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Some new results for single-valued and multi-valued mixed monotone operators of Rhoades type

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Abstract
In (2008), Zhang proved the existence of fixed points of mixed monotone operators along with certain convexity and concavity conditions. In this paper, mixed monotone single-valued and multi-valued operators of Rhoades type are defined and two fixed point theorems are proved.
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1 Introduction and preliminaries

In (1987), mixed monotone operators were introduced by Guo and Lakshmikantham [1]. Then many authors studied them in Banach spaces and obtained lots of interesting results (see [2, 3] and [4–8]).

On the other hand, in (2001), Rhoades [9] introduced a new fixed point theorem as a generalization of Banach fixed point theorem.

Theorem 1.1 (Rhoades [9]) *Let (X,d) be a complete metric space. Suppose that $T:X\to X$ is a single-valued mapping that satisfies*

$$d(Tx,Ty)\le d(x,y)-\psi\bigl(d(x,y)\bigr) \tag{1}$$

for each $x,y\in X$, where $\psi:[0,+\infty)\to[0,+\infty)$ is continuous, nondecreasing and $\psi^{-1}(0)=\{0\}$ (i.e., weakly contractive mappings). Then T has a fixed point.

In this paper, a weak mixed monotone single-valued and multi-valued operator of Rhoades type is defined. Then two fixed point theorems for this kind of operators are proved.

Let E be a real Banach space. The zero element of E is denoted by θ. A subset P of E is called a cone if and only if:

- P is closed, nonempty and $P\neq\{\theta\}$,
- $a,b\in\mathbb{R}$, $a,b\ge 0$ and $x,y\in P$ imply that $ax+by\in P$,
- $x\in P$ and $-x\in P$ imply that $x=\theta$.

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Given a cone $P \subset E$, a partial ordering $\leq$ with respect to P is defined by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ stands for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. The cone P is called normal if there exists a number $K > 0$ such that $\theta \leq x \leq y$ implies $\|x\| \leq K\|y\|$ for every $x, y \in E$. The least positive number satisfying this is called the normal constant of P.

Assume that $u_0, v_0 \in E$ and $u_0 \leq v_0$. The set $\{x \in E : u_0 \leq x \leq v_0\}$ is denoted by $[u_0, v_0]$.

Now, we recall the following definitions from [2, 3].

Definition 1.1 Let P be a cone of a real Banach space E. Suppose that $D \subset P$ and $\alpha \in (-\infty, +\infty)$. An operator $A : D \to D$ is said to be α-convex (α-concave) if it satisfies $A(tx) \leq t^\alpha Ax$ ($A(tx) \geq t^\alpha Ax$) for $(t, x) \in (0,1) \times D$.

Definition 1.2 Let E be an ordered Banach space and $D \subset E$. An operator is called mixed monotone on $D \times D$ if $A : D \times D \to E$ and $A(x_1, y_1) \leq A(x_2, y_2)$ for any $x_1, x_2, y_1, y_2 \in D$, where $x_1 \leq x_2$ and $y_2 \geq y_1$. Also, $x^* \in D$ is called a fixed point of A if $A(x^*, x^*) = x^*$.

Let $\mathcal{C}(E)$ be a collection of all closed subsets of E.

Definition 1.3 For two subsets X, Y of E, we write

- $X \preceq Y$ if for all $x \in X$, there exists $y \in Y$ such that $x \leq y$,
- $x \prec X$ if there exists $z \in X$ such that $x \ll z$,
- $X \prec x$ if for all $z \in X$, $z \ll x$.

Definition 1.4 Let D be a nonempty subset of E. $T : D \to \mathcal{C}(E)$ is called increasing (decreasing) upward if $u, v \in D$, $u \leq v$ and $x \in T(u)$ imply there exists $y \in T(v)$ such that $x \leq y$ ($x \geq y$). Similarly, $T : D \to \mathcal{C}(E)$ is called increasing (decreasing) downward if $u, v \in D$, $u \leq v$ and $y \in T(v)$ imply there exists $x \in T(u)$ such that $x \leq y$ ($x \geq y$). T is called increasing (decreasing) if T is an increasing (decreasing) upward and downward.

Definition 1.5 Let D be a nonempty subset of E. A multi-valued operator $T : D \times D \to \mathcal{C}(E)$ is said to be mixed monotone upward if $T(x, y)$ is increasing upward in x and decreasing upward in y, *i.e.*,

(A_1) for each $y \in D$ and any $x_1, x_2 \in D$ with $x_1 \leq x_2$, if $u_1 \in T(x_1, y)$, then there exists a $u_2 \in T(x_2, y)$ such that $u_1 \leq u_2$;

(A_2) for each $x \in D$ and any $y_1, y_2 \in D$ with $y_1 \leq y_2$, if $v_1 \in T(x, y_1)$, then there exists a $v_2 \in T(x, y_2)$ such that $v_1 \geq v_2$.

Definition 1.6 $x^* \in D$ is called a fixed point of T if $x^* \in T(x^*, x^*)$.

Definition 1.7 [10] A function $\Psi : [0,1) \times P \times P \times E \to E$ is called an $\mathcal{L}''$-function if $\Psi(t, x, y, 0) = 0$, $\Psi(t, x, y, s) \gg 0$ for $s \gg 0$, and $\Psi(t, x, y, z) < z$ for all $(t, x, y, z) \in [0,1) \times P \times P \times E$.

In 2011, Khojasteh and Razani [10] extended the results given by Zhang [6]. Also, in 2011 Khojasteh and Razani [11] introduced the concept of integral with respect to a cone. We recall the following definitions and lemmas of cone integration and refer to [11, 12] for their proofs.

Definition 1.8 [11] Suppose that P is a cone in E. Let $a, b \in E$ and $a < b$. Define

$$[a,b] := \{x \in E : x = tb + (1-t)a \text{ for some } t \in [0,1]\} \tag{2}$$

and

$$[a,b) := \{x \in E : x = tb + (1-t)a \text{ for some } t \in [0,1)\}. \tag{3}$$

Definition 1.9 [11] The set $\{a = x_0, x_1, \dots, x_n = b\}$ is called a partition for $[a,b]$ if and only if the intervals $\{[x_{i-1}, x_i)\}_{i=1}^n$ are pairwise disjoint and $[a,b] = \{\bigcup_{i=1}^n [x_{i-1}, x_i)\} \cup \{b\}$. Denote $\mathcal{P}[a,b]$ as the collection of all partitions of $[a,b]$.

Definition 1.10 [12] For each partition Q of $[a,b]$ and each increasing function $\phi : [a,b] \to E$, we define cone lower summation and cone upper summation as

$$L_n^{\mathrm{Con}}(\phi, Q) = \sum_{i=0}^{n-1} \phi(x_i) \|x_i - x_{i+1}\| \tag{4}$$

and

$$U_n^{\mathrm{Con}}(\phi, Q) = \sum_{i=0}^{n-1} \phi(x_{i+1}) \|x_i - x_{i+1}\|, \tag{5}$$

respectively. Also, we denote $\|\Delta(Q)\| = \sup\{\|x_i - x_{i-1}\|, x_i \in Q\}$.

Definition 1.11 [12] Suppose that P is a cone in E. $\phi : [a,b] \to E$ is called an integrable function on $[a,b]$ with respect to a cone P or, to put it simply, a cone integrable function if and only if for all partition Q of $[a,b]$,

$$\lim_{\|\Delta(Q)\| \to 0} L_n^{\mathrm{Con}}(\phi, Q) = S^{\mathrm{Con}} = \lim_{\|\Delta(Q)\| \to 0} U_n^{\mathrm{Con}}(\phi, Q),$$

where S^{Con} must be unique.

We show the common value S^{Con} by

$$\int_a^b \phi(x)\, d_P(x) \quad \text{or to simplicity} \quad \int_a^b \phi\, d_p.$$

We denote the set of all cone integrable functions $\phi : [a,b] \to E$ by $\mathcal{L}^1([a,b], E)$.

Lemma 1.1 [11] *Let M be a subset of P. The following conditions hold*:

(1) *If $[a,b] \subseteq [a,c] \subset M$, then $\int_a^b f\, d_p \leq \int_a^c f\, d_p$ for $f \in \mathcal{L}^1(M,P)$.*
(2) *$\int_a^b (\alpha f + \beta g)\, d_p = \alpha \int_a^b f\, d_p + \beta \int_a^b g\, d_p$ for $f, g \in \mathcal{L}^1(M,P)$ and $\alpha, \beta \in \mathrm{R}$.*

Remark 1.1 [13, Remark 1.2] Let P be a cone of E, and let $u \in P$. If for each $\epsilon \in \mathrm{int}(P)$, $0 \leq u \ll \epsilon$, then $u = 0$.

2 Main results

In this section, we introduce some new fixed point theorems in the class of mixed monotone operators. Due to this, the following definition is presented.

Definition 2.1 A mixed monotone operator $A : D \times D \to E$ is said to be a Weak Mixed Monotone single-valued operator of Rhoades type ($\mathrm{WM_2R}$ property for short) if

$$A(tx,y) \le A(x,ty) - \Psi\big(t,x,y,A(x,ty)\big) \tag{6}$$

for all $(x,y) \in D \times D$, where $\Psi : [0,1) \times P \times P \times E \to E$ is an $\mathcal{L}''$-function.

Theorem 2.1 *Let P be a cone of E, let S be a completely ordered closed subset of E with $S_0 = S\backslash\{\theta\} \subset \operatorname{int} P$ and let $\lambda S \subset S$ for all $\lambda \in [0,1]$. Let $u_0, v_0 \in S_0$, $A : P \times P \to E$ be a weak mixed monotone operator of Rhoades type with $A(([\theta,v_0] \cap S) \times ([\theta,v_0] \cap S)) \subset S$ satisfying the following conditions*:

(I) *there exists $r_0 > 0$ such that $u_0 \ge r_0 v_0$*,

(II) $A(u_0,v_0) \ll u_0 \ll v_0 \ll A(v_0,u_0)$,

(III) *for $u,v \in [u_0,v_0] \cap S$ with $A(u,v) \ll u \ll v$, there exists $u' \in S$ such that $u \le A(u',v) \ll u' \ll v$; similarly, for $u,v \in [u_0,v_0] \cap S$ with $u \ll v \ll A(v,u)$, there exists $v' \in S$ such that $u \ll v' \ll A(v',u) \le v$.*

Then A has at least one fixed point $x^* \in [u_0,v_0] \cap S$.

Proof By the above condition (III), there exists $u_1 \in S$ such that $u_0 \le A(u_1,v_0) \ll u_1 \ll v_0$. Then there exists $v_1 \in S$ such that $u_1 \ll v_1 \ll A(v_1,u_1) \le v_0$. Likewise, there exists $u_2 \in S$ such that $u_1 \le A(u_2,v_1) \ll u_2 \ll v_1$. Then there exists $v_2 \in S$ such that $u_2 \ll v_2 \ll A(v_2,u_2) \le v_1$. In general, there exists $u_n \in S$ such that $u_{n-1} \le A(u_n,v_{n-1}) \ll u_n \ll v_{n-1}$. Then there exists $v_n \in S$ such that $u_n \ll v_n \ll A(v_n,u_n) \le v_{n-1}$ $(n = 1,2,\ldots)$.

Take $r_n = \sup\{r \in (0,1) : u_n \ge r v_n\}$, thus $0 < r_0 < r_1 < \cdots < r_n < r_{n+1} < \cdots < 1$ and $\lim_{n\to\infty} r_n = \sup\{r_n : n = 0,1,2,\ldots\} = r^* \in (0,1]$. Since $r_{n+1} > r_n = \sup\{r \in (0,1) : u_n \ge r v_n\}$, thus $u_n \ngeq r_{n+1} v_n$. In addition, S is completely ordered and $\lambda S \subset S$ for all $\lambda \in [0,1]$, then $u_n < r_{n+1} v_n$. Now, one can prove $r^* = 1$. Otherwise, $r^* \in (0,1)$.

Since $u_n < r_{n+1} v_n$ and $r_{n+1} < r^*$, hence $u_n < r^* v_n$, and we have

$$\begin{aligned} A(u_{n+1},v_{n+1}) &\le A\left(\frac{1}{r^*}u_{n+1}, r^* v_{n+1}\right) \\ &\le A(u_{n+1},v_{n+1}) - \Psi\left(r^*, \frac{1}{r^*}u_{n+1}, v_{n+1}, A(u_{n+1},v_{n+1})\right) \\ &< A(u_{n+1},v_{n+1}), \end{aligned} \tag{7}$$

which is a contradiction. Thus, $r^* = 1$. Let $\epsilon \gg 0$ be given. Choose $\delta > 0$ such that $\epsilon + N_\delta(0) \subseteq P$, where $N_\delta(0) = \{y \in E : \|y\| < \delta\}$. Since $r_n \to 1$, one can choose a natural number N_1 such that $(1-r_n)v_1 \in N_\delta(0)$ for all $n \ge N_1$. Therefore $(1-r_n)v_1 \ll \epsilon$. Also, $v_n \le v_1$ and

$$0 < v_n - u_n \le (1-r_n)v_n \le (1-r_n)v_1 \ll \epsilon. \tag{8}$$

By Remark 1.1, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n$.

For all $n, p \geq 1$, applying the same argument, we have

$$0 < v_n - v_{n+p} \leq v_n - u_n \ll \epsilon. \tag{9}$$

Also,

$$0 < u_{n+p} - u_n \leq v_n - u_n \ll \epsilon. \tag{10}$$

Hence, $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences in E, then there exist $u^*, v^* \in E$ such that $u_n \to u^*$, $v_n \to v^*$ $(n \to \infty)$ and $u^* = v^*$. Write $x^* = u^* = v^*$.

It is easy to see $u_0 \leq u_n \leq u^* \leq v_n \leq v_0$ for all $n = 1, 2, \ldots$. In addition, S is closed, then $u^* \in [u_n, v_n] \cap S \subset [u_0, v_0] \cap S$ $(n = 0, 1, 2, \ldots)$.

Finally, by the mixed monotone property of A,

$$u_{n-1} \leq A(u_n, v_n) \leq A(x^*, x^*) \leq A(u_n, v_n) \leq u_{n-1}. \tag{11}$$

On taking limit on both sides of (11), when $n \to \infty$, we have $A(x^*, x^*) = x^*$. This means x^* is a fixed point of A in $[u_0, v_0] \cap S$. □

Corollary 2.1 *Let P be a cone of E, let S be a completely ordered closed subset of E with $S_0 = S \backslash \{\theta\} \subset \operatorname{int} P$ and let $\lambda S \subset S$ for all $\lambda \in [0, 1]$. Let $u_0, v_0 \in S_0$, $A : P \times P \to E$ satisfy*

$$\int_y^{tx} \phi \, d_P \leq \int_{ty}^{x} \phi \, d_P - \Psi\left(t, x, y, \int_{ty}^{x} \phi \, d_P\right) \tag{12}$$

for all $(x, y) \in D \times D$, where $\Psi : [0, 1) \times P \times P \times E \to E$ is an $\mathcal{L}''$-function, and let $\phi : P \to P$ be a non-vanishing, cone integrable mapping on each $[a, b] \subset P$ such that for each $\epsilon \gg 0$, $\int_0^{\epsilon} \phi \, d_p \gg 0$ and the mapping $\theta(x) = \int_0^x \phi \, d_P$ for $(x \geq 0)$ has a continuous inverse at zero. Also, $A(([\theta, v_0] \cap S) \times ([\theta, v_0] \cap S)) \subset S$ satisfies the following conditions:

(I) *there exists $r_0 > 0$ such that $u_0 \geq r_0 v_0$,*
(II) $A(u_0, v_0) \ll u_0 \ll v_0 \ll A(v_0, u_0)$,
(III) *for $u, v \in [u_0, v_0] \cap S$ with $A(u, v) \ll u \ll v$, there exists $u' \in S$ such that $u \leq A(u', v) \ll u' \ll v$; similarly, for $u, v \in [u_0, v_0] \cap S$ with $u \ll v \ll A(v, u)$, there exists $v' \in S$ such that $u \ll v' \ll A(v', u) \leq v$.*

Then A has at least one fixed point $x^* \in [u_0, v_0] \cap S$.

Proof Define

$$A(x, y) = \int_y^x \phi \, d_P.$$

A is a mixed monotone operator, and one can easily see that all conditions of Theorem 2.1 hold. Thus we obtain the desired result. □

3 $\mathbf{M_3R}$ property

In this section, we introduce a new fixed point theorem in the class of multi-valued mixed monotone operators. Due to this, the following definition is given.

Definition 3.1 A mixed monotone operator $T : D\times D \to \mathcal{C}(E)$ is said to be a Mixed Monotone Multi-valued operator of Rhoades type (M_3R property for short) if

$$T(tx,y) \preceq T(x,ty) - \Psi\bigl(t,x,y,T(tx,y)\bigr) \tag{13}$$

for each $(x,y) \in D \times D$, where $\Psi : [0,1) \times P \times P \times E \to E$ is an $\mathcal{L}''$-function.

Theorem 3.1 *Let P be a cone of E, let S be a completely ordered closed subset of E with $S_0 = S\backslash\{\theta\} \subset \operatorname{int} P$ and let $\lambda S \subset S$ for all $\lambda \in [0,1]$. Let $u_0, v_0 \in S_0$, $T : P \times P \to \mathcal{C}(E)$ be a mixed monotone multi-valued operator of Rhoades type with $T(([\theta,v_0] \cap S) \times ([\theta,v_0] \cap S)) \subset S$ satisfying the following conditions*:

(I) *there exists $r_0 > 0$ such that $u_0 \geq r_0 v_0$,*

(II) $T(u_0,v_0) \prec u_0 \ll v_0 \prec T(v_0,u_0)$,

(III) *for $u,v \in [u_0,v_0] \cap S$ with $T(u,v) \prec u \ll v$, there exists $u' \in S$ such that $u \preceq T(u',v) \prec u' \ll v$; similarly, for $u,v \in [u_0,v_0] \cap S$ with $u \ll v \prec T(v,u)$, there exists $v' \in S$ such that $u \ll v' \prec T(v',u) \preceq v$.*

Then T has at least one fixed point $x^* \in [u_0,v_0] \cap S$.

Proof By the above condition (III), there exists $u_1 \in S$ such that $u_0 \preceq T(u_1,v_0) \prec u_1 \ll v_0$. Then there exists $v_1 \in S$ such that $u_1 \ll v_1 \prec T(v_1,u_1) \preceq v_0$. Likewise, there exists $u_2 \in S$ such that $u_1 \preceq T(u_2,v_1) \prec u_2 \ll v_1$. Then there exists $v_2 \in S$ such that $u_2 \ll v_2 \prec T(v_2,u_2) \preceq v_1$. In general, there exists $u_n \in S$ such that $u_{n-1} \preceq T(u_n,v_{n-1}) \prec u_n \ll v_{n-1}$. Then there exists $v_n \in S$ such that $u_n \ll v_n \prec T(v_n,u_n) \preceq v_{n-1}$ ($n = 1,2,\ldots$).

Take $r_n = \sup\{r \in (0,1) : u_n \geq rv_n\}$, thus $0 < r_0 < r_1 < \cdots < r_n < r_{n+1} < \cdots < 1$, and $\lim_{n\to\infty} r_n = \sup\{r_n : n = 0,1,2,\ldots\} = r^* \in (0,1]$. Since $r_{n+1} > r_n = \sup\{r \in (0,1) : u_n \geq rv_n\}$, thus $u_n \ngeq r_{n+1}v_n$. In addition, S is completely ordered and $\lambda S \subset S$ for all $\lambda \in [0,1]$, then $u_n < r_{n+1}v_n$. Now, one can prove $r^* = 1$. Otherwise, $r^* \in (0,1)$. We claim

$$T(u_{n+1},v_{n+1}) \preceq T\bigl((1/r^*)u_{n+1},r^*v_{n+1}\bigr). \tag{14}$$

Suppose that $x \in T(u_{n+1},v_{n+1})$ is arbitrary. We have $u_{n+1} \leq (1/r^*)u_{n+1}$. If $x_1 = u_{n+1}$, $x_2 = (1/r^*)u_{n+1}$ and $y = v_{n+1}$, then by (A_1) of Definition 1.5, there exists $z \in T((1/r^*)u_{n+1},v_{n+1})$ such that $x \leq z$. Thus, $T(u_{n+1},v_{n+1}) \preceq T((1/r^*)u_{n+1},v_{n+1})$.

Also, if $y_1 = r^*v_{n+1}$, $y_2 = v_{n+1}$ and $x = (1/r^*)u_{n+1}$, then for $w \in T((1/r^*)u_{n+1},r^*v_{n+1})$, there exists $h \in T((1/r^*)u_{n+1},v_{n+1})$ such that $w \geq h$. It means that

$$T\bigl((1/r^*)u_{n+1},v_{n+1}\bigr) \preceq T\bigl((1/r^*)u_{n+1},r^*v_{n+1}\bigr). \tag{15}$$

Thus,

$$\begin{aligned} T(u_{n+1},v_{n+1}) &\preceq T\bigl((1/r^*)u_{n+1},r^*v_{n+1}\bigr) \\ &\preceq T(u_{n+1},v_{n+1}) - \Psi\left(\frac{1}{r^*},u_{n+1},r^*v_{n+1},T(u_{n+1},v_{n+1})\right) \\ &\prec T(u_{n+1},v_{n+1}), \end{aligned} \tag{16}$$

and this is a contradiction. Therefore, $r^* = 1$. Let $\epsilon \gg 0$ be given. Choose $\delta > 0$ such that $\epsilon + N_\delta(0) \subseteq P$, where $N_\delta(0) = \{y \in E : \|y\| < \delta\}$. Since $r_n \to 1$, one can choose a natural

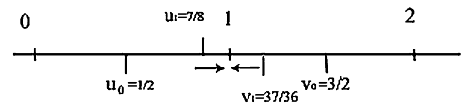

Figure 1 $A(u_0, v_0) \ll u_0 \ll u_1 \ll \cdots \ll u_n \ll \cdots \ll 1 \ll \cdots \ll v_n \ll \cdots \ll v_1 \ll v_0 \ll A(v_0, u_0)$.

number N_1 such that $(1-r_n)v_1 \in N_\delta(0)$ for all $n \geq N_1$. Therefore $(1-r_n)v_1 \ll \epsilon$. Also, $v_n \leq v_1$ and

$$0 < v_n - u_n \leq (1 - r_n)v_n \leq (1 - r_n)v_1 \ll \epsilon. \tag{17}$$

By Remark 1.1, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n$.

For all $n, p \geq 1$, applying the same argument, we have

$$0 < v_n - v_{n+p} \leq v_n - u_n \ll \epsilon. \tag{18}$$

Also,

$$0 < u_{n+p} - u_n \leq v_n - u_n \ll \epsilon. \tag{19}$$

Hence, $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences in E, then there exist $u^*, v^* \in E$ such that $u_n \to u^*$, $v_n \to v^*$ $(n \to \infty)$ and $u^* = v^*$. Write $x^* = u^* = v^*$.

It is easy to see that $u_n \preceq T(u_{n+1}, v_{n+1}) \preceq T(x^*, x^*) \preceq T(v_{n+1}, u_{n+1}) \preceq v_n$ for all $n = 1, 2, \ldots$. Thus, there exists $z_n \in T(x^*, x^*)$ such that $u_n \leq z_n \leq v_n$. By taking limit on both sides of (17),

$$0 < z_n - u_n \leq (1 - r_n)v_n \leq (1 - r_n)v_1 \ll \epsilon. \tag{20}$$

So, $z_n \to x^*$. Since T has closed values, then $x^* \in T(x^*, x^*)$ and

$$x^* \in [u_n, v_n] \cap S \subset [u_0, v_0] \cap S. \qquad \square$$

Remark 3.1 One can see easily that Theorem 2.1 should be included as a corollary of Theorem 3.1.

Example 3.1 Let $E = \mathrm{R}$, $P = [0, +\infty)$ and $S = P$. Then $S_0 = \operatorname{int}(P) = (0, +\infty)$.

Define $A : [0, +\infty) \times [0, +\infty) \to \mathrm{R}$ as

$$A(x,y) = \begin{cases} \frac{x}{y}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

A is a mixed monotone operator. Now suppose that $\Psi : [0,1) \times P \times P \times E \to E$ is as $\Psi(t,x,y,s) = (1-t^2)s$. Then Ψ is an $\mathcal{L}''$-function. Moreover,

$$A(tx,y) \leq A(x,ty) - \Psi\big(t,x,y,A(x,ty)\big)$$

for each $x, y \in S_0$. Also, by taking $u_0 = \frac{1}{2}$, $v_0 = \frac{3}{2}$ and $r_0 = \frac{1}{4}$, we have

(I) $u_0 \geq r_0 v_0$,

(II) $A(u_0, v_0) = \frac{1}{3} \ll u_0 \ll v_0 \ll A(v_0, u_0) = 3$,

(III) for $u, v \in [u_0, v_0] \cap S$ with $A(u, v) \ll u \ll v$, there exists $u' \in S$ such that $u \leq A(u', v) \ll u' \ll v$; similarly, for $u, v \in [u_0, v_0] \cap S$ with $u \ll v \ll A(v, u)$, there exists $v' \in S$ such that $u \ll v' \ll A(v', u) \leq v$.

For further explanation on (III), since $A(u_0, v_0) = \frac{1}{3} \ll u_0 \ll v_0$, by (III) there exists $u_1 \in S$ such that $u_0 \ll A(u_1, v_0) \ll u_1 \ll v_0$. It means that $\frac{1}{2} \ll \frac{u_1}{\frac{3}{2}} \ll u_1 \ll \frac{3}{2}$. Thus u_1 must be greater than $\frac{3}{4}$. Therefore we can set $u_1 = \frac{\frac{3}{4}+1}{2}$. Similarly, since $\frac{7}{8} = u_1 \ll v_0 = \frac{3}{2} \ll A(v_0, u_1) = \frac{12}{7}$, thus by (III) there exists $v_1 \in S$ such that $u_1 \ll v_1 \ll A(v_1, u_1) \leq v_0$. It means that v_1 must be less than $\frac{21}{16}$. We can set $v_1 = \frac{\frac{21}{16}+1}{2}$. By the continuity of such ways, we can consider the following reflexive sequences:

$$u_0 = \frac{1}{2}, \qquad v_0 = \frac{3}{2}, \qquad u_n = \frac{u_{n-1}v_{n-1} + 1}{2} \quad \text{and} \quad v_n = \frac{v_{n-1}u_n + 1}{2},$$

which satisfy (I), (II) and (III) (see Figure 1). Moreover, $u_n \to 1$ and $v_n \to 1$ and $A(1, 1) = 1$.

4 Application

The following result is given by Zhang [6] and is obtained by our main result.

Corollary 4.1 *Let P be a normal cone of E, let S be a completely ordered closed subset of E with $S_0 = S \backslash \{\theta\} \subset \operatorname{int} P$ and let $\lambda S \subset S$ for all $\lambda \in [0, 1]$. Let $u_0, v_0 \in S_0$, $A : P \times P \to E$ be a mixed monotone operator with $A(([\theta, v_0] \cap S) \times ([\theta, v_0] \cap S)) \subset S$ and $A(u_0, v_0) \ll u_0 \ll v_0 \ll A(v_0, u_0)$. Assume that there exists a function $\phi : (0, 1) \times ([u_0, v_0] \cap S) \times ([u_0, v_0] \cap S) \to (0, +\infty)$ such that $A(tx, y) \leq \phi(t, x, y)A(x, ty)$, where $0 < \phi(t, x, x) < t$ for all $(t, x, y) \in (0, 1) \times ([u_0, v_0] \cap S) \times ([u_0, v_0] \cap S)$. Suppose that*

(I) *for $u, v \in [u_0, v_0] \cap S$ with $A(u, v) \ll u \ll v$, there exists $u' \in S$ such that $u \leq A(u', v) \ll u' \ll v$; similarly, for $u, v \in [u_0, v_0] \cap S$ with $u \ll v \ll A(v, u)$, there exists $v' \in S$ such that $u \ll v' \ll A(v', u) \leq v$.*

(II) *there exists an element $w_0 \in [u_0, v_0] \cap S$ such that $\phi(t, x, x) \leq \phi(t, w_0, w_0)$ for all $(t, x) \in (0, 1) \times ([u_0, v_0] \cap S)$, and $\lim_{s \to t^-} \phi(s, w_0, w_0) < t$ for all $t \in (0, 1)$.*

Then A has at least one fixed point $x^* \in [u_0, v_0] \cap S$.

Proof Set $\Psi(t, x, y, z) = (1 - \phi(t, x, y))z$. Then Ψ is an $\mathcal{L}''$-function, and we have

$$A(tx, y) \leq \phi(t, x, y)A(x, ty) = A(x, ty) - \Psi\big(t, x, y, A(x, ty)\big).$$

Thus, by Theorem 2.1 the desired result is obtained. □

Competing interests
The author declares that they have no competing interests.

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