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The Mann algorithm in a complete geodesic space with curvature bounded above

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Abstract
The purpose of this paper is to prove two Δ-convergence theorems of the Mann algorithm to a common fixed point for a countable family of mappings in the case of a complete geodesic space with curvature bounded above by a positive number. The first one for nonexpansive mappings improves the recent result of He *et al.* (Nonlinear Anal. 75:445-452, 2012). The last one is proved for quasi-nonexpansive mappings and applied to the problem of finding a common fixed point of a countable family of quasi-nonexpansive mappings.

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1 Introduction

For a real number κ, a CAT(κ) space is defined by a geodesic metric space whose geodesic triangle is sufficiently thinner than the corresponding comparison triangle in a model space with curvature κ. The concept of these spaces has been studied by a large number of researchers. We know that any CAT(κ) space is a CAT(κ') space for $\kappa' > \kappa$ (see [1]), thus all results for CAT(0) spaces can immediately be applied to any CAT(κ) with $\kappa \leq 0$. Moreover, CAT(κ) spaces with positive κ can be treated as CAT(1) spaces by changing the scale of the space. So we are interested in CAT(1) spaces.

One of the most important analytical problems is the existence of fixed points for nonlinear mappings. In the case for nonexpansive mappings in a CAT(κ) space was proved by Kirk [2, 3] for $\kappa \leq 0$, and by Espánola and Fernández-León [4] for $\kappa > 0$. In the cases when at least one fixed point exists, it is natural to wonder whether such a fixed point can be approximated by iterations. There are many methods for approximating fixed points of a nonexpansive mapping T. One of the most successful methods is the Mann algorithm [5] which is defined in a geodesic space X by $x_1 \in X$ and

$$x_{n+1} = t_n x_n \oplus (1 - t_n) T x_n \quad \text{for all } n \in \mathbb{N}, \tag{1.1}$$

where $\{t_n\}$ is a sequence in $[0, 1]$. By using this algorithm, He *et al.* [6] proved the following result.

Theorem 1.1 *Let X be a complete* CAT(1) *space and $T : X \to X$ be a mapping. Let $\{x_n\}$ be the Mann algorithm* (1.1) *in X. Suppose that*

(C0′) *T is nonexpansive with* $\mathrm{F}(T) \neq \emptyset$;
(C1′) $d(x_1, \mathrm{F}(T)) < \pi/4$;
(C2) $\sum_{n=1}^{\infty} t_n(1-t_n) = \infty$.

Then the sequence $\{x_n\}$ Δ*-converges to a fixed point of* T.

Motivated by these results, we prove two Δ-convergence theorems of the Mann algorithm to a common fixed point for a countable family of mappings in complete CAT(1) spaces. The first one for nonexpansive mappings improves Theorem 1.1. The last one is proved for quasi-nonexpansive mappings and applied to the problem of finding a common fixed point of a countable family of quasi-nonexpansive mappings.

2 Preliminaries

Let X be a metric space with a metric d and let $x, y \in X$ with $d(x,y) = l$. A *geodesic path* from x to y is an isometry $c : [0,l] \to X$ such that $c(0) = x$ and $c(l) = y$. The image of a geodesic path from x to y is called a *geodesic segment* joining x and y. Let $r \in (0,\infty]$. If for every $x, y \in X$ with $d(x,y) < r$, a geodesic from x to y exists, then we say that X is *r-geodesic*. Moreover, if such a geodesic is unique for each pair of points, then X is said to be *r-uniquely geodesic*.

A geodesic segment joining x and y is not necessarily unique in general. When it is unique, this geodesic segment is denoted by $[x,y]$. We write $z \in [x,y]$ if and only if there exists $t \in [0,1]$ such that $d(z,x) = (1-t)d(x,y)$ and $d(z,y) = td(x,y)$. In this case, we will write $z = tx \oplus (1-t)y$ for simplicity. A *geodesic triangle* $\triangle(x,y,z)$ consists of three points $x,y,z \in X$ and geodesic segments $[y,z]$, $[z,x]$ and $[x,z]$ joining two of them. We write $w \in \triangle(x,y,z)$ if $w \in [y,z] \cup [z,x] \cup [x,y]$.

To define a CAT(κ) space, we use the following notation called *model space*. For $\kappa = 0$, the two-dimensional model space $M_\kappa^2 = M_0^2$ is the Euclidean space $\mathbb{R}^2$ with the metric induced from the Euclidean norm. For $\kappa > 0$, M_κ^2 is the two-dimensional sphere $(1/\sqrt{\kappa})\mathbb{S}^2$ whose metric is a length of a minimal great arc joining each two points. For $\kappa < 0$, M_κ^2 is the two-dimensional hyperbolic space $(1/\sqrt{-\kappa})\mathbb{H}^2$ with the metric defined by a usual hyperbolic distance.

The diameter of M_κ^2 is denoted by D_κ, that is, $D_\kappa = \pi/\sqrt{\kappa}$ if $\kappa > 0$ and $D_\kappa = \infty$ if $\kappa \leq 0$. We know that M_κ^2 is a D_κ-uniquely geodesic space for each $\kappa \in \mathbb{R}$.

Let $\kappa \in \mathbb{R}$. For $\triangle(x,y,z)$ in a geodesic space X satisfying that $d(x,y) + d(y,z) + d(x,z) < 2D_\kappa$, there exist points $\bar{x}, \bar{y}, \bar{z} \in M_\kappa^2$ such that $d(x,y) = d_{M_\kappa^2}(\bar{x},\bar{y})$, $d(y,z) = d_{M_\kappa^2}(\bar{y},\bar{z})$ and $d(x,z) = d_{M_\kappa^2}(\bar{x},\bar{z})$. We call the triangle having vertices $\bar{x}$, $\bar{y}$ and $\bar{z}$ in M_κ^2 a *comparison triangle* of $\triangle(x,y,z)$. Notice that it is unique up to an isometry of M_κ^2. For a specific choice of comparison triangles, we denote it by $\triangle(\bar{x},\bar{y},\bar{z})$. A point $\bar{p} \in [\bar{x},\bar{y}]$ is called a *comparison point* for $p \in [x,y]$ if $d(x,p) = d_{M_\kappa^2}(\bar{x},\bar{p})$.

Let $\kappa \in \mathbb{R}$ and X be a D_κ-geodesic space. If for any $x,y,z \in X$ with $d(x,y) + d(y,z) + d(x,z) < 2D_\kappa$, for any $p, q \in \triangle(x,y,z)$, and for their comparison points $\bar{p}, \bar{q} \in \triangle(\bar{x},\bar{y},\bar{z})$, the inequality

$$d(p,q) \leq d_{M_\kappa^2}(\bar{p},\bar{q})$$

holds, then we call X a CAT(κ) *space*. It is easy to see that all CAT(κ) spaces are D_κ-uniquely geodesic; consider the triangle such that two of its vertices are identical.

Let $\{x_n\}$ be a bounded sequence in a metric space X. For $x \in X$, let $r(x, \{x_n\}) := \limsup_{n\to\infty} d(x, x_n)$ and define the *asymptotic radius* $r(\{x_n\})$ of $\{x_n\}$ by

$$r(\{x_n\}) := \inf_{x\in X} r(x, \{x_n\}).$$

An element z of X is said to be an *asymptotic center* of $\{x_n\}$ if $r(z, \{x_n\}) = r(\{x_n\})$. We say that $\{x_n\}$ is Δ-*convergent* to $x \in X$ if x is the unique asymptotic center of any subsequence of $\{x_n\}$. The concept of Δ-convergence introduced by Lim in 1976 was shown by Kirk and Panyanak [7] in CAT(0) spaces to be very similar to the weak convergence in Banach space setting.

Remark 2.1

(1) If $\{x_n\}$ is a sequence in a complete CAT(κ) space such that $r(\{x_n\}) < D_\kappa/2$, then its asymptotic center consists of exactly one point.

(2) Every sequence $\{x_n\}$ whose asymptotic radius is less than $D_\kappa/2$ has a Δ-convergent subsequence (see [4, 7, 8]), that is, $\omega_\Delta(\{x_n\}) := \{x \in X : \text{there exists } \{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ Δ-converges to $x\} \neq \emptyset$.

We note that Remark 2.1(1) was proved by Dhompongsa *et al.* [9] for the case $\kappa \leq 0$, and by Espánola and Fernández-León [4] for the case $\kappa > 0$.

Let X be a metric space with a metric d. A mapping $T : X \to X$ is called *nonexpansive* if

$$d(Tx, Ty) \leq d(x, y) \quad \text{for all } x, y \in X.$$

A point $x \in X$ is called a *fixed point* of T if $x = Tx$. We denote by $\mathrm{F}(T)$ the set of fixed points of T. The mapping T is called *quasi-nonexpansive* if $\mathrm{F}(T) \neq \emptyset$ and

$$d(Tx, p) \leq d(x, p) \quad \text{for all } x \in X \text{ and } p \in \mathrm{F}(T).$$

The following lemmas are essentially needed for our main results.

Lemma 2.2 ([10, Lemma 1]) *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers such that*

$$a_{n+1} \leq a_n + b_n \quad \textit{for all } n \in \mathbb{N}.$$

If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.3 ([11, Lemma 5.4]) *Let $\triangle(x, y, z)$ be a geodesic triangle in a* CAT(1) *space such that $d(x, y) + d(x, z) + d(y, z) < 2\pi$. Let $u = tz \oplus (1 - t)x$ and $v = tz \oplus (1 - t)y$ for some $t \in [0, 1]$. If $d(x, z) \leq M$, $d(y, z) \leq M$, and $\sin((1 - t)M) \leq \sin M$ for some $M \in (0, \pi)$, then*

$$d(u, v) \leq \frac{\sin(1 - t)M}{\sin M} d(x, y).$$

Lemma 2.4 ([12, Corollary 2.2]) *Let $\triangle(x, y, z)$ be a geodesic triangle in a* CAT(1) *space such that $d(x, y) + d(x, z) + d(y, z) < 2\pi$. Let $u = tx \oplus (1 - t)y$ for some $t \in [0, 1]$. Then*

$$\cos d(u, z) \sin d(x, y) \geq \cos d(x, z) \sin\big(td(x, y)\big) + \cos d(y, z) \sin\big((1 - t)d(x, y)\big).$$

Lemma 2.5 ([11, Lemma 3.1]) *Let $\triangle(x,y,z)$ be a geodesic triangle in a* CAT(1) *space such that $d(x,y) + d(x,z) + d(y,z) < 2\pi$, and let $t \in [0,1]$. Then*

$$\cos d\bigl(tx \oplus (1-t)y, z\bigr) \geq t \cos d(x,z) + (1-t) \cos d(y,z).$$

3 Main results

We start with some propositions which are common tools for proving the main results in the next two subsections.

Proposition 3.1 *Let $\{x_n\}$ be a sequence of a complete* CAT(1) *space X such that $r(\{x_n\}) < \pi/2$. Suppose that $\lim_{n\to\infty} d(x_n, z)$ exists for all $z \in \omega_\Delta(\{x_n\})$. Then $\{x_n\}$ Δ-converges to an element of $\omega_\Delta(\{x_n\})$. Moreover, $\omega_\Delta(\{x_n\})$ consists of exactly one point.*

Proof Let x be the asymptotic center of $\{x_n\}$ and let $\{x_{n_k}\}$ be any subsequence of $\{x_n\}$ with the asymptotic center y. We show that $y = x$ and hence $\{x_n\}$ Δ-converges to x as desired. Since $r(\{x_{n_k}\}) \leq r(\{x_n\}) < \pi/2$, there exists a subsequence $\{x_{n_{k_l}}\}$ of $\{x_{n_k}\}$ such that $\{x_{n_{k_l}}\}$ Δ-converges to z for some $z \in X$. Clearly, $z \in \omega_\Delta(\{x_n\})$ and it follows from the assumption that $\lim_{n\to\infty} d(x_n, z)$ exists. Let $u \in X$. Then

$$\begin{aligned}
\lim_{n\to\infty} d(x_n, z) &= \lim_{k\to\infty} d(x_{n_k}, z) \\
&= \lim_{l\to\infty} d(x_{n_{k_l}}, z) \\
&\leq \limsup_{l\to\infty} d(x_{n_{k_l}}, u) \\
&\leq \limsup_{k\to\infty} d(x_{n_k}, u) \\
&\leq \limsup_{n\to\infty} d(x_n, u).
\end{aligned}$$

Since

$$\lim_{n\to\infty} d(x_n, z) \leq \limsup_{n\to\infty} d(x_n, u) \quad \text{for all } u \in X,$$

we have $z = x$. Since

$$\lim_{k\to\infty} d(x_{n_k}, z) \leq \limsup_{k\to\infty} d(x_{n_k}, u) \quad \text{for all } u \in X,$$

we have $z = y$. This implies that $y = x$. □

Proposition 3.2 *Let X be a complete* CAT(1) *space and $\{T_n\} : X \to X$ be a countable family of quasi-nonexpansive mappings with $\mathrm{F} := \bigcap_{n=1}^\infty \mathrm{F}(T_n) \neq \emptyset$. Let $\{x_n\}$ be a sequence in X such that $d(x_1, \mathrm{F}) < \pi/2$ and*

$$x_{n+1} = t_n x_n \oplus (1-t_n) T_n x_n \quad \textit{for all } n \in \mathbb{N},$$

where $\{t_n\}$ is a sequence in $[0,1]$. Then

(i) $\{x_n\}$ *is well defined and* $r(\{x_n\}) < \pi/2$. *In particular,* $\omega_\Delta(\{x_n\}) \neq \emptyset$ *and* $d(x_{n+1}, p) \leq d(x_n, p)$ *for all* $n \in \mathbb{N}$ *whenever* $d(x_1, p) < \pi/2$ *and* $p \in \mathrm{F}$.

(ii) *If* $\omega_\Delta(\{x_n\}) \subset \mathrm{F}$, *then* $\{x_n\}$ Δ*-converges to an element of* F.

Proof (i) Let $p \in \mathrm{F}$ be such that $d(x_1, p) < \pi/2$. Since $d(T_1x_1, p) \leq d(x_1, p) < \pi/2$, we have that $d(x_1, T_1x_1) < \pi$. This implies that x_2 is well defined. It follows from Lemma 2.5 that

$$\begin{aligned}\cos d(x_2, p) &= \cos d\bigl(t_1x_1 \oplus (1 - t_1)T_1x_1, p\bigr) \\ &\geq t_1 \cos d(x_1, p) + (1 - t_1) \cos d(T_1x_1, p) \\ &\geq t_1 \cos d(x_1, p) + (1 - t_1) \cos d(x_1, p) \\ &= \cos d(x_1, p),\end{aligned}$$

and we have

$$\begin{aligned}d(x_2, p) &\leq \min\bigl\{d(x_1, x_2) + d(x_1, p), d(Tx_1, x_2) + d(Tx_1, p)\bigr\} \\ &= \min\bigl\{(1 - t_1)d(x_1, Tx_1) + d(x_1, p), t_1d(x_1, Tx_1) + d(Tx_1, p)\bigr\} \\ &\leq \min\{1 - t_1, t_1\}d(x_1, Tx_1) + d(x_1, p) \\ &\leq \frac{1}{2}d(x_1, Tx_1) + d(x_1, p) \\ &< \frac{\pi}{2} + \frac{\pi}{2} = \pi.\end{aligned}$$

Therefore, $d(x_2, p) \leq d(x_1, p) < \pi/2$. Using mathematical induction, we can conclude that the sequence $\{x_n\}$ is well defined and

$$d(x_{n+1}, p) \leq d(x_n, p) \leq d(x_1, p) < \pi/2 \quad \text{for all } n \in \mathbb{N}.$$

Then $\lim_{n\to\infty} d(x_n, p)$ exists which is less than $\pi/2$, and so $r(\{x_n\}) < \pi/2$. This implies that $\omega_\Delta(\{x_n\}) \neq \emptyset$.

(ii) Let $u \in \omega_\Delta(\{x_n\}) \subset \mathrm{F}$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ Δ-converges to u. Notice that $r(u, \{x_{n_k}\}) = r(\{x_{n_k}\}) \leq r(\{x_n\}) < \pi/2$. Thus there is $N \in \mathbb{N}$ such that $d(x_N, u) < \pi/2$. Similar to the first step, we have that $d(x_{n+1}, u) \leq d(x_n, u)$ for all $n \geq N$. This implies that $\lim_{n\to\infty} d(x_n, u)$ exists. It follows immediately from Proposition 3.1 that $\{x_n\}$ Δ-converges to an element of F and the proof is finished. □

3.1 Countable nonexpansive mappings

The following concept is introduced by Aoyama *et al.* [13]. Let X be a complete metric space and $\{T_n\}$ be a countable family of mappings from X into itself with $\mathrm{F} := \bigcap_{n=1}^{\infty} \mathrm{F}(T_n) \neq \emptyset$. We say that $(\{T_n\}, T)$ satisfies *AKTT-condition* if

- $\sum_{n=1}^{\infty} \sup\{d(T_{n+1}y, T_ny) : y \in Y\} < \infty$ for each bounded subset Y of X;
- $Tx := \lim_{n\to\infty} T_nx$ for all $x \in X$ and $\mathrm{F}(T) = \mathrm{F}$.

Remark 3.3 Assume that $(\{T_n\}, T)$ satisfies AKTT-condition.

(1) For each $x \in X$, we have $\{T_nx\}$ is a Cauchy sequence and hence the mapping T above is well defined.

(2) If $\{x_n\}$ is bounded, then $\sum_{n=1}^{\infty} d(T_{n+1}x_n, T_nx_n) < \infty$.

Theorem 3.4 *Let X be a complete* CAT(1) *space and $\{T_n\}$ be a countable family of mappings from X into itself. Let $\{x_n\}$ be a sequence in X defined by $x_1 \in X$ and*

$$x_{n+1} = t_n x_n \oplus (1 - t_n) T_n x_n \quad \text{for all } n \in \mathbb{N},$$

where $\{t_n\}$ is a sequence in $[0,1]$. *Suppose that*

(C0n') T_n *is nonexpansive for all $n \in \mathbb{N}$ and* $\mathrm{F} := \bigcap_{n=1}^{\infty} \mathrm{F}(T_n) \neq \emptyset$;
(C1) $d(x_1, \mathrm{F}) < \pi/2$;
(C2) $\sum_{n=1}^{\infty} t_n(1 - t_n) = \infty$;
(C3$'$) $(\{T_n\}, T)$ *satisfies AKTT-condition.*

Then the sequence $\{x_n\}$ Δ-converges to a common fixed point of $\{T_n\}$.

Proof We first show that $\lim_{n\to\infty} d(x_n, T_n x_n)$ exists. Using the nonexpansiveness of T_n and the definition of $\{x_n\}$, we obtain that

$$\begin{aligned}
&d(x_{n+1}, T_{n+1}x_{n+1}) \\
&\quad \leq d(x_{n+1}, T_n x_n) + d(T_n x_n, T_n x_{n+1}) + d(T_n x_{n+1}, T_{n+1}x_{n+1}) \\
&\quad \leq d(x_{n+1}, T_n x_n) + d(x_n, x_{n+1}) + d(T_n x_{n+1}, T_{n+1}x_{n+1}) \\
&\quad = d(x_n, T_n x_n) + d(T_n x_{n+1}, T_{n+1}x_{n+1})
\end{aligned}$$

for all $n \in \mathbb{N}$. It follows from Lemma 2.2 and $\sum_{n=1}^{\infty} d(T_n x_{n+1}, T_{n+1}x_{n+1}) < \infty$ that

$$\lim_{n\to\infty} d(x_n, T_n x_n)$$

exists.

Next, we show that $\lim_{n\to\infty} d(x_n, T_n x_n) = 0$. Assume that $\lim_{n\to\infty} d(x_n, T_n x_n) > 0$. Thus, without loss of generality, there is a positive real number A such that

$$A \leq d(x_n, T_n x_n) < \pi \quad \text{for all } n \in \mathbb{N}.$$

To get the right inequality of the preceding expression, let $p \in \mathrm{F}$ be such that $d(x_1, p) < \pi/2$. By Proposition 3.2(i), we have

$$d(x_n, T_n x_n) \leq d(x_n, p) + d(T_n x_n, p) \leq 2d(x_n, p) < 2d(x_1, p) < \pi.$$

Put $A_n := d(x_n, T_n x_n)$ for all $n \in \mathbb{N}$. By elementary trigonometry and Lemma 2.4, we get that

$$\begin{aligned}
&\cos d(x_{n+1}, p) \sin A_n \\
&\quad = \cos d\big(t_n x_n \oplus (1 - t_n) T_n x_n, p\big) \sin A_n \\
&\quad \geq \cos d(x_n, p) \sin(t_n A_n) + \cos d(T_n x_n, p) \sin\big((1 - t_n) A_n\big) \\
&\quad \geq \cos d(x_n, p) \big(\sin(t_n A_n) + \sin\big((1 - t_n) A_n\big)\big),
\end{aligned}$$

and it follows that

$$\begin{aligned}
&\cos d(x_{n+1},p) - \cos d(x_n,p) \\
&\quad \geq \cos d(x_n,p)\left(\frac{\sin(t_nA_n) + \sin((1-t_n)A_n)}{\sin A_n} - 1\right) \\
&\quad = \frac{2\cos d(x_n,p)\sin(t_nA_n/2)\sin((1-t_n)A_n/2)}{\cos(A_n/2)} \\
&\quad \geq \frac{2\cos d(x_1,p)\sin(t_nA/2)\sin((1-t_n)A/2)}{\cos(A/2)} \\
&\quad \geq \frac{2t_n(1-t_n)\cos d(x_1,p)\sin^2(A/2)}{\cos(A/2)}
\end{aligned}$$

for all $n \in \mathbb{N}$. Notice that $\cos d(x_1,p)$, $\cos(A/2)$, and $\sin(A/2)$ are positive. Consequently,

$$\sum_{n=1}^{\infty} t_n(1-t_n) \leq \frac{\cos(A/2)}{2\cos d(x_1,p)\sin^2(A/2)} \sum_{n=1}^{\infty}\bigl(\cos d(x_{n+1},p) - \cos d(x_n,p)\bigr) < \infty,$$

which is a contradiction. Then we get that $\lim_{n\to\infty} d(x_n, T_nx_n) = 0$ and hence

$$\begin{aligned}
d(x_n, Tx_n) &\leq d(x_n, T_nx_n) + d(T_nx_n, Tx_n) \\
&\leq d(x_n, T_nx_n) + \sup\bigl\{d(T_ny, Ty) : y \in Y\bigr\} \to 0,
\end{aligned}$$

where $Y := \{x_n\}$.

Finally, we show that $\{x_n\}$ Δ-converges to an element of $\mathrm{F}(T)$. To apply Proposition 3.2(ii), we show that $\omega_\Delta(\{x_n\}) \subset \mathrm{F}(T)$. Let $u \in \omega_\Delta(\{x_n\})$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ Δ-converges to u. Clearly, u is the unique asymptotic center of $\{x_{n_k}\}$. Using the nonexpansiveness of T and $d(x_n, Tx_n) \to 0$, we get that

$$\begin{aligned}
\limsup_{k\to\infty} d(x_{n_k}, Tu) &\leq \limsup_{k\to\infty} d(x_{n_k}, Tx_{n_k}) + \limsup_{k\to\infty} d(Tx_{n_k}, Tu) \\
&\leq \limsup_{k\to\infty} d(x_{n_k}, u).
\end{aligned}$$

This implies that $Tu = u$, that is, $\omega_\Delta(\{x_n\}) \subset \mathrm{F}(T)$. This completes the proof. □

As an immediate consequence of Theorem 3.4, we obtain the following result.

Corollary 3.5 *Let X be a complete* CAT(1) *space and $T : X \to X$ be a mapping. Let $\{x_n\}$ be the Mann algorithm* (1.1) *in X. Suppose that*

(C0′) *T is nonexpansive with* $\mathrm{F}(T) \neq \emptyset$;
(C1) $d(x_1, \mathrm{F}(T)) < \pi/2$;
(C2) $\sum_{n=1}^{\infty} t_n(1-t_n) = \infty$.

Then the sequence $\{x_n\}$ Δ-converges to a fixed point of T.

Remark 3.6 Our Corollary 3.5 improves Theorem 3.1 of He *et al.* [6] (see Theorem 1.1) because (C1′) of Theorem 1.1 implies (C1) of Corollary 3.5. Moreover, (C1) is sharp in the sense that if $d(x_1, \mathrm{F}(T)) = \pi/2$, then we may construct the Mann algorithm for a nonexpansive mapping which is not Δ-convergent.

Example 3.7 Let $\mathbb{S}^2$ be the unit sphere of the Euclidean space $\mathbb{R}^3$ with the geodesic metric. Let $T : \mathbb{S}^2 \to \mathbb{S}^2$ be defined by

$$T(x,y,z) = (-y,x,z) \quad \text{for all } (x,y,z) \in \mathbb{S}^2.$$

Then T is nonexpansive and $\mathrm{F}(T) = \{(0,0,1),(0,0,-1)\}$. Let $\{x_n\}$ be a sequence in $\mathbb{S}^2$ defined by $x_1 = (1,0,0)$ and

$$x_{n+1} = \frac{1}{2}x_n \oplus \frac{1}{2}Tx_n \quad \text{for all } n \in \mathbb{N}.$$

Then $d(x_1, \mathrm{F}(T)) = \pi/2$ and

$$x_{n+1} = \left(\cos\frac{n\pi}{4}, \sin\frac{n\pi}{4}, 0\right) \quad \text{for all } n \in \mathbb{N}.$$

It is easy to see that $\{x_{8n+1}\}$ has the unique asymptotic center which is $\{(1,0,0)\}$ and $\{x_{8n+3}\}$ has the unique asymptotic center which is $\{(0,1,0)\}$. Hence, $\{x_n\}$ is not Δ-convergent.

3.2 Countable quasi-nonexpansive mappings

In this subsection, we give a supplement result to Theorem 3.4. Obviously, every nonexpansive mapping with a fixed point is quasi-nonexpansive. Moreover, if T is nonexpansive, then T is Δ-*demiclosed* [11], that is, if for any Δ-convergent sequence $\{x_n\}$ in X, its Δ-limit belongs to $\mathrm{F}(T)$ whenever $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

In the following theorem, we deal with quasi-nonexpansive mappings satisfying Δ-demiclosedness. This interesting class of mappings includes the metric projections [11]. However, there are many metric projections such that they are not nonexpansive.

Theorem 3.8 *Let X be a complete* CAT(1) *space and $\{T_n\}$ be a countable family of mappings from X into itself. Let $\{x_n\}$ be a sequence in X defined by $x_1 \in X$ and*

$$x_{n+1} = t_n x_n \oplus (1-t_n)T_n x_n \quad \text{for all } n \in \mathbb{N},$$

where $\{t_n\}$ is a sequence in $(0,1)$. *Suppose that*

- (C0n) *T_n is quasi-nonexpansive for all $n \in \mathbb{N}$ and* $\mathrm{F} = \bigcap_{n=1}^{\infty} \mathrm{F}(T_n) \neq \emptyset$;
- (C1) $d(x_1, \mathrm{F}) < \pi/2$;
- (C2′) $\liminf_{n\to\infty} t_n(1-t_n) > 0$;
- (C3) *there exists a mapping $T : X \to X$ such that*
  - *$\{T_n\}$ converges uniformly to T on each bounded subset of X*;
  - $\mathrm{F}(T) = \mathrm{F}$;
- (C4) *T is Δ-demiclosed.*

Then the sequence $\{x_n\}$ Δ-converges to a common fixed point of $\{T_n\}$.

Remark 3.9 Let us compare Theorems 3.4 and 3.8:

(1) (C0n') $\Rightarrow$ (C0n);
(2) (C3$'$) $\Rightarrow$ (C3);
(3) (C0$'$) and (C3$'$) $\Rightarrow$ (C4);
(4) (C2$'$) $\Rightarrow$ (C2).

Proof of Theorem 3.8 We first show that $\lim_{n\to\infty} d(x_n, T_nx_n) = 0$. Let $p \in \mathrm{F}(T)$ be such that $d(x_1,p) < \pi/2$. We have that $A := \lim_{n\to\infty} d(x_n,p)$ exists which is less than $\pi/2$ and $r(\{x_n\}) < \pi/2$ by Proposition 3.2(i). Put $B := \limsup_{n\to\infty} d(x_n, T_nx_n)$. Notice that $B < \pi$. Since $\liminf_{n\to\infty} t_n(1 - t_n) > 0$, we may assume that there exists a subsequence $\{n_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} d(x_{n_k}, T_{n_k}x_{n_k}) = B$ and $t_{n_k} \to t \in (0,1)$. Using the quasi-nonexpansiveness of T_n and Lemma 2.4, we get that

$$\begin{aligned}
&\cos d(x_{n_k+1},p)\sin d(x_{n_k}, T_{n_k}x_{n_k}) \\
&\quad = \cos d\big(t_{n_k}x_{n_k} \oplus (1 - t_{n_k})T_{n_k}x_{n_k}, p\big) \sin d(x_{n_k}, T_{n_k}x_{n_k}) \\
&\quad \geq \cos d(x_{n_k},p) \sin\big(t_{n_k} d(x_{n_k}, T_{n_k}x_{n_k})\big) + \cos d(T_{n_k}x_{n_k},p) \sin\big((1 - t_{n_k})d(x_{n_k}, T_{n_k}x_{n_k})\big) \\
&\quad \geq \cos d(x_{n_k},p)\big(\sin\big(t_{n_k} d(x_{n_k}, T_{n_k}x_{n_k})\big) + \sin\big((1 - t_{n_k})d(x_{n_k}, T_{n_k}x_{n_k})\big)\big).
\end{aligned}$$

Letting $k \to \infty$ yields

$$\cos A \sin B \geq \cos A\big(\sin tB + \sin(1 - t)B\big).$$

Using elementary trigonometry, we get that $B = 0$. Hence it follows that

$$\lim_{n\to\infty} d(x_n, T_nx_n) = 0.$$

By condition (C3), we get that

$$\begin{aligned}
d(x_n, Tx_n) &\leq d(x_n, T_nx_n) + d(T_nx_n, Tx_n) \\
&\leq d(x_n, T_nx_n) + \sup\big\{d(T_ny, Ty) : y \in Y\big\} \to 0,
\end{aligned}$$

where $Y := \{x_n\}$.

Finally, we show that $\omega_\Delta(\{x_n\}) \subset \mathrm{F}(T)$. Let $u \in \omega_\Delta(\{x_n\})$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ Δ-converges to u. It follows from the Δ-demiclosedness of T and $d(x_n, Tx_n) \to 0$ that $Tu = u$, that is, $\omega_\Delta(\{x_n\}) \subset \mathrm{F}(T)$. Hence the result follows from Proposition 3.2(ii). The proof is now finished. □

As an immediate consequence of Theorem 3.8, we obtain the following result.

Corollary 3.10 *Let X be a complete* CAT(1) *space and $T : X \to X$ be a mapping with $\mathrm{F}(T) \neq \emptyset$. Let $\{x_n\}$ be the Mann algorithm* (1.1) *in X. Suppose that*

(C0) *T is quasi-nonexpansive and Δ-demiclosed*;
(C1) $d(x_1, \mathrm{F}(T)) < \pi/2$;
(C2$'$) $\liminf_{n\to\infty} t_n(1 - t_n) > 0$.

Then the sequence $\{x_n\}$ Δ-converges to a fixed point of T.

Question 3.11 We do not know whether the conclusion of Corollary 3.10 holds if (C2$'$) is replaced by the more general condition (C2).

Let $\{T_n\} : X \to X$ be a countable family of quasi-nonexpansive mappings with $\mathrm{F} := \bigcap_{n=1}^{\infty} \mathrm{F}(T_n) \neq \emptyset$. We next show how to generate a family $\{W_n\}$ and a mapping W satisfying (C3′) and (C4), and hence Theorem 3.8 is applicable.

Theorem 3.12 *Let X be a complete* CAT(1) *space such that $d(u,v) < \pi/2$ for all $u, v \in X$, and let $\{T_n\} : X \to X$ be a countable family of quasi-nonexpansive mappings with* $\mathrm{F} := \bigcap_{n=1}^{\infty} \mathrm{F}(T_n) \neq \emptyset$. *Then there exist a family of quasi-nonexpansive mappings $\{W_n\} : X \to X$ and a quasi-nonexpansive mapping $W : X \to X$ such that*

(i) *$(\{W_n\}, W)$ satisfies AKTT-condition and* $\mathrm{F}(W) = \bigcap_{n=1}^{\infty} \mathrm{F}(W_n) = \mathrm{F}$;

(ii) *W is Δ-demiclosed whenever T_n is Δ-demiclosed for all $n \in \mathbb{N}$.*

To prove Theorem 3.12, we need the following lemmas.

Lemma 3.13 ([14]) *Let X be a complete* CAT(1) *space such that $d(u,v) < \pi/2$ for all $u, v \in X$, and let $S, T : X \to X$ be quasi-nonexpansive mappings with* $\mathrm{F}(S) \cap \mathrm{F}(T) \neq \emptyset$. *Then, for each* $0 < t < 1$, $\mathrm{F}(S) \cap \mathrm{F}(T) = \mathrm{F}(tS \oplus (1-t)T)$ *and the mapping $tS \oplus (1-t)T$ is quasi-nonexpansive.*

The following lemma is essentially proved in [11]. For the sake of completeness, we show the proof.

Lemma 3.14 *Let X be a complete* CAT(1) *space such that $d(u,v) < \pi/2$ for all $u, v \in X$, and let $S, T : X \to X$ be quasi-nonexpansive mappings with* $\mathrm{F}(S) \cap \mathrm{F}(T) \neq \emptyset$. *Let $\{x_n\}$ be a sequence of X. If $d(x_n, \frac{1}{2}Sx_n \oplus \frac{1}{2}Tx_n) \to 0$, then $d(x_n, Sx_n) \to 0$ and $d(x_n, Tx_n) \to 0$.*

Proof Put $W := \frac{1}{2}S \oplus \frac{1}{2}T$. By Lemma 3.13, we have that W is quasi-nonexpansive and $\mathrm{F}(W) = \mathrm{F}(S) \cap \mathrm{F}(T)$. Let $p \in \mathrm{F}(W)$. By Lemma 2.4 and the quasi-nonexpansiveness of S and T, we get that

$$\begin{aligned}
&2\cos d(Wx_n, p) \sin \frac{d(Sx_n, Tx_n)}{2} \cos \frac{d(Sx_n, Tx_n)}{2} \\
&\quad = \cos d\left(\frac{1}{2}Sx_n \oplus \frac{1}{2}Tx_n, p\right) \sin d(Sx_n, Tx_n) \\
&\quad \geq \cos d(Sx_n, p) \sin \frac{d(Sx_n, Tx_n)}{2} + \cos d(Tx_n, p) \sin \frac{d(Sx_n, Tx_n)}{2} \\
&\quad \geq 2\cos d(x_n, p) \sin \frac{d(Sx_n, Tx_n)}{2}.
\end{aligned}$$

This implies that

$$d(Sx_n, Tx_n) = 0 \quad \text{or} \quad \cos \frac{d(Sx_n, Tx_n)}{2} \geq \frac{\cos d(x_n, p)}{\cos d(Wx_n, p)}.$$

It follows from $d(x_n, Wx_n) \to 0$ that $\frac{\cos d(x_n,p)}{\cos d(Wx_n,p)} \to 1$, that is, $d(Sx_n, Tx_n) \to 0$. Thus

$$\begin{aligned}
d(x_n, Sx_n) &\leq d\left(x_n, \frac{1}{2}Sx_n \oplus \frac{1}{2}Tx_n\right) + d\left(\frac{1}{2}Sx_n \oplus \frac{1}{2}Tx_n, Sx_n\right) \\
&= d\left(x_n, \frac{1}{2}Sx_n \oplus \frac{1}{2}Tx_n\right) + \frac{d(Sx_n, Tx_n)}{2} \to 0.
\end{aligned}$$

Hence $d(x_n, Sx_n) \to 0$. Similarly, $d(x_n, Tx_n) \to 0$ and the proof is finished. □

We are now ready to prove Theorem 3.12.

Proof of Theorem 3.12 Put $S_n := \frac{1}{2}I \oplus \frac{1}{2}T_n$ for all $n \in \mathbb{N}$. We define a family of mappings $\{W_n\} : X \to X$ by

$$
\begin{aligned}
&W_1 x = S_1 x;\\
&W_2 x = \frac{1}{2}S_1 x \oplus \frac{1}{2}S_2 x;\\
&W_3 x = \frac{1}{2}S_1 x \oplus \frac{1}{2}\left(\frac{1}{2}S_2 x \oplus \frac{1}{2}S_3 x\right);\\
&\vdots\\
&W_n x = \frac{1}{2}S_1 x \oplus \frac{1}{2}\left(\frac{1}{2}S_2 x \oplus \frac{1}{2}\left(\cdots \oplus \frac{1}{2}\left(\frac{1}{2}S_{n-1} x \oplus \frac{1}{2}S_n x\right)\cdots\right)\right);\\
&\vdots
\end{aligned}
$$

It follows from Lemma 3.13 that W_n is quasi-nonexpansive for all $n \in \mathbb{N}$ and $\bigcap_{n=1}^{\infty} \mathrm{F}(W_n) = \bigcap_{n=1}^{\infty} \mathrm{F}(T_n)$.

We first show that $\sum_{n=1}^{\infty} \sup\{d(W_{n+1}x, W_n x) : x \in X\} < \infty$. For $k \in \mathbb{N}$, put $V_k^{(k)} := S_k$ and

$$
V_n^{(k)} := \frac{1}{2}S_k \oplus \frac{1}{2}\left(\frac{1}{2}S_{k+1} \oplus \frac{1}{2}\left(\cdots \oplus \frac{1}{2}\left(\frac{1}{2}S_{n-1} \oplus \frac{1}{2}S_n\right)\cdots\right)\right)
$$

for all $n > k$. By Lemma 2.3, we have that

$$
\begin{aligned}
d(W_{n+1}x, W_n x) &= d\left(\frac{1}{2}S_1 x \oplus \frac{1}{2}V_{n+1}^{(2)}x, \frac{1}{2}S_1 x \oplus \frac{1}{2}V_n^{(2)}x\right)\\
&\le \left(\frac{\sqrt{2}}{2}\right) d\left(\frac{1}{2}S_2 x \oplus \frac{1}{2}V_{n+1}^{(3)}x, \frac{1}{2}S_2 x \oplus \frac{1}{2}V_n^{(3)}x\right)\\
&\le \left(\frac{\sqrt{2}}{2}\right)^2 d\left(\frac{1}{2}S_3 x \oplus \frac{1}{2}V_{n+1}^{(4)}x, \frac{1}{2}S_3 x \oplus \frac{1}{2}V_n^{(4)}x\right)\\
&\le \cdots\\
&\le \left(\frac{\sqrt{2}}{2}\right)^{n-1} d\left(\frac{1}{2}S_n x \oplus \frac{1}{2}S_{n+1}x, S_n x\right)\\
&\le \left(\frac{\sqrt{2}}{2}\right)^{n-1} \cdot \frac{\pi}{2}
\end{aligned}
$$

for all $x \in X$ and $n \in \mathbb{N}$. Then

$$
\sup\bigl\{d(W_{n+1}x, W_n x) : x \in X\bigr\} \le \left(\frac{\sqrt{2}}{2}\right)^{n-1} \cdot \frac{\pi}{2}
$$

and the result follows. In particular, $\{W_n x\}$ is a Cauchy sequence for each $x \in X$. We now define the mapping $W : X \to X$ by

$$
Wx := \lim_{n\to\infty} W_n x \quad \text{for all } x \in X.
$$

Next, we show that $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$. It is easy to see that $\bigcap_{n=1}^{\infty} F(T_n) \subset F(W)$. On the other hand, let $q \in \bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{n=1}^{\infty} F(S_n)$ and $p \in F(W)$. We prove that $p \in F(T_k)$ for all $k \in \mathbb{N}$. Let $k \in \mathbb{N}$ be given. For any $n > k$, it follows from Lemma 2.5 that

$$\begin{aligned}
&\cos d(q, W_n p) \\
&\quad = \cos d\left(q, \frac{1}{2}S_1 p \oplus \frac{1}{2}\left(\frac{1}{2}S_2 p \oplus \frac{1}{2}\left(\cdots \oplus \frac{1}{2}\left(\frac{1}{2}S_{n-1}p \oplus \frac{1}{2}S_n p\right)\cdots\right)\right)\right) \\
&\quad \geq \frac{1}{2}\cos d(q, S_1 p) + \frac{1}{2}\cos d\left(q, \frac{1}{2}S_2 p \oplus \frac{1}{2}\left(\cdots \oplus \frac{1}{2}\left(\frac{1}{2}S_{n-1}p \oplus \frac{1}{2}S_n p\right)\cdots\right)\right) \\
&\quad \geq \frac{1}{2}\cos d(q, S_1 p) + \cdots + \frac{1}{2^k}\cos d(q, S_k p) + \cdots + \frac{1}{2^{n-1}}\cos d(q, S_{n-1}p) \\
&\qquad + \frac{1}{2^{n-1}}\cos d(q, S_n p) \\
&\quad \geq \left(\frac{1}{2} + \cdots + \frac{1}{2^{k-1}} + \frac{1}{2^{k+1}} + \cdots + \frac{1}{2^{n-1}} + \frac{1}{2^{n-1}}\right)\cos d(q,p) + \frac{1}{2^k}\cos d(q, S_k p) \\
&\quad = \left(1 - \frac{1}{2^k}\right)\cos d(q,p) + \frac{1}{2^k}\cos d(q, S_k p).
\end{aligned}$$

Letting $n \to \infty$ yields $W_n p \to Wp = p$ and

$$\cos d(q,p) = \cos d(q, Wp) \geq \left(1 - \frac{1}{2^k}\right)\cos d(q,p) + \frac{1}{2^k}\cos d(q, S_k p).$$

This implies that $\cos d(q,p) = \cos d(q, S_k p)$. It follows from Lemma 2.4 that

$$\begin{aligned}
\cos d(q,p)\sin d(p, T_k p) &= \cos d(q, S_k p)\sin d(p, T_k p) \\
&= \cos d\left(q, \frac{1}{2}p \oplus \frac{1}{2}T_k p\right)\sin d(p, T_k p) \\
&\geq \cos d(q,p)\sin\frac{d(p, T_k p)}{2} + \cos d(q, T_k p)\sin\frac{d(p, T_k p)}{2} \\
&\geq 2\cos d(q,p)\sin\frac{d(p, T_k p)}{2}.
\end{aligned}$$

Using elementary trigonometry, we get that $d(p, T_k p) = 0$. Since k is arbitrary, we have $p \in \bigcap_{n=1}^{\infty} F(T_n)$. Hence (i) is proved.

Finally, we prove (ii). We assume that T_n is Δ-demiclosed for all $n \in \mathbb{N}$. We show that W is Δ-demiclosed. Let $\{x_n\} \subset X$ be such that $\lim_{n\to\infty} d(x_n, Wx_n) = 0$ and $\{x_n\}$ Δ-converges to $x \in X$. It follows from the definitions of $\{W_n\}$ and $\{V_n^{(2)}\}$ that

$$W_n = \frac{1}{2}S_1 \oplus \frac{1}{2}V_n^{(2)} \quad \text{for all } n \geq 2.$$

Similar to the proof of the first and the second steps, we can define the quasi-nonexpansive mapping $V : X \to X$ by

$$Vx := \lim_{n\to\infty} V_n^{(2)} x \quad \text{for all } x \in X$$

and $\mathrm{F}(V) = \bigcap_{n=2}^{\infty} \mathrm{F}(T_n)$. This implies that

$$W = \frac{1}{2}S_1 \oplus \frac{1}{2}V \quad \text{and} \quad \mathrm{F}(W) = \mathrm{F}(S_1) \cap \mathrm{F}(V).$$

Then, by Lemma 3.14, we obtain that $d(x_n, S_1x_n) \to 0$ and $d(x_n, Vx_n) \to 0$. Thus

$$d(x_n, T_1x_n) = 2d\left(x_n, \frac{1}{2}x_n \oplus \frac{1}{2}T_1x_n\right) = 2d(x_n, S_1x_n) \to 0.$$

Since T_1 is Δ-demiclosed, we have $x \in \mathrm{F}(T_1)$. Continuing this procedure gives $x \in \bigcap_{n=1}^{\infty} \mathrm{F}(T_n) = \mathrm{F}(W)$. This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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References
1. Bridson, MR, Haefliger, A: Metric Spaces of Non-Positive Curvature. Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 319. Springer, Berlin (1999)
2. Kirk, WA: Geodesic geometry and fixed point theory. In: Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003). Colecc. Abierta, vol. 64, pp. 195-225. Univ. Sevilla Secr. Publ., Seville (2003)
3. Kirk, WA: Geodesic geometry and fixed point theory II. In: International Conference on Fixed Point Theory and Applications, pp. 113-142. Yokohama Publ., Yokohama (2004)
4. Espínola, R, Fernández-León, A: CAT(k)-Spaces, weak convergence and fixed points. J. Math. Anal. Appl. **353**, 410-427 (2009)
5. Mann, WR: Mean value methods in iteration. Proc. Am. Math. Soc. **4**, 506-510 (1953)
6. He, JS, Fang, DH, López, G, Li, C: Mann's algorithm for nonexpansive mappings in CAT(κ) spaces. Nonlinear Anal. **75**, 445-452 (2012)
7. Kirk, WA, Panyanak, B: A concept of convergence in geodesic spaces. Nonlinear Anal. **68**, 3689-3696 (2008)
8. Goebel, K, Kirk, WA: Topics in Metric Fixed Point Theory. Cambridge Studies in Advanced Mathematics, vol. 28. Cambridge University Press, Cambridge (1990)
9. Dhompongsa, S, Kirk, WA, Sims, B: Fixed points of uniformly Lipschitzian mappings. Nonlinear Anal. **65**, 762-772 (2006)
10. Tan, K-K, Xu, HK: Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process. J. Math. Anal. Appl. **178**, 301-308 (1993)
11. Kimura, Y, Satô, K: Halpern iteration for strongly quasinonexpansive mappings on a geodesic space with curvature bounded above by one. Fixed Point Theory Appl. **2013**, 14 (2013)
12. Kimura, Y, Satô, K: Convergence of subsets of a complete geodesic space with curvature bounded above. Nonlinear Anal. **75**, 5079-5085 (2012)
13. Aoyama, K, Kimura, Y, Takahashi, W, Toyoda, M: Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space. Nonlinear Anal. **67**, 2350-2360 (2007)
14. Kimura, Y, Satô, K: Image recovery problem on a geodesic space with curvature bounded above by one (submitted)

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