Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Fixed points of mappings with a contractive iterate at a point in partial metric spaces

Dejan Ilić, Vladimir Pavlović* and Vladimir Rakočević

*Correspondence:
vlada@pmf.ni.ac.rs
Department of Mathematics, Faculty of Sciences and Mathematics, University of Niš, Višegradska 33, Niš, 18000, Serbia

Abstract

In 1994, Matthews introduced and studied the concept of partial metric space and obtained a Banach-type fixed point theorem on complete partial metric spaces. In this paper we study fixed point results of new mappings with a contractive iterate at a point in partial metric spaces. Our results generalize and unify some results of Sehgal, Guseman and Ćirić for mappings with a generalized contractive iterate at a point to partial metric spaces. We give some generalized versions of the fixed point theorem of Matthews. The theory is illustrated by some examples.

Keywords: fixed point; partial metric space; contractive iterate at a point

1 Introduction

In 1922, Banach proved the following famous fixed point theorem [1]. Let (X,d) be a complete metric space. Let T be a contractive mapping on X, that is, one for which exists $q \in [0,1)$ satisfying

$$d(Tx,Ty) \le q \cdot d(x,y) \tag{1.1}$$

for all $x, y \in X$. Then there exists a unique fixed point $x_0 \in X$ of T. This theorem, called the Banach contraction principle, is a forceful tool in nonlinear analysis. This principle has many applications and has been extended by a great number of authors. For the convenience of the reader, let us recall the following results [2–4].

In 1969, Sehgal [4] proved the following interesting generalization of the contraction mapping principle.

Theorem 1.1 ([4]) *Let (X,d) be a complete metric space, $q \in [0,1)$ and $T : X \mapsto X$ be a continuous mapping. If for each $x \in X$ there exists a positive integer $n = n(x)$ such that*

$$d\left(T^n x, T^n y\right) \le q \cdot d(x,y) \tag{1.2}$$

for all $y \in X$, then T has a unique fixed point $u \in X$. Moreover, for any $x \in X$, $u = \lim_m T^m x$.

In 1970, Guseman [3] (see also [5]) generalized the result of Sehgal to mappings which are both necessarily continuous and which have a contractive iterate at each point in a (possibly proper) subset of the space.

In 1983, Ćirić [2], among other things, proved the following interesting generalization of Sehgal's result.

Theorem 1.2 ([2]) *Let (X,d) be a complete metric space, $q \in [0,1)$ and $T : X \mapsto X$. If for each $x \in X$ there exists a positive integer $n = n(x)$ such that*

$$d(T^n x, T^n y) \leq q \cdot \max\{d(x,y), d(x,Ty), \dots, d(x,T^n y), d(x,T^n x)\} \tag{1.3}$$

holds for all $y \in X$, then T has a unique fixed point $u \in X$. Moreover, for every $x \in X$, $u = \lim_m T^m x$.

Partial metric spaces were introduced in [6] by Matthews as part of the study of denotational semantics of dataflow networks and served as a device to solve some difficulties in domain theory of computer science (see also [7, 8]), in particular the ones which arose in the modeling of a parallel computation program given in [9]. The concept has since proved extremely useful in domain theory (see, *e.g.*, [10–13]) and in constructing models in the theory of computation (see, *e.g.*, [14–17]).

On the other hand, fixed point theory of mappings defined on partial metric spaces since the first results obtained in [6] has flourished in the meantime, a fact evidenced by quite a number of papers dedicated to this subject (see, *e.g.*, [18–47]). The task seems to have been set forth of determining what known fixed point results from the usual metric setting remain valid - after adequate modifications that should as much as possible reflect the nature of the concept of partial metric - when formulated in the partial metric setting.

The potentially nonzero self-distance, built into Matthew's definition of partial metrics, was taken into account in [29] in an essential way by a rather mild variation of the classical Banach contractive condition, and in [30] further considerations in this direction were carried out which were in turn generalized by Chi *et al.* in [27]. This paper represents a continuation of the previous work by the authors. Now we study fixed point results of new mappings with a contractive iterate at a point in partial metric spaces. Our results generalize and unify some results of Sehgal, Guseman and Ćirić for mappings with a generalized contractive iterate at a point to partial metric spaces. We give some generalized versions of the fixed point theorem of Matthews. The theory is illustrated by some examples.

2 Preliminaries

Throughout this paper the letters $\mathbb{R}$ and $\mathbb{N}$ will denote the set of real numbers and positive integers, respectively.

Let us recall [7] that a nonnegative mapping $p : X \times X \to \mathbb{R}$, where X is a nonempty set, is said to be a *partial metric on X* if for any $x, y, z \in X$ the following four conditions hold true:

(P1) $p(x,y) = p(y,x)$,
(P2) $p(x,x) \leq p(x,y)$,
(P3) if $p(x,x) = p(y,y) = p(x,y)$, then $x = y$,
(P4) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

The pair (X,p) is then called a *partial metric space*. A sequence $\{x_m\}_{m=0}^{\infty}$ of elements of X is called *p-Cauchy* if the limit $\lim_{m,n} p(x_n, x_m)$ exists and is finite. The partial metric space (X,p) is called *complete* if for each p-Cauchy sequence $\{x_m\}_{m=0}^{\infty}$ there is some $z \in X$ such that

$$p(z,z) = \lim_n p(z,x_n) = \lim_{n,m} p(x_n,x_m). \tag{2.1}$$

Observe that condition (P4) is a strengthening of the triangle inequality and that $p(x,y) = 0$ implies $x = y$ as in the case of an ordinary metric.

It can be shown that if (X,p) is a partial metric space, then by $p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$, for $x,y \in X$, a metric p^s is defined on the set X such that $\{x_n\}_{n\geq 1}$ converges to $z \in X$ with respect to p^s if and only if (2.1) holds. Also (X,p) is a complete partial metric space if and only if (X,p^s) is a complete metric space. For proofs of these facts, see [7, 44].

A paradigm for partial metric spaces is the pair (X,p) where $X = [0,+\infty)$ and $p(x,y) = \max\{x,y\}$ for $x,y \geq 0$. Below we give two more examples of partial metrics both of which are taken from [7].

Example 2.1 If $X := \{[a,b] \mid a,b \in \mathbb{R}, a \leq b\}$, then $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,b\}$ defines a partial metric p on X.

Example 2.2 Let $X := \mathbb{R}^{\mathbb{N}_0} \cup \bigcup_{n\geq 1} \mathbb{R}^{\{0,1,\dots,n-1\}}$, where $\mathbb{N}_0$ is the set of nonnegative integers.

By $L(x)$ denote the set $\{0,1,\dots,n\}$ if $x \in \mathbb{R}^{\{0,1,\dots,n-1\}}$ for some $n \in \mathbb{N}$, and the set $\mathbb{N}_0$ if $x \in \mathbb{R}^{\mathbb{N}_0}$. Then a partial metric is defined on X by

$$p(x,y) = \inf\bigl\{2^{-i} \mid i \in L(x) \cap L(y) \text{ and } \forall j \in \mathbb{N}_0 \ \bigl(j < i \Rightarrow x(j) = y(j)\bigr)\bigr\}.$$

For applications of partial metrics to problems in theoretical computer science, the reader is referred to [8, 11, 14, 16].

In [7] Matthews proved the following extension of the Banach contraction principle to the setting of partial metric spaces.

Theorem 2.1 *Let (X,p) be a complete partial metric space, $\alpha \in [0,1)$ and $T : X \to X$ be a given mapping. Suppose that for each $x,y \in X$ the following condition holds*

$$p(Tx,Ty) \leq \alpha p(x,y). \tag{2.2}$$

Then there is a unique $z \in X$ such that $Tz = z$. Also $p(z,z) = 0$ and for each $x \in X$ the sequence $\{T^n x\}_{n\geq 1}$ converges with respect to the metric p^s to z.

A variant of the result above concerning the so-called dualistic partial metric spaces was later given in [44]. Altun *et al.* [23] further generalized the result of Matthews as well as extended to partial metric spaces several other well-known results about fixed points of mappings on metric spaces.

Taking a different approach to the way in which contractive condition (2.2) can be generalized for partial metrics, we [29, 30] have obtained other extensions of Theorem 2.1. To state one of them, we will use the following notation. Given a partial metric space (X,p) set $r_p := \inf\{p(x,y) : x,y \in X\} = \inf\{p(x,x) : x \in X\}$ and $R_p := \{x \in X : p(x,x) = r_p\}$. Notice that R_p may be empty and that if p is a metric, then, clearly, $r_p = 0$ and $R_p = X$.

Theorem 2.2 (Theorem 3.1 of [29]) *Let (X,p) be a complete partial metric space, $\alpha \in [0,1)$ and $T : X \to X$ be a given mapping. Suppose that for each $x,y \in X$ the following condition holds*

$$p(Tx,Ty) \leq \max\bigl\{\alpha p(x,y), p(x,x), p(y,y)\bigr\}. \tag{2.3}$$

Then the set R_p is nonempty. There is a unique $u \in R_p$ such that $Tu = u$. For each $x \in R_p$, the sequence $\{T^n x\}_{n\geq 1}$ converges with respect to the metric p^s to u.

Remark 2.1 Although Theorem 2.2 does not imply uniqueness of the fixed point, it is easy to see that, under the assumptions made, if u and v are both fixed points satisfying $p(u,u) = p(v,v)$, then $u = v$.

Remark 2.2 Completeness of a partial metric does not necessarily entail that R_p is nonempty. A (class of) counterexample(s) is easily constructed as follows.

Let (X,d) be a partial metric space, $a > 0$ and $f : X \to [0,a)$ be an arbitrary mapping. If $x,y \in X$ are such that $x \neq y$, define $p(x,y) = d(x,y) + a$ and $p(x,x) = f(x)$. Then (X,p) is a partial metric space, as is easily verified.

Now if $b := \sup f[X] < a$, then, given a sequence $\{x_n\}_{n\geq 1}$, we have $\limsup_n p(x_n,x_n) \leq b < a$ and $p(x_n,x_m) \geq a$ whenever $x_n \neq x_m$. Thus there are no nonstationary p-Cauchy sequences. Hence (X,p) is complete. But $R_p = \emptyset$ whenever $\inf f[X] \notin f[X]$.

If condition (2.3) is replaced by the somewhat stronger condition below, then the uniqueness of the fixed point is guaranteed.

Theorem 2.3 (Theorem 3.2 of [29]) *Let (X,p) be a complete partial metric space, $\alpha \in [0,1)$ and $T : X \to X$ be a given mapping. Suppose that for each $x,y \in X$ the following condition holds*

$$p(Tx,Ty) \leq \max\left\{\alpha p(x,y), \frac{p(x,x) + p(y,y)}{2}\right\}. \tag{2.4}$$

Then there is a unique $z \in X$ such that $Tz = z$. Furthermore, $z \in R_p$ and for each $x \in R_p$ the sequence $\{T^n x\}_{n\geq 1}$ converges with respect to the metric p^s to z.

3 Auxiliary results

We now introduce the two types of contractive conditions that we shall be considering in this paper. Let us remark that if $T : X \to X$, then we write $T^0 = I$ for the identity mapping $I : X \to X$, *i.e.*, $I(x) = x$, $x \in X$.

Definition 3.1 Let (X,p) be a partial metric space, $\alpha \in (0,1)$ and $T : X \to X$.

(i) We say that T is a C_1-operator on X if for each $x \in X$ there is some $n(x) \in \mathbb{N}$ such that for each $y \in X$ there holds

$$\begin{aligned} &p\bigl(T^{n(x)}x, T^{n(x)}y\bigr) \\ &\quad\leq \max\bigl\{\alpha p\bigl(x, T^j y\bigr), \alpha p\bigl(x, T^{n(x)}x\bigr), p(x,x), p\bigl(T^{n(x)-1}y, T^{n(x)-1}y\bigr)\bigr\} \end{aligned} \tag{3.1}$$

for some $j \in \{0,1,\ldots,n(x)\}$.

(ii) We say that T is a C-operator on X if for each $x \in X$ there is some $n(x) \in \mathbb{N}$ such that for each $y \in X$ there holds

$$p\bigl(T^{n(x)}x, T^{n(x)}y\bigr) \leq \alpha \max\bigl\{p\bigl(x, T^j y\bigr), p\bigl(x, T^{n(x)}x\bigr)\bigr\} \tag{3.2}$$

for some $j \in \{0,1,\ldots,n(x)\}$.

For a C_1-operator T on a partial metric space (X,p) and $x \in X$ define the supporting sequence at the point x as the sequence $\{s_k\}_{k\geq 0}$, where $s_0 = 0$ and $s_{k+1} = s_k + n(T^{s_k}x)$. Clearly, this is a strictly increasing sequence. Also set $R_T(X) := \{x \in X \mid T^m x = T^{m+1}x$ for some $m \in \mathbb{N}\}$.

Lemma 3.1 *Let T be a C_1-operator on a partial metric space (X,p), $x \in X \setminus R_T(X)$, $\{s_k\}_{k\geq 0}$ be the supporting sequence at x and $k \geq 1$ and $i \geq s_k$ be given integers. Then we must have*

$$p\bigl(T^{s_k}x, T^i x\bigr) \leq \max\bigl\{\alpha p\bigl(T^{s_{k-1}}x, T^j x\bigr), p\bigl(T^{s_{k-1}}x, T^{s_{k-1}}x\bigr)\bigr\} \quad \text{for some } j \geq s_{k-1}. \tag{3.3}$$

Proof *Case* 1. Suppose $i = s_k + 2$. By (3.1) we know that if

$$p\bigl(T^{s_k}x, T^{s_k+2}x\bigr) > \max\bigl\{\alpha p\bigl(T^{s_{k-1}}x, T^j x\bigr), p\bigl(T^{s_{k-1}}x, T^{s_{k-1}}x\bigr)\bigr\} \quad \text{for all } j \in \{s_{k-1}, \dots, s_k + 2\}, \tag{3.4}$$

then

$$p\bigl(T^{s_k}x, T^{s_k+2}x\bigr) \leq p\bigl(T^{s_k+1}x, T^{s_k+1}x\bigr). \tag{3.5}$$

Likewise, if

$$p\bigl(T^{s_k}x, T^{s_k+1}x\bigr) > \max\bigl\{\alpha p\bigl(T^{s_{k-1}}x, T^j x\bigr), p\bigl(T^{s_{k-1}}x, T^{s_{k-1}}x\bigr)\bigr\} \quad \text{for all } j \in \{s_{k-1}, \dots, s_k + 1\}, \tag{3.6}$$

then

$$p\bigl(T^{s_k}x, T^{s_k+1}x\bigr) \leq p\bigl(T^{s_k}x, T^{s_k}x\bigr). \tag{3.7}$$

Now if (3.4) were to hold, then (3.5) and (P2) would imply that (3.6) is true as well. So (3.7) also holds and thus

$$\begin{aligned} p\bigl(T^{s_k}x, T^{s_k+2}x\bigr) &\leq p\bigl(T^{s_k+1}x, T^{s_k+1}x\bigr) \leq p\bigl(T^{s_k}x, T^{s_k+1}x\bigr) \\ &\leq p\bigl(T^{s_k}x, T^{s_k}x\bigr) \leq p\bigl(T^{s_k}x, T^{s_k+2}x\bigr) \end{aligned}$$

meaning (by (P3)) that $T^{s_k}x = T^{s_k+1}x$. But this contradicts the assumption $x \notin R_T(X)$, so (3.4) must be true.

Case 2. Suppose $i = s_k$. Since $p(T^{s_k}x, T^{s_k}x) \leq p(T^{s_k}x, T^{s_k+2}x)$, the assertion here follows from the previous case.

Case 3. Suppose now $i = s_k + 1$. Assume that

$$p\bigl(T^{s_k}x, T^{s_k+1}x\bigr) > \max\bigl\{\alpha p\bigl(T^{s_{k-1}}x, T^j x\bigr), p\bigl(T^{s_{k-1}}x, T^{s_{k-1}}x\bigr)\bigr\} \quad \text{for all } j \in \{s_{k-1}, \dots, s_k + 1\}$$

since otherwise there is nothing to prove. Then by (3.1) we must have $p(T^{s_k}x, T^{s_k+1}x) \leq p(T^{s_k}x, T^{s_k}x)$. But then, by the previous case, there is some $j \geq s_{k-1}$ such that

$$p\left(T^{s_k}x, T^{s_k+1}x\right) \leq p\left(T^{s_k}x, T^{s_k}x\right) \leq \max\left\{\alpha p\left(T^{s_{k-1}}x, T^{j}x\right), p\left(T^{s_{k-1}}x, T^{s_{k-1}}x\right)\right\}$$

and we are done.

For $i > s_k + 2$, the argument carries on by induction. Suppose that (3.3) holds for some $i \geq s_k$. If $p(T^{s_k}x, T^{i+1}x) > \{\alpha p(T^{s_{k-1}}x, T^{j}x), p(T^{s_{k-1}}x, T^{s_{k-1}}x)\}$ for all $j \in \{s_{k-1}, \ldots, i+1\}$, then we must have $p(T^{s_k}x, T^{i+1}x) \leq p(T^{i}x, T^{i}x)$. By the induction hypothesis, there is some $j \geq s_{k-1}$ such that

$$p\left(T^{s_k}x, T^{i}x\right) \leq \left\{\alpha p\left(T^{s_{k-1}}x, T^{j}x\right), p\left(T^{s_{k-1}}x, T^{s_{k-1}}x\right)\right\}.$$

But since $p(T^{s_k}x, T^{i+1}x) \leq p(T^{i}x, T^{i}x) \leq p(T^{s_k}x, T^{i}x)$, the last inequality clashes with our assumption. □

To shorten the foregoing considerations, we introduce some auxiliary notions as follows. Fix $x \in X \setminus R_T(X)$. For integers $k \geq 1$ and $i \geq s_k$, use Lemma 3.1 repeatedly to fix integers $l_j \geq s_j$, $0 \leq j < k$ and $t_1, \ldots, t_k \in \{0,1\}$ such that, putting $l_k := i$, there holds

$$p\left(T^{s_j}x, T^{l_j}x\right) \leq \alpha^{t_j} p\left(T^{s_{j-1}}x, T^{l_{j-1}}x\right)$$

for all $1 \leq j \leq k$, where

$$t_j = \begin{cases} 1 & \text{if } s_{j-1} < l_{j-1}, \\ 0 & \text{if } s_{j-1} = l_{j-1}. \end{cases}$$

We shall refer to $(l_0, \ldots, l_{k-1})$ and $(t_1, \ldots, t_k)$ as the (k,i)-descent and the (k,i)-signature at x, respectively. Set $S^x_{k,i} := \{j \in \{1, \ldots, k\} \mid t_j = 1\}$. We shall say that x is of type 1 if there are sequences of positive integers $\{k_m\}_{m\geq 0}$ and $\{i_m\}_{m\geq 0}$, the first one strictly increasing, such that for all $m \geq 0$ we have $i_m \geq s_{k_m}$ and $\operatorname{card}(S^x_{k_m,i_m}) < \operatorname{card}(S^x_{k_{m+1},i_{m+1}})$; here and henceforth, for a finite set P, we denote by $\operatorname{card}(P)$ the number of its elements. We shall say that x is of type 2 if x is not of type 1, *i.e.*, if there are $k_0, D \in \mathbb{N}$ such that for all $k \geq k_0$ and all $i \geq s_k$ there holds $\operatorname{card}(S^x_{k,i}) < D$.

To make the proof of our main result more transparent, we have extracted from it several parts and presented them first in form of the next five lemmas.

Lemma 3.2 *Let T be a* C_1*-operator on a partial metric space* (X,p), $x \notin R_T(X)$, *and let* $\{s_k\}_{k\geq 0}$ *be the supporting sequence at* x. *Then*:

(a) *If* $(l_0, \ldots, l_{k-1})$ *is the* (k, i_0)*-descent at* x, *then*

$$p\left(T^{s_k}x, T^{i_0}x\right) \leq \alpha^{\operatorname{card}(S^x_{k,i_0})} p\left(x, T^{l_0}x\right) \quad \textit{and}$$
$$p\left(T^{s_k}x, T^{i_0}x\right) \leq p\left(T^{s_j}x, T^{l_j}x\right) \quad \textit{for all } 0 \leq j \leq k,$$

where we set $l_k := i_0$.

(b) *If* $P \subseteq \{0,\dots,k-1\}$ *is such that* $\operatorname{card}(S^x_{k,i_0}) < \operatorname{card}(P)$, *then for some* $j_0 \in P$ *there must hold*

$$p\bigl(T^{s_k}x, T^{i_0}x\bigr) \le p\bigl(T^{s_{j_0}}x, T^{s_{j_0}}x\bigr).$$

Proof Regarding Lemma 3.1, we get $p(T^{s_k}x, T^{i_0}x) \le \alpha^{\sum_{i=j+1}^{k} t_i} p(T^{s_j}x, T^{l_j}x)$ for all j such that $k > j \ge 0$, recursively. Now (a) follows directly.

To prove (b) simply observe that $\operatorname{card}(S^x_{k,i_0}) < \operatorname{card}(P)$ implies that the set $\{j+1 \mid j \in P\}$ is a subset of $\{1,\dots,k\}$ with $\operatorname{card}(P) > \operatorname{card}(S^x_{k,i_0})$ elements so that there must be some $j_0 \in P$ with $t_{j_0+1} = 0$. Hence

$$p\bigl(T^{s_k}x, T^{i_0}x\bigr) \le p\bigl(T^{s_{j_0+1}}x, T^{l_{j_0+1}}x\bigr) \le \alpha^{t_{j_0+1}} p\bigl(T^{s_{j_0}}x, T^{l_{j_0}}x\bigr) = p\bigl(T^{s_{j_0}}x, T^{s_{j_0}}x\bigr),$$

where we used (a) and the fact that $l_{j_0} = s_{j_0}$. □

Lemma 3.3 *If T is a* C_1*-operator on a partial metric space (X,p) and $x \in X$, then there is some $M_x > 0$ such that for all $i \ge 0$ we must have $p(x, T^i x) \le M_x$.*

Proof If $x \in R_T(X)$, then this is obvious. Thus suppose $x \notin R_T(X)$ and set $M_x := \frac{1}{1-\alpha}\max\{p(x, T^j x) \mid 0 < j \le n(x)\} + p(x,x)$. If $k = n(x)$, then it is certainly true that $p(x, T^i x) \le M_x$ for all $0 \le i \le k$. Now suppose that the same is valid for some $k \ge n(x)$.

If $p(x, T^{k+1}x) \le p(x, T^i x)$ for some $0 \le i \le k$, then by the induction hypothesis there holds $p(x, T^{k+1}x) \le p(x, T^i x) \le M_x$. Otherwise, we must have

$$p\bigl(x, T^{k+1}x\bigr) > \max\bigl\{p\bigl(x, T^i x\bigr) \mid 0 \le i \le k\bigr\}. \tag{3.8}$$

Now using (3.1)

$$\begin{aligned} p\bigl(x, T^{k+1}x\bigr) &\le p\bigl(x, T^{n(x)}x\bigr) + p\bigl(T^{n(x)}x, T^{k+1}x\bigr) \\ &\le p\bigl(x, T^{n(x)}x\bigr) + \max\bigl\{\alpha p\bigl(x, T^j x\bigr), p(x,x), p\bigl(T^k x, T^k x\bigr), \alpha p\bigl(x, T^{n(x)}x\bigr)\bigr\} \end{aligned}$$

for some $k+1-n(x) \le j \le k+1$. Hence we either have

$$p\bigl(x, T^{k+1}x\bigr) \le p\bigl(x, T^{n(x)}x\bigr) + p(x,x) \le M_x$$

or, using (3.8) and the fact that $p(T^k x, T^k x) \le p(x, T^k x)$, it must be that

$$\begin{aligned} &p\bigl(x, T^{k+1}x\bigr) \le p\bigl(x, T^{n(x)}x\bigr) + \alpha p\bigl(x, T^{k+1}x\bigr), \quad \textit{i.e.}, \\ &p\bigl(x, T^{k+1}x\bigr) \le \frac{1}{1-\alpha} p\bigl(x, T^{n(x)}x\bigr) \le M_x. \end{aligned}$$

By induction the desired conclusion follows. □

Lemma 3.4 *Let T be a* C_1*-operator on a partial metric space (X,p) and $x \in X \setminus R_T(X)$. If x is of type* 1, *then* $\lim_{i,j} p(T^i x, T^j x) = 0$.

Proof Fix $m \geq 0$. If $(l_0, \dots, l_{k_m-1})$ is the (s_{k_m}, i_m)-descent, then by (a) of Lemma 3.2 we have

$$p(T^{s_{k_m}}x, T^{i_m}x) \leq \alpha^{\operatorname{card}(S^x_{k_m,i_m})} p(x, T^{l_0}x) \leq \alpha^{\operatorname{card}(S^x_{k_m,i_m})} M_x.$$

Since $\lim_m \operatorname{card}(S^x_{k_m,i_m}) = \infty$, this implies $\lim_m p(T^{s_{k_m}}x, T^{i_m}x) = \lim_m p(T^{s_{k_m}}x, T^{s_{k_m}}x) = 0$.

Now given $\varepsilon > 0$ choose $m_0 \geq 1$ such that $\alpha^{m_0} M_x < \varepsilon$ and such that for all $m \geq m_0$ it holds $p(T^{s_{k_m}}x, T^{s_{k_m}}x) < \varepsilon$. Let $i \geq s_{k_{2m_0}}$ be arbitrary.

Suppose first that $\operatorname{card}(S^x_{k_{2m_0},i}) \geq m_0$. Then

$$p(T^{s_{k_{2m_0}}}x, T^i x) \leq \alpha^{\operatorname{card}(S^x_{k_{2m_0},i})} M_x \leq \alpha^{m_0} M_x < \varepsilon.$$

Suppose now that $\operatorname{card}(S^x_{k_{2m_0},i}) < m_0$. For $P := \{k_{m_0}, \dots, k_{2m_0-1}\} \subseteq \{0, 1, \dots, k_{2m_0} - 1\}$, we have $\operatorname{card}(P) > \operatorname{card}(S^x_{k_{2m_0},i})$, so by (b) of Lemma 3.2 there must be some $m_0 \leq j \leq 2m_0 - 1$ such that $p(T^{s_{k_{2m_0}}}x, T^i x) \leq p(T^{s_{k_j}}x, T^{s_{k_j}}x) < \varepsilon$.

We have thus shown that $p(T^{s_{k_{2m_0}}}x, T^i x) < \varepsilon$ must hold for all $i \geq s_{k_{2m_0}}$. Therefore if $i, j \geq s_{k_{2m_0}}$, then

$$p(T^i x, T^j x) \leq p(T^{s_{k_{2m_0}}}x, T^i x) + p(T^{s_{k_{2m_0}}}x, T^j x) < 2\varepsilon.$$

The previous analysis proves $\lim_{i,j} p(T^i x, T^j x) = 0$. □

Lemma 3.5 *Let T be a C_1-operator on a partial metric space (X, p) and $x \in X \setminus R_T(X)$. If x is of type 2, then the sequence $\{T^i x\}_{i \geq 0}$ is p-Cauchy.*

Proof Let $\{s_k\}_{k \geq 0}$ be the supporting sequence at x.

We first show $\liminf_m p(T^{s_m}x, T^{s_m}x) = \limsup_m p(T^{s_m}x, T^{s_m}x)$. Suppose that this is not true and pick a real θ with $\liminf_m p(T^{s_m}x, T^{s_m}x) < \theta < \limsup_m p(T^{s_m}x, T^{s_m}x)$. Let $k_1 < k_2 < \cdots < k_D < k_{D+1}$ and $i > k_{D+1}$ be positive integers, where $k_{D+1} \geq k_0$, such that

$$p(T^{s_{k_j}}x, T^{s_{k_j}}x) < \theta \quad \text{for all } 1 \leq j \leq D \quad \text{and} \quad p(T^{s_i}x, T^{s_i}x) > \theta.$$

The fact that $s_i > s_{k_{D+1}}$ implies that $S^x_{k_{D+1},s_i}$ is defined and since $k_{D+1} \geq k_0$, we have $\operatorname{card}(S^x_{k_{D+1},s_i}) < D$. For $P := \{k_1, \dots, k_D\} \subseteq \{0, 1, \dots, k_{D+1} - 1\}$, we have $\operatorname{card}(P) = D > \operatorname{card}(S^x_{k_{D+1},s_i})|$ so, by (b) of Lemma 3.2, there is some $j \in \{1, \dots, D\}$ such that

$$\theta < p(T^{s_i}x, T^{s_i}x) \leq p(T^{s_{k_{D+1}}}x, T^{s_i}x) \leq p(T^{s_{k_j}}x, T^{s_{k_j}}x) < \theta,$$

a contradiction.

By the preceding part and since $0 \leq p(T^{s_m}x, T^{s_m}x) \leq 2M_x$, with M_x as in Lemma 3.3, we have $r_x := \lim_m p(T^{s_m}x, T^{s_m}x) \in \mathbb{R}$.

Let us prove

$$\forall \varepsilon > 0\ \exists m_0\ \forall m \geq m_0\ \forall i \geq s_m \quad p(T^{s_m}x, T^i x) \in (r_x - \varepsilon, r_x + \varepsilon). \tag{3.9}$$

Given $\varepsilon > 0$, take $m_1 \geq k_0$ such that $p(T^{s_m}x, T^{s_m}x) \in (r_x - \varepsilon, r_x + \varepsilon)$ for all $m \geq m_1$. Let $m \geq m_1 + D$ and $i \geq s_m$ be arbitrary. For $P := \{m_1, \dots, m_1 + D - 1\} \subseteq \{0, 1, \dots, m - 1\}$, we have

$\mathrm{card}(P) = D > \mathrm{card}(S^x_{m,i})$, so for some $m_1 \le j \le m_1 + D - 1$ it must be

$$r_x - \varepsilon < p\bigl(T^{s_m}x, T^{s_m}x\bigr) \le p\bigl(T^{s_m}x, T^i x\bigr) \le p\bigl(T^{s_j}x, T^{s_j}x\bigr) < r_x + \varepsilon$$

and we are done.

From (3.9) it now immediately follows that

$$\forall \varepsilon > 0\ \exists k\ \forall i, j \ge k \quad p\bigl(T^i x, T^j x\bigr) < r_x + \varepsilon. \tag{3.10}$$

Indeed, given $\varepsilon > 0$, consider m_0 as in (3.9) and let $i, j \ge s_{m_0}$ be arbitrary. Then

$$\begin{aligned} p\bigl(T^i x, T^j x\bigr) &\le p\bigl(T^{s_{m_0}}x, T^i x\bigr) + \bigl[p\bigl(T^{s_{m_0}}x, T^j x\bigr) - p\bigl(T^{s_{m_0}}x, T^{s_{m_0}}x\bigr)\bigr] \\ &< r_x + \varepsilon + 2\varepsilon = r_x + 3\varepsilon. \end{aligned}$$

To prove

$$\lim_{i,j} p\bigl(T^i x, T^j x\bigr) = r_x, \tag{3.11}$$

we now only need to show that

$$\forall \varepsilon > 0\ \exists k\ \forall i, j \ge k \quad r_x - \varepsilon < p\bigl(T^i x, T^j x\bigr). \tag{3.12}$$

Let $\varepsilon \in (0, \frac{r_x(1-\alpha)}{1+\alpha})$ be arbitrary and let $k \in \mathbb{N}$ be as in (3.10). We claim that $r_x - \varepsilon < p(T^i x, T^i x)$ holds for all $i \ge k$. This would prove (3.12) since $p(T^i x, T^i x) \le p(T^i x, T^j x)$.

Suppose to the contrary that there is some $i_0 \ge k$ with $p(T^{i_0}x, T^{i_0}x) \le r_x - \varepsilon$. Put $z := T^{i_0}x$. $x \notin R_T(X)$ implies $z \notin R_T(X)$. If z is of type 1, then by Lemma 3.4 we have $0 = \lim_{i,j} p(T^i z, T^j z) = \lim_{i,j} p(T^i x, T^j x)$, so $\{T^i x\}_{i\ge 0}$ is p-Cauchy and we are finished. Suppose now that z is of type 2, so that by what we have proved thus far we know that $r_z = \lim_m p(T^{s_m}z, T^{s_m}z) \in \mathbb{R}$ and also that (3.10) holds with z taken instead of x. It cannot be $p(T^{n(z)}z, T^{n(z)}z) > r_x - \varepsilon$ because this would mean that $p(z,z) < p(T^{n(z)}z, T^{n(z)}z)$, so using Lemma 3.1, it would follow $r_x - \varepsilon < p(T^{n(z)}z, T^{n(z)}z) \le \alpha p(z, T^j z)$ for some $j \in \mathbb{N}_0$, *i.e.*, $r_x - \varepsilon < \alpha p(T^{i_0}x, T^{i_0+j}x) \le \alpha(r_x + \varepsilon)$, giving $\frac{r_x(1-\alpha)}{1+\alpha} < \varepsilon$, a contradiction. So $p(T^{n(z)}z, T^{n(z)}z) \le r_x - \varepsilon$. The argument actually shows that $p(T^{q_m}z, T^{q_m}z) \le r_x - \varepsilon$ holds for every $m \ge 0$, where $\{q_m\}_{m\ge 0}$ is the supporting sequence at the point z. So $r_z = \lim_m p(T^{q_m}z, T^{q_m}z) \le r_x - \varepsilon$. Now use the fact that $r_z < \frac{r_z + r_x}{2}$ and (3.10) (with z taken instead of x and $\frac{r_x - r_z}{2}$ instead of ε of course) to find $j_0 \in \mathbb{N}$ such that

$$p\bigl(T^j z, T^j z\bigr) < \frac{r_z + r_x}{2} \quad \text{for all } j \ge j_0. \tag{3.13}$$

As $\lim_m p(T^{s_m}x, T^{s_m}x) = r_x$ and $\frac{r_z + r_x}{2} < r_x$, there is some $m \ge i_0 + j_0$ with $p(T^{s_m}x, T^{s_m}x) > \frac{r_z + r_x}{2}$. Now, using $s_m - i_0 \ge m - i_0 \ge j_0$, we obtain

$$\frac{r_z + r_x}{2} < p\bigl(T^{s_m}x, T^{s_m}x\bigr) = p\bigl(T^{s_m - i_0}z, T^{s_m - i_0}z\bigr) < \frac{r_z + r_x}{2},$$

which is not possible. □

Lemma 3.6 *Let $T : X \to X$ and let $p : X \times X \to \mathbb{R}$ be any mapping satisfying* (P3). *Suppose that $x \in X$ is such that $T^k x = x$ holds for some positive integer k, and that there exists $y \in X$ such that*

$$p(y,y) = \lim_i p\bigl(y, T^i x\bigr) = \lim_{i,j} p\bigl(T^i x, T^j x\bigr). \tag{3.14}$$

Then $Tx = x$.

Proof From $T^{ki}x = x$, $i \geq 0$, we have

$$p(y,y) = \lim_i p\bigl(y, T^{ki}x\bigr) = p(y,x) \quad \text{and} \quad p(y,y) = \lim_i p\bigl(T^{ki}x, T^{ki}x\bigr) = p(x,x),$$

hence $y = x$. But (3.14) now gives

$$p(x,x) = \lim_i p\bigl(x, T^{ki+1}x\bigr) = p(x, Tx) \quad \text{and} \quad p(x,x) = \lim_i p\bigl(T^{ki+1}x, T^{ki+1}x\bigr) = p(Tx, Tx)$$

so $Tx = x$. □

4 Main results

Having made the necessary preparations, we are now able to prove fixed point results for C_1-operators on complete partial metric spaces. But first we prove a proposition giving some insight into the structure of this type of mappings.

Proposition 4.1 *If T is a C_1-operator on a complete partial metric space (X,p), then*

(1) *for each $x \in X$, the sequence $\{T^i x\}_{i\geq 0}$ p^s-converges to some $v_x \in X$;*
(2) *for all $x, y \in X$, there holds $p(v_x, v_y) = \max\{p(v_x, v_x), p(v_y, v_y)\}$.*

Proof The existence of such points v_x is assured by Lemmas 3.4 and 3.5 and completeness if $x \in X \setminus R_T(X)$, and is self-evident if $x \in R_T(X)$.

To prove (2), let $x, y \in X$ be arbitrary and suppose that $p(v_x, v_x) \geq p(v_y, v_y)$. If $p(v_x, v_y) = 0$, then $v_x = v_y$ (by (P2) and (P3)) and we are done. Thus assume that $p(v_x, v_y) > 0$ and let $\varepsilon > 0$ be arbitrary such that $\frac{2\varepsilon(2+\alpha)}{1-\alpha} < p(v_x, v_y)$. There is some $m_0 \in \mathbb{N}$ such that for all $i, j \geq m_0$ there holds

$$\begin{aligned}
&\max\bigl\{\bigl|p\bigl(T^i y, T^j y\bigr) - p(v_y, v_y)\bigr|, \bigl|p\bigl(v_y, T^j y\bigr) - p(v_y, v_y)\bigr|\bigr\} < \varepsilon,\\
&\max\bigl\{\bigl|p\bigl(T^i x, T^j x\bigr) - p(v_x, v_x)\bigr|, \bigl|p\bigl(v_x, T^j x\bigr) - p(v_x, v_x)\bigr|\bigr\} < \varepsilon.
\end{aligned}$$

For $i, j \geq m_0$ we have

$$p\bigl(T^i y, T^j x\bigr) \leq p\bigl(T^i y, v_y\bigr) - p(v_y, v_y) + p(v_y, v_x) - p(v_x, v_x) + p\bigl(v_x, T^j x\bigr) < 2\varepsilon + p(v_y, v_x)$$

and, similarly,

$$\begin{aligned}
p(v_y, v_x) &\leq p\bigl(v_y, T^i y\bigr) - p\bigl(T^i y, T^i y\bigr) + p\bigl(T^i y, T^j x\bigr) - p\bigl(T^j x, T^j x\bigr) + p\bigl(T^j x, v_x\bigr)\\
&< 4\varepsilon + p\bigl(T^i y, T^j x\bigr).
\end{aligned}$$

Fix any $i_1 \geq m_0$ and set $i_1 := n(T^{i_0}x)$. By (3.1) there is some $j_0 \in \{i_0, \dots, i_0 + i_1\}$ such that

$$\begin{aligned}
&p(v_y, v_x) - 4\varepsilon \\
&\quad < p\bigl(T^{i_0+i_1}x, T^{i_0+i_1}y\bigr) \\
&\quad \leq \max\bigl\{\alpha p\bigl(T^{i_0}x, T^{j_0}y\bigr), \alpha p\bigl(T^{i_0}x, T^{i_0+i_1}x\bigr), p\bigl(T^{i_0}x, T^{i_0}x\bigr), p\bigl(T^{i_0+i_1-1}y, T^{i_0+i_1-1}y\bigr)\bigr\} \\
&\quad \leq \max\bigl\{\alpha\bigl[2\varepsilon + p(v_y, v_x)\bigr], \alpha\bigl[p(v_x, v_x) + \varepsilon\bigr], p(v_x, v_x) + \varepsilon, p(v_y, v_y) + \varepsilon\bigr\} \\
&\quad = \max\bigl\{\alpha\bigl[2\varepsilon + p(v_y, v_x)\bigr], p(v_x, v_x) + \varepsilon\bigr\}.
\end{aligned}$$

Now $p(v_y, v_x) - 4\varepsilon < \alpha[2\varepsilon + p(v_y, v_x)]$ is just $\frac{2\varepsilon(2+\alpha)}{1-\alpha} > p(v_x, v_y)$, which is false by our choice of ε. This leaves us with the only other possibility: $p(v_y, v_x) - 4\varepsilon < p(v_x, v_x) + \varepsilon$, *i.e.*, $p(v_y, v_x) < p(v_x, v_x) + 5\varepsilon$.

From the preceding analysis it follows that $p(v_y, v_x) \leq p(v_x, v_x)$, which by (P2) actually means that $p(v_y, v_x) = p(v_x, v_x) = \max\{p(v_x, v_x), p(v_y, v_y)\}$. □

Theorem 4.1 *If T is a* C_1*-operator on a complete partial metric space (X, p), then there is a fixed point $z \in X$ of T such that $p(z, z) = \inf_{x \in X} p(v_x, v_x)$, where v_x are as in Proposition* 4.1.

Proof For $x \in X$ put $r_x := p(v_x, v_x)$ (this is consistent with the notation of Lemma 3.5). Set $I := \inf_{x \in X} r_x$. For $m \geq 1$ pick $x_m \in X$ such that for all $i, j \geq 0$ it holds

$$p\bigl(T^i x_m, T^j x_m\bigr) \in (I - 1/m, I + 1/m). \tag{4.1}$$

(You can first pick $x'_m \in X$ such that $\lim_{i,j} p(T^i x'_m, T^j x'_m) = v_{x'_m} \in [I, I + \frac{1}{m})$, then choose $k(m) \in \mathbb{N}$ such that for all $i, j \geq k(m)$ there holds $I - \frac{1}{m} < p(T^i x'_m, T^j x'_m) < I + \frac{1}{m}$ and finally put $x_m := T^{k(m)} x'_m$.)

Notice that if $i(m)$ and $j(m)$ are nonnegative integers for $m \in \mathbb{N}$, then we have

$$\lim_m p\bigl(T^{i(m)} x_m, T^{j(m)} x_m\bigr) = 0.$$

First we prove that $\lim_{m,k} p(x_m, x_k) = I$.

For $m, k \geq 2$ let $C_{m,k} > 0$ be such that $p(T^i x_m, T^j x_k) < C_{m,k}$ holds for all $i, j \geq 0$.

Fix $m, k \geq 2$ and let $\{s_q\}_{q \in \mathbb{N}}$ be the supporting sequence at x_m. Let $l \geq 1$ be any integer such that $\alpha^l C_k < \frac{1}{k+m}$. We have

$$\begin{aligned}
p(x_m, x_k) \leq{}& p\bigl(x_m, T^{s_l} x_m\bigr) - p\bigl(T^{s_l} x_m, T^{s_l} x_m\bigr) \\
&+ p\bigl(T^{s_l} x_m, T^{s_l} x_k\bigr) + p\bigl(x_k, T^{s_l} x_k\bigr) - p\bigl(T^{s_l} x_k, T^{s_l} x_k\bigr).
\end{aligned}$$

Now

$$\delta_{m,k} := p\bigl(x_m, T^{s_l} x_m\bigr) - p\bigl(T^{s_l} x_m, T^{s_l} x_m\bigr) < 2/m$$

and

$$\mu_{m,k} := p\bigl(x_k, T^{s_l} x_k\bigr) - p\bigl(T^{s_l} x_k, T^{s_l} x_k\bigr) < 2/k.$$

First suppose that $p(T^{s_l}x_m, T^{s_l}x_k) > p(T^i x_k, T^i x_k)$ for all $i \in \{0,\dots,s_l\}$, and $p(T^{s_l}x_m, T^{s_l}x_k) > p(T^i x_m, T^j x_m)$ for all $i,j \in \{0,\dots,s_l\}$. Then, by repeated use of (3.1), we obtain

$$p\bigl(T^{s_l}x_m, T^{s_l}x_k\bigr) \le \alpha p\bigl(T^{s_{l-1}}x_m, T^{i_1}x_k\bigr) \quad \text{for some } i_1 \ge s_{l-1},$$
$$p\bigl(T^{s_l}x_m, T^{s_l}x_k\bigr) \le \alpha^2 p\bigl(T^{s_{l-2}}x_m, T^{i_2}x_k\bigr) \quad \text{for some } i_2 \ge s_{l-2},$$

and continuing in this manner finally

$$p\bigl(T^{s_l}x_m, T^{s_l}x_k\bigr) \le \alpha^l p\bigl(x_m, T^{i_l}x_k\bigr) \quad \text{for some } i_l \ge 0.$$

Thus $p(T^{s_l}x_m, T^{s_l}x_k) \le \alpha^l C_{m,k} < \frac{1}{k+m}$.

On the other hand, if $p(T^{s_l}x_m, T^{s_l}x_k) \le p(T^i x_k, T^i x_k)$ for some $i \in \{0,\dots,s_l\}$, or $p(T^{s_l}x_m, T^{s_l}x_k) \le p(T^i x_m, T^j x_m)$ for some $i,j \in \{0,\dots,s_l\}$, then by (4.1) we must have $p(T^{s_l}x_m, T^{s_l}x_k) < I + \max\{\frac{1}{m}, \frac{1}{k}\}$.

Therefore

$$\begin{aligned} p(x_m, x_k) &\le \delta_{m,k} + \mu_{m,k} + p\bigl(T^{s_l}x_m, T^{s_l}x_k\bigr) \\ &< 2\left(\frac{1}{m} + \frac{1}{k}\right) + I + \max\left\{\frac{1}{m}, \frac{1}{k}\right\}. \end{aligned}$$

From the above considerations and from $I - 1/m < p(x_m, x_m) \le p(x_m, x_k)$, it is now clear that $\lim_{m,k} p(x_m, x_k) = I$.

So by completeness there is some $u \in X$ such that

$$I = \lim_{m,k} p(x_m, x_k) = \lim_k p(u, x_k) = p(u, u). \tag{4.2}$$

Let $\{s_m\}_{m\ge 0}$ be the supporting sequence at u.

Let us show by induction on k that if $f : \mathbb{N} \to \mathbb{N}_0$ is such that $f(m) \ge s_k$, for all $m \in \mathbb{N}$, then

$$I = \lim_m p\bigl(T^{s_k}u, T^{f(m)}x_m\bigr) = p(u, u). \tag{4.3}$$

Suppose first that $k = 0$.

$$\begin{aligned} p(u, u) &\le p\bigl(u, T^{f(m)}x_m\bigr) \le p(u, x_m) + p\bigl(x_m, T^{f(m)}x_m\bigr) - p(x_m, x_m) \\ &< p(u, x_m) + \frac{2}{m}, \end{aligned}$$

so the desired conclusion immediately follows. Now suppose that the assertion is true for some $k \ge 0$, take any $f : \mathbb{N} \to \mathbb{N}_0$ such that $f(m) \ge s_{k+1}$, $m \ge 1$, and proceed as follows.

We have

$$p\bigl(T^{s_k}u, T^{s_k}u\bigr) \le 2p\bigl(T^{s_k}u, T^{f(m)}x_m\bigr) - p\bigl(T^{f(m)}x_m, T^{f(m)}x_m\bigr) \le 2p\bigl(T^{s_k}u, T^{f(m)}x_m\bigr) - I + \frac{1}{m}$$

for all $m \in \mathbb{N}$, so that taking the limit above as m approaches infinity and using (4.3) (which is justified since $f(m) \ge s_{k+1} > s_k$) it follows that $p(T^{s_k}u, T^{s_k}u) \le I$.

Now for each $m \in \mathbb{N}$, since $f(m) - (s_{k+1} - s_k) \geq s_k$, there must be some $h(m) \in \{s_k, \dots, f(m)\}$ such that

$$p\bigl(T^{s_{k+1}}u, T^{f(m)}x_m\bigr) \leq \max\bigl\{\alpha p\bigl(T^{s_k}u, T^{h(m)}x_m\bigr), p\bigl(T^{s_k}u, T^{s_k}u\bigr),$$
$$p\bigl(T^{f(m)-1}x_m, T^{f(m)-1}x_m\bigr), \alpha p\bigl(T^{s_k}u, T^{s_{k+1}}u\bigr)\bigr\}.$$

Using $p(T^{s_k}u, T^{s_{k+1}}u) \leq p(T^{s_k}u, T^{f(m)}x_m) - p(T^{f(m)}x_m, T^{f(m)}x_m) + p(T^{s_{k+1}}u, T^{f(m)}x_m)$, we proceed to obtain

$$p\bigl(T^{f(m)}x_m, T^{f(m)}x_m\bigr) \leq p\bigl(T^{s_{k+1}}u, T^{f(m)}x_m\bigr) \leq \max\biggl\{\alpha p\bigl(T^{s_k}u, T^{h(m)}x_m\bigr), I, I + \frac{1}{m},$$
$$\frac{\alpha}{1-\alpha}\bigl[p\bigl(T^{s_k}u, T^{f(m)}x_m\bigr) - p\bigl(T^{f(m)}x_m, T^{f(m)}x_m\bigr)\bigr]\biggr\}.$$

Now we have $I - 1/m < p(T^{f(m)}x_m, T^{f(m)}x_m) < I + 1/m$ and also $h(m) \geq s_k$ and $f(m) \geq s_{k+1} > s_k$ for all $m \geq 1$. Hence, in view of the induction hypothesis, we finally arrive at $\lim_m p(T^{s_{k+1}}u, T^{f(m)}x_m) = I$.

Using (4.3) it is straightforward to see that for all $k_1, k_2 \geq 0$ there holds $p(T^{s_{k_1}}u, T^{s_{k_2}}u) \leq I = p(u,u)$: indeed this follows by letting $m \in \mathbb{N}$ tend to infinity in

$$p\bigl(T^{s_{k_1}}u, T^{s_{k_2}}u\bigr) \leq p\bigl(T^{s_{k_1}}u, T^{s_{k_2}}x_m\bigr) + p\bigl(T^{s_{k_2}}u, T^{s_{k_2}}x_m\bigr) - p\bigl(T^{s_{k_2}}x_m, T^{s_{k_2}}x_m\bigr).$$

Thus $r_u \leq I$. But by definition of I we must actually have $I = r_u$.

We now claim that there are positive integers $k_1 < k_2$ such that

$$p\bigl(T^{s_{k_1}}u, T^{s_{k_1}}u\bigr) = p\bigl(T^{s_{k_2}}u, T^{s_{k_2}}u\bigr) = I.$$

Assume this is not the case. Then $p(T^{s_k}u, T^{s_k}u) = I$ can hold for at most one $k \in \mathbb{N}$. As we have $0 \leq p(T^{s_k}u, T^{s_k}u) \leq I$ for all $k \in \mathbb{N}$, our assumption implies in particular that $r_u = I > 0$. Thus we can take some $\varepsilon > 0$ such that $r_u - \varepsilon > \alpha(r_u + \varepsilon)$. The assumption also allows us to find some $m_0 \in \mathbb{N}$ such that for all k with $s_k \geq m_0$ we have $p(T^{s_k}u, T^{s_k}u) < I$, and such that for all $i, j \geq m_0$ it holds $p(T^i u, T^j u) \in (r_u - \varepsilon, r_u + \varepsilon)$ (remember that $r_u = p(v_u, v_u) = \lim_{i,j} p(T^i u, T^j u)$).

Take any k with $s_k \geq m_0$. Then $L := \max\{p(T^{s_k}u, T^{s_k}u), p(T^{s_{k+1}}u, T^{s_{k+1}}u)\} < I = r_y$. There is some positive $\varepsilon_1 < \varepsilon$ such that $L < r_u - \varepsilon_1$. Let i be the smallest integer with $i > s_{k+1}$ such that $p(T^i u, T^i u) > r_u - \varepsilon_1$, and let $m \in \mathbb{N}$ be the greatest integer such that $s_m \leq i$. So $m \geq k+1$. By (3.1) there is some $j \geq s_{m-1}$ such that

$$p\bigl(T^i u, T^i u\bigr) \leq p\bigl(T^{s_m}u, T^i u\bigr)$$
$$\leq \max\bigl\{\alpha p\bigl(T^{s_{m-1}}u, T^j u\bigr), p\bigl(T^{s_{m-1}}u, T^{s_{m-1}}u\bigr), p\bigl(T^{i-1}u, T^{i-1}u\bigr)\bigr\}.$$

Clearly, we have $s_{m-1} \geq s_k \geq m_0$ and $i - 1 \geq s_{k+1} \geq m_0$. The minimality of i and m and the fact that $L < r_u - \varepsilon_1$ can now easily be used to deduce that $p(T^i u, T^i u) > \max\{p(T^{s_{m-1}}u, T^{s_{m-1}}u), p(T^{i-1}u, T^{i-1}u)\}$. Therefore $r_u - \varepsilon < r_u - \varepsilon_1 < p(T^i u, T^i u) \leq \alpha p(T^{s_{m-1}}u, T^j u) \leq \alpha(r_u + \varepsilon)$ and this cannot be true by the choice of ε.

So we have proved that there are positive integers $k_1 < k_2$ such that $p(T^{s_{k_1}}u, T^{s_{k_1}}u) = p(T^{s_{k_2}}u, T^{s_{k_2}}u) = I$. Since $p(T^{s_{k_1}}u, T^{s_{k_2}}u) \leq I$, we must have $T^{s_{k_1}}u = T^{s_{k_2}}u$ (by (P2) and (P3)), *i.e.*, $T^{s_{k_2}-s_{k_1}}z = z$ for $z := T^{s_{k_1}}u$. As $p(v_z, v_z) = \lim_i p(v_z, T^i z) = \lim_{i,j} p(T^i z, T^j z)$ and $s_{k_2} - s_{k_1} \in \mathbb{N}$, by Lemma 3.6 it follows $Tz = z$. Of course $\lim_{i,j} p(T^i z, T^j z) = \lim_{i,j} p(T^i u, T^j u) = r_u = I = \inf_{x\in X} p(v_x, v_x)$. $\square$

Remark 4.1 To ensure uniqueness of the fixed point, we can strengthen condition (3.1) as follows. Given a partial metric space (X,p), call $T: X \to X$ a C_2-operator if for each $x \in X$ there is some $n(x) \in \mathbb{N}$ such that for each $y \in X$ there holds

$$p\big(T^{n(x)}x, T^{n(x)}y\big) \leq \max\left\{\alpha p\big(x, T^j y\big), \alpha p\big(x, T^{n(x)}x\big), \frac{p(x,x) + p(T^{n(x)-1}y, T^{n(x)-1}y)}{2}\right\} \tag{4.4}$$

for some $j \in \{0,1,\dots,n(x)\}$. Evidently, each C_2-operator is a C_1-operator as well so that if (X,p) is complete, the conclusion of Theorem 4.1 holds. But now, in addition, if $Ta = a$ and $Tb = b$, then

$$p(a,b) = p\big(T^{n(a)}a, T^{n(b)}b\big) \leq \max\left\{\alpha p(a,b), \alpha p(a,a), \frac{p(a,a) + p(b,b)}{2}\right\}$$

so that either $(1-\alpha)p(a,b) \leq 0$ or $p^s(a,b) = 2p(a,b) - p(a,a) - p(b,b) = 0$, meaning that in any case we must have $a = b$.

Recall that a sequence x_n in a partial metric space (X,p) is called 0-Cauchy with respect to p (see, *e.g.*, [29]) if $\lim_{m,n} p(x_n, x_m) = 0$. We say that (X,p) is 0-complete if every 0-Cauchy sequence in X p^s-converges to some $x \in X$ (for which we then necessarily must have $p(x,x) = 0$). Note that every 0-Cauchy sequence in (X,p) is Cauchy in (X,p^s), and that every complete partial metric space is 0-complete.

Remark 4.2 Recently a very interesting paper by Haghi, Rezapour and Shahzad [45] showed up in which the authors associated to each partial metric space (X,p) a metric space (X,d) by setting $d(x,x) = 0$ and $d(x,y) = p(x,y)$ if $x \neq y$, and proved that (X,p) is 0-complete if and only if (X,d) is complete. They then proceeded to demonstrate how using the associated metric d some of the fixed point results in partial metric spaces can easily be deduced from the corresponding known results in metric spaces.

Let us point out that these considerations cannot apply to C_1-operators since the terms $p(x,x)$ and $p(T^{n(x)-1}y, T^{n(x)-1}y)$ on the right-hand side of (3.1) are not multiplied by α. Thus our Theorem 4.1 cannot follow from the result of Ćirić it generalizes.

If we completely neglect the role of self-distances in (3.1), we can easily verify that the statement of Theorem 1.2 remains valid upon substituting the words '*partial metric*' for '*metric*' and '0-*complete*' for '*complete*'. We will prove this using the approach of Haghi, Rezapour and Shahzad [45] that will allow us to deduce Theorem 4.2 directly from Ćirić's result (Theorem 1.2).

Theorem 4.2 *If T is a* C*-operator on a* 0*-complete partial metric space* (X,p), *then there is a unique fixed point z of T. Furthermore, we have $p(z,z) = 0$ and for each $x \in X$ the sequence $\{T^i x\}_{i\geq 0}$ p^s-converges to z.*

Proof Let d be defined as in Remark 4.2. So (X,d) is a complete metric space (see Proposition 2.1 of [45]). Observe that we have $d(x,y) \le p(x,y)$ for all $x,y \in X$. For $x \in X$ let $n(x) \in \mathbb{N}$ be as in (3.2), and for $x,y \in X$ set

$$S(x,y) = \{y, Ty, T^2y, \dots, T^{n(x)}y, T^{n(x)}x\}$$

and $\mathrm{M}_p = \max\{p(x,z) \mid z \in S(x,y)\}$, $\mathrm{M}_d = \max\{d(x,z) \mid z \in S(x,y)\}$. We thus have that

$$p(T^{n(x)}x, T^{n(x)}y) \le \alpha \mathrm{M}_p(x,y)$$

for all $x,y \in X$. We check that for all $x,y \in X$ it holds that $d(T^{n(x)}x, T^{n(x)}y) \le \alpha \mathrm{M}_d(x,y)$, so that Theorem 1.2 can immediately be applied.

Case 1. There is some $z \in S(x,y)$ such that $x \neq z$ and $\mathrm{M}_p(x,y) = p(x,z)$. Here we have

$$d(T^{n(x)}x, T^{n(x)}y) \le p(T^{n(x)}x, T^{n(x)}y) \le \alpha \mathrm{M}_p(x,y) = \alpha p(x,z) = \alpha d(x,z) \le \alpha \mathrm{M}_d(x,y).$$

Case 2. For all $z \in S(x,y)$ we have that $\mathrm{M}_p(x,y) = p(x,z) \Rightarrow x = z$. So it must be $\mathrm{M}_p(x,y) = p(x,x)$, in particular, and hence $p(x, T^{n(x)}x) \le \mathrm{M}_p(x,y) = p(x,x) \le p(x, T^{n(x)}x)$, *i.e.*, $\mathrm{M}_p(x,y) = p(x, T^{n(x)}x)$. But by our assumption it now follows that $x = T^{n(x)}x$. Similarly, from $p(x, T^{n(x)}y) \le \mathrm{M}_p(x,y) = p(x,x) \le p(x, T^{n(x)}y)$, we obtain $\mathrm{M}_p(x,y) = p(x, T^{n(x)}y)$ and consequently $x = T^{n(x)}y$. Now $d(T^{n(x)}x, T^{n(x)}y) = d(x,x) = 0 \le \alpha \mathrm{M}_d(x,y)$. □

Remark 4.3 It should be pointed out, however, that even though the results of Haghi *et al.* can deduce the same fixed point as the corresponding partial metric fixed point result, using the partial metric version computers evaluate faster since many nonsense terms are omitted. This is very important from the aspect of computer science due to its cost and explains the vast body of partial metric fixed point results found in literature.

Given a C_1-operator and a point x, one may ask what the minimal value of $n(x)$ is for which inequality (3.1) holds true. In the following example, for an arbitrary positive integer m, we construct a C_1-operator on a complete partial metric space (X,p) such that for some $x \in X$ it must be $n(x) > m$.

Example 4.1 Denote by X_∞ the set of all sequences $x : \mathbb{N} \to \mathbb{N}$ and for $n \in \mathbb{N}$ by X_n the set of all n-tuples $x : \{1,\dots,n\} \to \mathbb{N}$ of positive integers. Put $X := X_\infty \cup \bigcup_{n\in\mathbb{N}} X_n$. For $x,y \in X$ set

$$I(x,y) = \{i \in \mathbb{N} \cup \{0\} \mid [j \in \mathrm{dom}(x) \cap \mathrm{dom}(y) \wedge j \le i] \Rightarrow x(j) = y(j)\}$$

and define $p(x,y) := \inf\{\frac{1}{2^i} \mid i \in I(x,y)\}$ (thus if $x(1) \neq y(1)$, then $I(x,y) = \{0\}$ and $p(x,y) = 1$). Here 'dom(x)' stands for the domain of the function x. Then (X,p) is a partial metric space (see [7]) and a complete one as can easily be verified.

Fix $l_0 \in \mathbb{N}$ and define $T : X \to X$ as follows. For $x \in X$ let $I_x = \{i \in \mathbb{N} \mid x(i) \neq i\}$.

If $I_x = \emptyset$, then set $Tx = x$. If $I_x \neq \emptyset$, then define $Tx = y$ by $x \in X^n \Leftrightarrow y \in X^n$, $x \in X_\infty \Leftrightarrow y \in X_\infty$ and the following two conditions:

- if I_x is finite and has at most l_0 elements, then $y(i) = i$ if $i = \max I_x$, and $y(i) = x(i)$ else;
- if I_x is either infinite or finite with more than l_0 elements, then $y(i) = i$ if $i = \min I_x$, and $y(i) = x(i)$ else.

Let us show that T is a C_1-operator with $n(x) = l_0$ for all $x \in \mathbb{N}$. Before we proceed, observe that if k is a nonnegative integer such that $k+1 \in \operatorname{dom}(x)$ and $x(i) = i$ for $1 \leq i \leq k$, then for $y = T^{l_0}x$ we must have $y(i) = i$ for all $1 \leq i \leq k+1$.

Case 1. There is a nonnegative integer i with $i+1 \in \operatorname{dom}(x) \cap \operatorname{dom}(y)$ such that $x(i+1) \neq i+1 \vee y(i+1) \neq i+1$. Denote by k the least such nonnegative integer.

If $x(k+1) \neq y(k+1)$, then $p(T^{l_0}x, T^{l_0}y) \leq \frac{1}{2^{k+1}} = \frac{1}{2}\frac{1}{2^k} = \frac{1}{2}p(x,y)$.

If $x(k+1) = y(k+1)$, then since $x(k+1) \neq k+1 \vee y(i+1) \neq i+1$ we must actually have $x(k+1) = y(k+1) \neq k+1$ and thus $p(x, T^{l_0}x) = \frac{1}{2^k}$. Hence $p(T^{l_0}x, T^{l_0}y) \leq \frac{1}{2^{k+1}} = \frac{1}{2}p(x, T^{l_0}x)$.

Case 2. $x = (1,2,\dots,k)$ for some $k \in \mathbb{N}$ and $x \subseteq y$. Here $p(T^{l_0}x, T^{l_0}y) = \frac{1}{2^k} = p(x,x)$.

Case 3. $y = (1,2,\dots,k)$ for some $k \in \mathbb{N}$ and $y \subseteq x$. Here $p(T^{l_0}x, T^{l_0}y) = \frac{1}{2^k} = p(y,y) = p(T^{l_0-1}y, T^{l_0-1}y)$.

Condition (4.4) fails because the fixed point is not unique. So T is not a C_2-operator, hence not a C-operator either.

Now suppose that $l_1 < l_0$ is an arbitrary positive integer and take $x, y \in \mathbb{N}^{l_1+1}$ such that $x(i) = 1$ for all $1 \leq i \leq l_1+1$, and $y(1) = 2$, $y(i) = 1$ for all $2 \leq i \leq l_1+1$.

We have $p(T^{l_1}x, T^{l_1}y) = 1 = p(x, T^j y)$, for $0 \leq j \leq l_1$, $p(x, T^{l_1}x) = \frac{1}{2}$ and $p(T^{l_1-1}y, T^{l_1-1}y) = p(x,x) = \frac{1}{2^{l_1+1}}$. So we see that for this particular choice of x and y, substituting l_1 for $n(x)$ in (3.1) makes the inequality false.

Let us use this very example to illustrate Proposition 4.1. Let $t \in X_\infty$ be defined by $t(i) = i$ for all $i \in \mathbb{N}$. For $n \in \mathbb{N}$ let $t_n \in X^n$ be defined by $t_n(i) = i$ for $i = \overline{1,n}$.

If $x \in X_\infty$, we clearly have $\nu_x = t$. Similarly, if $x \in X^n$, then $\nu_x = t_n$. So $p(t, t_n) = \frac{1}{2^n} = \max\{p(t_n,t_n), p(t,t)\}$ because $p(t_n,t_n) = \frac{1}{2^n}$ and $p(t,t) = 0$. Also

$$p(t_m, t_n) = \begin{cases} \frac{1}{2^n} = p(t_n, t_n) & \text{if } n \leq m, \\ \frac{1}{2^m} = p(t_m, t_m) & \text{if } m \leq n, \end{cases}$$

thus $p(t_m, t_n) = \max\{p(t_n,t_n), p(t_m,t_m)\}$.

Remark 4.4 It should be noted that if in Theorem 4.1 we require $n(x)$ to be equal to 1 for all $x \in X$, then Theorem 2.2 is obtained as a corollary. On the other hand, as already pointed out, if in Theorem 4.2 p is a complete (ordinary) metric on X, then the result of Ćirić (Theorem 1.2) is recovered.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors read and approved the final manuscript.

Acknowledgements

This work was supported by Grant No. 174025 of the Ministry of Science, Technology and Development, Republic of Serbia.

Received: 30 September 2013 Accepted: 20 November 2013 Published: 13 Dec 2013

References

1. Banach, S: Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fundam. Math. **3**, 133-181 (1922)
2. Ćirić, L: On Sehgal's maps with a contractive iterate at a point. Publ. Inst. Math. **33**(47), 59-62 (1983)
3. Guseman, LF: Fixed point theorems for mappings with a contractive iterate at a point. Proc. Am. Math. Soc. **26**, 615-618 (1970)
4. Sehgal, VM: A fixed point theorem for mappings with a contractive iterate. Proc. Am. Math. Soc. **23**, 631-634 (1969)

5. Pavlović, M, Radenović, S, Radojević, S: Abstract metric spaces and Sehgal-Guseman-type theorems. Comput. Math. Appl. **60**, 865-872 (2010)
6. Matthews, SG: Partial metric topology. Research report 212, Dept. of Computer Science, University of Warwick (1992)
7. Matthews, SG: Partial metric topology. In: Proc. 8th Summer Conference on General Topology and Applications. Ann. New York Acad. Sci., vol. 728, pp. 183-197 (1994)
8. Matthews, SG: An extensional treatment of lazy data flow deadlock. Theor. Comput. Sci. **151**, 195-205 (1995)
9. Kahn, G: The semantics of a simple language for parallel processing. In: Proc. IFIP Congress, pp. 471-475. North-Holland, Amsterdam (1974)
10. Bukatin, M, Kopperman, R, Matthews, S, Pajoohesh, H: Partial metric spaces. Am. Math. Mon. **116**, 708-718 (2009)
11. Bukatin, MA, Shorina, SY: Partial metrics and co-continuous valuations. In: Nivat, M (ed.) Foundations of Software Science and Computation Structure. Lecture Notes in Computer Science, vol. 1378, pp. 125-139. Springer, Berlin (1998)
12. O' Neill, SJ: Partial metrics, valuations and domain theory. In: Proceedings Eleventh Summer Conference on General Topology and Applications. Ann. New York Acad. Sci., vol. 806, pp. 304-315 (1996)
13. Waszkiewicz, P: Distance and measurement in domain theory. Electron. Notes Theor. Comput. Sci. **40**, 448-462 (2001)
14. Romaguera, S, Schellekens, M: Partial metric monoids and semivaluation spaces. Topol. Appl. **153**(5-6), 948-962 (2005)
15. Romaguera, S, Valero, O: A quantitative computational model for complete partial metric spaces via formal balls. Math. Struct. Comput. Sci. **19**(3), 541-563 (2009)
16. Schellekens, MP: The correspondence between partial metrics and semivaluations. Theor. Comput. Sci. **315**, 135-149 (2004)
17. Waszkiewicz, P: Partial metrisability of continuous posets. Math. Struct. Comput. Sci. **16**(2), 359-372 (2006)
18. Abbas, M, Ali, B, Vetro, C: A Suzuki type fixed point theorem for a generalized multivalued mapping on partial Hausdorff metric spaces. Topol. Appl. **160**, 553-563 (2013)
19. Abdeljawad, T, et al.: Coupled fixed points for Meir-Keeler contractions in ordered partial metric spaces. Math. Probl. Eng. **2012**, Article ID 327273 (2012)
20. Abdeljawad, T, et al.: Existence and uniqueness of a common fixed point on partial metric spaces. Appl. Math. Lett. **24**(11), 1894-1899 (2011). doi:10.1016/j.aml.2011.05.014
21. Abdeljawad, T: Fixed points for generalized weakly contractive mappings in partial metric spaces. Math. Comput. Model. **54**(11-12), 2923-2927 (2011)
22. Abdeljawad, T, Karapinar, E, Tas, K: A generalized contraction principle with control functions on partial metrics paces. Comput. Math. Appl. **63**, 716-719 (2012)
23. Altun, I, Sola, F, Simsek, H: Generalized contractions on partial metric spaces. Topol. Appl. **157**(18), 2778-2785 (2010)
24. Altun, I, Erduran, A: Fixed point theorems for monotone mappings on partial metric spaces. Fixed Point Theory Appl. **2011**, Article ID 508730 (2011). doi:10.1155/2011/508730.
25. Aydi, H, Abbas, M, Vetro, C: Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces. Topol. Appl. **159**, 3234-3242 (2012)
26. Aydi, H, et al.: Berinde type generalized contractions on partial metric spaces. Abstr. Appl. Anal. **2013**, Article ID 312479 (2013)
27. Chi, KP, Karapinar, E, Thanh, TD: A generalized contraction principle in partial metric spaces. Math. Comput. Model. **55**, 1673-1681 (2012)
28. Ćirić, L, Samet, B, Aydi, H, Vetro, C: Common fixed points of generalized contractions on partial metric spaces and an application. Appl. Math. Comput. **218**, 2398-2406 (2011)
29. Ilić, D, Pavlović, V, Rakočević, V: Some new extensions of Banach's contraction principle to partial metric space. Appl. Math. Lett. **24**, 1326-1330 (2011)
30. Ilić, D, Pavlović, V, Rakočević, V: Extensions of the Zamfirescu theorem to partial metric spaces. Math. Comput. Model. (2011). doi:10.1016/j.mcm.2011.09.005
31. Kadelburg, Z, Nashine, HK, Radenović, S: Fixed point results under various contractive conditions in partial metric spaces. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. (2013). doi:10.1007/s13398-012-0066-6
32. Karapinar, E: Generalizations of Caristi Kirk's theorem on partial metric spaces. Fixed Point Theory Appl. **2011**, 4 (2011). doi:10.1186/1687-1812-2011-4
33. Karapinar, E, Erhan, IM: Fixed point theorems for operators on partial metric spaces. Appl. Math. Lett. **24**(11), 1894-1899 (2011). doi:10.1016/j.aml.2011.05.013
34. Karapinar, E: A note on common fixed point theorems in partial metric spaces. Miskolc Math. Notes **12**(2), 185-191 (2011)
35. Kumam, P, Vetro, C, Vetro, F: Fixed points for weak α-ψ-contractions in partial metric spaces. Abstr. Appl. Anal. **2013**, Article ID 986028 (2013)
36. Nashine, HK, et al.: Fixed point results in orbitally complete partial metric spaces. Bull. Malays. Math. Sci. Soc. **36**(4), 1185-1193 (2013)
37. Paesano, D, Vetro, P: Suzuki's type characterizations of completeness for partial metric spaces and fixed points for partially ordered metric spaces. Topol. Appl. **159**, 911-920 (2012)
38. Rakočevic, V, et al.: On cyclic generalized weakly C-contractions on partial metric spaces. Abstr. Appl. Anal. **2013**, Article ID 831491 (2013)
39. Roldan, A, et al.: Multidimensional fixed point theorems in partially ordered complete partial metric spaces under (ψ, φ)-contractivity conditions. Abstr. Appl. Anal. **2013**, Article ID 634371 (2013)
40. Tas, K, *et al.*: Fixed point theorem on partial metric spaces involving rational expressions. Miskolc Math. Notes **14**(1), 135-142 (2013)
41. Vaezpour, M, et al.: A common fixed point theorem for cyclic operators on partial metric spaces. Filomat **26**(2), 407-414 (2012)
42. Valero, O: On Banach's fixed point theorems for partial metric spaces. Appl. Gen. Topol. **6**, 229-240 (2005)
43. Vetro, F, Radenović, S: Nonlinear ψ-quasi-contractions of Ćirić-type in partial metric spaces. Appl. Math. Comput. **219**, 1594-1600 (2012)

44. Oltra, S, Valero, O: Banach's fixed theorem for partial metric spaces. Rend. Ist. Mat. Univ. Trieste **36**, 17-26 (2004)
45. Haghi, RH, Rezapour, S, Shahzad, N: Be careful on partial metric fixed point results. Topol. Appl. **160**, 450-454 (2013)
46. Aydi, H, Vetro, C, Sintunavarat, W, Kumam, P: Coincidence and fixed points for contractions and cyclical contractions in partial metric spaces. Fixed Point Theory Appl. **2012**, 124 (2012)
47. Di Bari, C, Milojević, M, Radenović, S, Vetro, P: Common fixed points for self-mappings on partial metric spaces. Fixed Point Theory Appl. **2012**, 140 (2012)

10.1186/1687-1812-2013-335
Cite this article as: Ilić et al.: Fixed points of mappings with a contractive iterate at a point in partial metric spaces. *Fixed Point Theory and Applications* 2013, 2013:335