem is easier to solve when q is odd.

2 The constant term of $P_q^*(x)$

In this section, we calculate the constant term for all values of q. Let c denote the constant term of the minimal polynomial $P_q^*(x)$ of λ_q, *i.e.*,

$$c = P_q^*(0). \tag{10}$$

We know from [4, Lemma, p.473] that the roots of $P_q^*(x)$ are $2\cos\frac{h\pi}{q}$ with $(h,q) = 1$, h odd and $1 \leq h \leq q-1$. Being the constant term, c is equal to the product of all roots of $P_q^*(x)$:

$$c = \prod_{\substack{h=1 \\ (h,q)=1 \\ h \text{ odd}}}^{q-1} 2\cos\frac{h\pi}{q}. \tag{11}$$

Therefore we need to calculate the product on the right-hand side of (11). To do this, we need the following result given in [11].

Lemma 1 $\prod_{h=0}^{q-1} 2\sin(\frac{h\pi}{q} + \theta) = 2\sin q\theta$.

We now want to obtain a similar formula for cosine. Replacing θ by $\frac{\pi}{2} - \theta$, we get

$$\prod_{h=0}^{q-1} 2\cos\left(\frac{h\pi}{q} - \theta\right) = 2\sin q\left(\frac{\pi}{2} - \theta\right). \tag{12}$$

Let now μ denote the Möbius function defined by

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ is not square-free,} \\ 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n \text{ has } k \text{ distinct prime factors,} \end{cases} \tag{13}$$

for $n \in \mathbb{N}$. It is known that

$$\sum_{d|n} \mu(d) = \begin{cases} 0 & \text{if } n > 1, \\ 1 & \text{if } n = 1. \end{cases} \tag{14}$$

Using this last fact, we obtain

$$\begin{aligned} \ln & \prod_{h=0,(h,q)=1}^{q-1} 2\cos\left(\frac{h\pi}{q} - \theta\right) \\ &= \sum_{h=0}^{q-1} \ln\left(2\cos\left(\frac{h\pi}{q} - \theta\right)\right) \sum_{d|(h,q)} \mu(d) \\ &= \sum_{d|q} \mu(d) \sum_{k=0}^{\frac{q}{d}-1} \ln\left(2\cos\left(\frac{kd\pi}{q} - \theta\right)\right) \\ &= \sum_{d|q} \mu(d) \left(\ln \prod_{k=0}^{\frac{q}{d}-1} 2\cos\left(\frac{kd\pi}{q} - \theta\right)\right) \\ &= \sum_{d|q} \mu(d) \cdot \left(\ln 2 \sin \frac{q}{d}\left(\frac{\pi}{2} - \theta\right)\right) \quad \text{by (12)} \\ &= \ln \prod_{d|q} \sin d\left(\frac{\pi}{2} - \theta\right)^{\mu(q/d)}. \end{aligned} \tag{15}$$

Therefore

$$\prod_{\substack{h=0 \\ (h,q)=1}}^{q-1} 2\cos\left(\frac{h\pi}{q} - \theta\right) = \prod_{d|q} \left(\sin d\left(\frac{\pi}{2} - \theta\right)\right)^{\mu(q/d)}. \tag{16}$$

Finally, as $(0,q) \neq 1$, we can write (16) as

$$\prod_{\substack{h=1 \\ (h,q)=1}}^{q-1} 2\cos\left(\frac{h\pi}{q} - \theta\right) = \prod_{d|q} \left(\sin d\left(\frac{\pi}{2} - \theta\right)\right)^{\mu(q/d)}. \tag{17}$$

Note that if q is even, then

$$\prod_{\substack{h=1 \\ (h,q)=1}}^{q-1} 2\cos\left(\frac{h\pi}{q}\right) = \prod_{\substack{h=1 \\ (h,q)=1 \\ h \text{ odd}}}^{q-1} 2\cos\frac{h\pi}{q} = c, \tag{18}$$

while if q is odd, then

$$\left| \prod_{\substack{h=1 \\ (h,q)=1}}^{q-1} 2\cos\left(\frac{h\pi}{q}\right) \right| = c^2, \tag{19}$$

as $\cos(h-i)\frac{\pi}{q} = -\cos\frac{i\pi}{q}$. Also note that

$$\sin d\left(\frac{\pi}{2}-\theta\right) = \begin{cases} \cos d\theta & \text{if } d \equiv 1 \bmod 4, \\ \sin d\theta & \text{if } d \equiv 2 \bmod 4, \\ -\cos d\theta & \text{if } d \equiv 3 \bmod 4, \\ -\sin d\theta & \text{if } d \equiv 0 \bmod 4. \end{cases} \tag{20}$$

To compute c, we let $\theta \to 0$ in (17). If d is odd, then $\sin d(\frac{\pi}{2}-\theta) \to \pm 1$ as $\theta \to 0$ by (20). So, we are only concerned with even d. Indeed, if q is odd, then the left-hand side at $\theta = 0$ is equal to ± 1. Therefore we have the following result.

Theorem 2 *Let q be odd. Then*

$$|c| = 1. \tag{21}$$

Proof It follows from (19) and (20). □

Let us now investigate the case of even q. As $(h,q) = 1$, h must be odd. So, by a similar discussion, we get the following.

Theorem 3 *Let q be even. Then*

$$c = \lim_{\theta \to 0} \prod_{d|q} \left(\sin d\left(\frac{\pi}{2}-\theta\right)\right)^{\mu(q/d)}. \tag{22}$$

Proof Note that by (20), the right-hand side of (22) becomes a product of $\pm(\cos d\theta)^{\pm 1}$'$s$ and $\pm(\sin d\theta)^{\pm 1}$'$s$. Above we saw that we can omit the former ones as they tend to ± 1 as θ tends to 0. Now, as $\sum_{d|n} \mu(d) = 0$, there are equal numbers of the latter kind factors in the numerator and denominator, *i.e.*, if there is a factor $\sin d\theta$ in the numerator, then there is a factor $\sin d'\theta$ in the denominator. Then using the fact that

$$\lim_{\theta \to 0} \frac{\sin k\theta}{\sin l\theta} = \frac{k}{l}, \tag{23}$$

we can calculate c.

In fact the calculations show that there are three possibilities:

(i) Let $q = 2^{\alpha_0}$, $\alpha_0 \geq 2$. Then the only divisors of q such that $\mu(q/d) \neq 0$ are $d = 2^{\alpha_0}$ and $2^{\alpha_0 - 1}$. Therefore

$$\begin{aligned} c &= \lim_{\theta \to 0} \frac{\sin 2^{\alpha_0}(\frac{\pi}{2}-\theta)}{\sin 2^{\alpha_0-1}(\frac{\pi}{2}-\theta)} \\ &= \begin{cases} 2 & \text{if } \alpha_0 > 2, \\ -2 & \text{if } \alpha_0 = 2. \end{cases} \end{aligned} \tag{24}$$

(ii) Secondly, let $q = 2p^{\alpha}$, $\alpha \geq 1$, p odd prime. Then the only divisors of q such that $\mu(q/d) \neq 0$ are $d = 2p^{\alpha}, 2p^{\alpha-1}, p^{\alpha}$ and $p^{\alpha-1}$. Therefore

$$\begin{aligned} c &= \lim_{\theta\to 0} \frac{\sin 2p^{\alpha}(\frac{\pi}{2}-\theta)\cdot \sin p^{\alpha-1}(\frac{\pi}{2}-\theta)}{\sin p^{\alpha}(\frac{\pi}{2}-\theta)\cdot \sin 2p^{\alpha-1}(\frac{\pi}{2}-\theta)} \\ &= \lim_{\theta\to 0} \epsilon \cdot \frac{\sin 2p^{\alpha}\theta \cdot \cos p^{\alpha-1}\theta}{\cos p^{\alpha}\theta \cdot \sin 2p^{\alpha-1}\theta} \\ &= \epsilon \cdot p, \end{aligned} \tag{25}$$

where

$$\epsilon = \begin{cases} 1 & \text{if } p \equiv 1 \bmod 4, \\ -1 & \text{if } p \equiv -1 \bmod 4. \end{cases} \tag{26}$$

(iii) Let q be different from above. Then q can be written as

$$q = 2^{\alpha_0} p_1^{\alpha_1} \cdots p_k^{\alpha_k}, \tag{27}$$

where p_i are distinct odd primes and $\alpha_i \geq 1$, $0 \leq i \leq k$.

Here we consider the first two cases $k = 1$ and $k = 2$.

Let $k = 1$, *i.e.*, let $q = 2^{\alpha_0} p_1^{\alpha_1}$. We have already discussed the case $\alpha_0 = 1$. Let $\alpha_0 > 1$. Then the only divisors d of q with $\mu(q/d) \neq 0$ are $d = 2^{\alpha_0} p_1^{\alpha_1}, 2^{\alpha_0-1} p_1^{\alpha_1}, 2^{\alpha_0} p_1^{\alpha_1-1}$ and $2^{\alpha_0-1} p_1^{\alpha_1-1}$. Therefore

$$\begin{aligned} c &= \lim_{\theta\to 0} \frac{\sin 2^{\alpha_0} p_1^{\alpha_1}(\frac{\pi}{2}-\theta)\cdot \sin 2^{\alpha_0-1} p_1^{\alpha_1-1}(\frac{\pi}{2}-\theta)}{\sin 2^{\alpha_0-1} p_1^{\alpha_1}(\frac{\pi}{2}-\theta)\cdot \sin 2^{\alpha_0} p_1^{\alpha_1-1}(\frac{\pi}{2}-\theta)} \\ &= 1. \end{aligned} \tag{28}$$

Now let $k = 2$, *i.e.*, let $q = 2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2}$, $(p_1 < p_2)$. Similarly, all divisors d of q such that $\mu(q/d) \neq 0$ are $d = 2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2}$, $2^{\alpha_0-1} p_1^{\alpha_1} p_2^{\alpha_2}$, $2^{\alpha_0} p_1^{\alpha_1-1} p_2^{\alpha_2}$, $2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2-1}$, $2^{\alpha_0} p_1^{\alpha_1-1} p_2^{\alpha_2-1}$, $2^{\alpha_0-1} p_1^{\alpha_1-1} p_2^{\alpha_2-1}$, $2^{\alpha_0-1} p_1^{\alpha_1} p_2^{\alpha_2-1}$ and $2^{\alpha_0-1} p_1^{\alpha_1-1} p_2^{\alpha_2}$. Therefore

$$\begin{aligned} c &= \lim_{\theta\to 0} \frac{\sin 2^{\alpha_0-1} p_1^{\alpha_1-1} p_2^{\alpha_2-1}(\frac{\pi}{2}-\theta)\cdot \sin 2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2-1}(\frac{\pi}{2}-\theta)}{\sin 2^{\alpha_0} p_1^{\alpha_1-1} p_2^{\alpha_2-1}(\frac{\pi}{2}-\theta)\cdot \sin 2^{\alpha_0-1} p_1^{\alpha_1} p_2^{\alpha_2-1}(\frac{\pi}{2}-\theta)} \\ &\quad \times \lim_{\theta\to 0} \frac{\sin 2^{\alpha_0} p_1^{\alpha_1-1} p_2^{\alpha_2}(\frac{\pi}{2}-\theta)\cdot \sin 2^{\alpha_0-1} p_1^{\alpha_1} p_2^{\alpha_2}(\frac{\pi}{2}-\theta)}{\sin 2^{\alpha_0-1} p_1^{\alpha_1-1} p_2^{\alpha_2}(\frac{\pi}{2}-\theta)\cdot \sin 2^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2}(\frac{\pi}{2}-\theta)} \\ &= 1. \end{aligned} \tag{29}$$

Finally, $k \geq 3$, i.e., let

$$q = 2^{\alpha_0} p_1^{\alpha_1} \cdots p_k^{\alpha_k} \quad \text{with } p_1 < p_2 < \cdots < p_k.$$

In this case the proof is similar, but rather more complicated. In fact, the number of all divisors d of q such that $\mu(q/d) \neq 0$ is 2^{k+1}. There is $\binom{k+1}{0} = 1$ divisor of the form

$$d = 2^{\alpha_0} p_1^{\alpha_1} \cdots p_k^{\alpha_k}.$$

There are $\binom{k+1}{1} = k+1$ divisors of the form

$$d = 2^{\alpha_0-1}p_1^{\alpha_1}\cdots p_k^{\alpha_k}, 2^{\alpha_0}p_1^{\alpha_1-1}\cdots p_k^{\alpha_k},\dots,2^{\alpha_0}p_1^{\alpha_1}\cdots p_k^{\alpha_k-1}.$$

There are $\binom{k+1}{2} = \frac{k(k+1)}{2}$ divisors of the form

$$\begin{aligned} d &= 2^{\alpha_0-1}p_1^{\alpha_1-1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}, 2^{\alpha_0-1}p_1^{\alpha_1}p_2^{\alpha_2-1}\cdots p_k^{\alpha_k},\dots,2^{\alpha_0-1}p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k-1}, \\ &\quad 2^{\alpha_0}p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_k^{\alpha_k},\dots,2^{\alpha_0}p_1^{\alpha_1-1}p_2^{\alpha_2}\cdots p_k^{\alpha_k-1},\dots,2^{\alpha_0}p_1^{\alpha_1}\cdots p_{k-1}^{\alpha_{k-1}-1}p_k^{\alpha_k-1}. \end{aligned}$$

If we continue, we can find other divisors d of q, similarly. Finally, there is $\binom{k+1}{k+1} = 1$ divisor of the form $2^{\alpha_0-1}p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_k^{\alpha_k-1}$. Thus, the product of all coefficients d in the factors $\sin d(\frac{\pi}{2}-\theta)$ in the numerator is equal to the product of all coefficients e in the factors $\sin e(\frac{\pi}{2}-\theta)$ in the denominator implying $c = 1$. Therefore the proof is completed. □

Now we give an example for all possible even q cases.

Example 1 (i) Let $q = 8 = 2^3$. The only divisors of 8 such that $\mu(8/d) \neq 0$ are $d = 8$ and 4. Therefore

$$\begin{aligned} c &= \lim_{\theta\to 0}\frac{\sin 8(\frac{\pi}{2}-\theta)}{\sin 4(\frac{\pi}{2}-\theta)} \\ &= 2. \end{aligned}$$

(ii) Let $q = 14 = 2\cdot 7$. The only divisors of 14 such that $\mu(14/d) \neq 0$ are $d = 14, 2, 7$ and 1. Therefore

$$\begin{aligned} c &= \epsilon\cdot\lim_{\theta\to 0}\frac{\sin 14(\frac{\pi}{2}-\theta)\cdot\sin(\frac{\pi}{2}-\theta)}{\sin 7(\frac{\pi}{2}-\theta)\cdot\sin 2(\frac{\pi}{2}-\theta)} \\ &= -7, \end{aligned}$$

since $p \equiv -1 \bmod 4$.

(iii) Let $q = 24 = 2^3\cdot 3$. The only divisors of 24 such that $\mu(24/d) \neq 0$ are $d = 24, 12, 8$ and 4. Therefore

$$\begin{aligned} c &= \lim_{\theta\to 0}\frac{\sin 24(\frac{\pi}{2}-\theta)\cdot\sin 4(\frac{\pi}{2}-\theta)}{\sin 12(\frac{\pi}{2}-\theta)\cdot\sin 8(\frac{\pi}{2}-\theta)} \\ &= 1. \end{aligned}$$

(iv) Let $q = 30 = 2\cdot 3\cdot 5$. The only divisors of 30 such that $\mu(30/d) \neq 0$ are $d = 30, 15, 10, 6, 5, 3, 2$ and 1. Therefore

$$\begin{aligned} c &= \lim_{\theta\to 0}\frac{\sin(\frac{\pi}{2}-\theta)\cdot\sin 6(\frac{\pi}{2}-\theta)\cdot\sin 10(\frac{\pi}{2}-\theta)\cdot\sin 15(\frac{\pi}{2}-\theta)}{\sin 2(\frac{\pi}{2}-\theta)\cdot\sin 3(\frac{\pi}{2}-\theta)\cdot\sin 5(\frac{\pi}{2}-\theta)\cdot\sin 30(\frac{\pi}{2}-\theta)} \\ &= 1. \end{aligned}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors completed the paper alone and they read and approved the final manuscript.

Acknowledgements
Dedicated to Professor Hari M Srivastava.
Both authors are supported by the Scientific Research Fund of Uludag University under the project number F2012/15 and the second author is supported under F2012/19.

Received: 21 January 2013 Accepted: 11 March 2013 Published: 29 March 2013

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doi:10.1186/1687-1812-2013-77
Cite this article as: Demirci and Cangül: **The constant term of the minimal polynomial of** $\cos(2\pi/n)$ **over** $\mathbb{Q}$. *Fixed Point Theory and Applications* 2013 **2013**:77.

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