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Convergence and stability of Jungck-type iterative procedures in convex b-metric spaces

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Abstract

The purpose of this paper is to investigate some strong convergence as well as stability results of some iterative procedures for a special class of mappings. First, this class of mappings called weak Jungck (φ, ψ)-contractive mappings, which is a generalization of some known classes of Jungck-type contractive mappings, is introduced. Then, using an iterative procedure, we prove the existence of coincidence points for such mappings. Finally, we investigate the strong convergence of some iterative Jungck-type procedures and study stability and almost stability of these procedures. Our results improve and extend many known results in other spaces.

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Keywords: weak Jungck (φ, ψ)-contractive mapping; iterative procedure; coincidence point; stability; convex b-metric space

1 Introduction

Czerwik [1] initiated the study of multivalued contractions in b-metric spaces.

Definition 1.1 Let X be a set and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

Then the pair (X, d) is called a b-metric space.

It is clear that normed linear spaces, l^p (or L^p) spaces ($p > 0$), l^∞ (or L^∞) spaces, Hilbert spaces, Banach spaces, hyperbolic spaces, $\mathbb{R}$-trees and CAT(0) spaces are examples of b-metric spaces.

Throughout this paper, $\mathbb{R}^+$ is the set of nonnegative real numbers and Y is a nonempty arbitrary subset of a b-metric space (X, d). Moreover, $F(T) = \{x \in Y : Tx = x\}$ will be denoted as the set of fixed points of $T : Y \to X$. Approximately, all the concepts and results in metric spaces are extended to the setting of b-metric spaces (for more details, see [1]).

The first result on stability of T-stable mappings was introduced by Ostrowski [2] for the Banach contraction principle. Harder and Hicks [3] proved that the sequence $\{x_n\}$

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generated by the Picard iterative process in a complete metric space converges strongly to the fixed point of T and is stable with respect to T, provided that T is a Zamfirescu mapping. Rhoades [4] extended the stability results of [3] to more general classes of contractive mappings. Ding [5] constructed the Ishikawa-type iterative process in a convex metric space. He showed that this process converges to the fixed point of T, provided that T belongs in the class which is defined by Rhoades.

A mapping T is said to be a φ-quasinonexpansive if $F(T) \neq \emptyset$ and there exists a function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$d(Tx,p) \leq \varphi\big(d(x,p)\big)$$

for all $x \in X$ and $p \in F(T)$.

Osilike [6] considered a mapping T from a metric space X into itself satisfying the condition $d(Tx,Ty) \leq \delta d(x,y) + Ld(x,Tx)$ for some $\delta \in [0,1)$ and $L \geq 0$ for all $x,y \in X$. Furthermore, he extended some of the stability results in [4]. Indeed, he proved T-stability for such a mapping with respect to Picard, Kirk, Mann, and Ishikawa iterations. Thereafter, Olatinwo [7] improved this concept to the context of multivalued weak contraction for the Jungck iteration in a complete b-metric space. In [8] this contractive condition was generalized by replacing this condition with $d(Tx,Ty) \leq \delta d(x,y) + \varphi(d(x,Tx))$, where $0 \leq \delta < 1$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is monotone increasing with $\varphi(0) = 0$, and some stability results were proved. Recently, Olatinwo [9] extended this condition to $d(Tx,Ty) \leq \varphi(d(x,y)) + \psi(d(x,Tx))$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a subadditive comparison function and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is monotone increasing with $\psi(0) = 0$. He studied this contractive condition as a particular case of the class of φ-quasinonexpansive mappings (see [10]). Also, he proved some stability results as well as strong convergence results for the pair of nonself mappings in a complete metric space.

In 1968, Goebel [11] generalized the well-known Banach contraction principle by taking a continuous mapping S in place of the identity mapping, where S commuted with T and $T(X) \subset S(X)$. In fact, he used two mappings $S,T : Y \to X$ for introducing the contractive condition as follows.

A mapping T is called a Jungck contraction if there exists a real number $0 \leq \alpha < 1$ such that

$$\text{(JC)}\quad d(Tx,Ty) \leq \alpha d(Sx,Sy)$$

for all $x,y \in Y$. In addition, Jungck [12], using a constructive method, proved the existence of a unique common fixed point of S and T, where $Y = X$.

A mapping T is said to be a Jungck-Zamfirescu contraction (JZ) if there exist real numbers α, β, and γ satisfying $0 \leq \alpha < 1$, $0 \leq \beta,\gamma < \frac{1}{2}$ such that for each $x,y \in Y$, one has at least one of the following:

(z_1) $d(Tx,Ty) \leq \alpha d(Sx,Sy)$;
(z_2) $d(Tx,Ty) \leq \beta[d(Sx,Tx) + d(Sy,Ty)]$;
(z_3) $d(Tx,Ty) \leq \gamma[d(Sx,Ty) + d(Sy,Tx)]$.

A mapping T is said to be a contractive mapping satisfying (JS), (JR) or (JQC) if there exists a constant $q \in [0,1)$ such that for any $x, y \in Y$,

$$\text{(JS)} \quad d(Tx,Ty) \le q \max\left\{ d(Sx,Sy), \frac{1}{2}\big[d(Sx,Ty) + d(Sy,Tx)\big], d(Sx,Tx), d(Sy,Ty) \right\},$$

$$\text{(JR)} \quad d(Tx,Ty) \le q \max\left\{ d(Sx,Sy), \frac{1}{2}\big[d(Sx,Tx) + d(Sy,Ty)\big], d(Sx,Ty), d(Sy,Tx) \right\},$$

$$\text{(JQC)} \quad d(Tx,Ty) \le q \max\big\{ d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), d(Sx,Ty), d(Sy,Tx) \big\}.$$

A mapping T is said to be a weak Jungck contraction if there exist two constants $a \in [0,1)$ and $L \ge 0$ such that for all $x, y \in Y$,

$$\text{(WJC)} \quad d(Tx,Ty) \le ad(Sx,Sy) + Ld(Sx,Tx).$$

It is worth mentioning that a Jungck-Zamfirescu mapping is a (JR) mapping. In [13, Proposition 3.3], a comparison of the above contractive conditions is established as follows.

Proposition 1.2
- (i) (JC) $\Rightarrow$ (JS) $\Rightarrow$ (JQC);
- (ii) (JC) $\Rightarrow$ (JR) $\Rightarrow$ (JQC);
- (iii) (JS) *and* (JR) *are independent*;
- (iv) (JR) $\Rightarrow$ (WJC);
- (v) (JS) *and* (WJC) *are independent*;
- (vi) (JQC) *and* (WJC) *are independent*;
- (vii) *reverse implications of* (i), (ii), *and* (iv) *are not true*.

In this paper, a special class of mappings called a weak Jungck (φ, ψ)-contraction is introduced, and it is shown that it contains other known classes of Jungck-type contractive mappings. Then, using a Jungck-Picard iterative procedure, we investigate the existence of coincidence points and the uniqueness of the coincidence value of weak Jungck (φ, ψ)-contractive mappings. Also, some strong convergence as well as stability results of some Jungck-type iterative procedures (such as Jungck-Ishikawa *etc.*) are studied. These results play a crucial role in numerical computations for approximation of coincidence values of two nonlinear mappings.

2 Preliminary

In [14], Berinde introduced the concepts of comparison function and (c)-comparison function with respect to the function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$. A function φ is called a comparison function if it satisfies the following:

- (i_φ) φ is monotone increasing, *i.e.*, $t_1 < t_2 \Rightarrow \varphi(t_1) \le \varphi(t_2)$;
- (ii_φ) The sequence $\{\varphi^n(t)\} \to 0$ for all $t \in \mathbb{R}^+$, where φ^n stands for the nth iterate of φ.

If φ satisfies (i_φ) and

(iii_φ) $\sum_{n=0}^{\infty} \varphi^n(t)$ converges for all $t \in \mathbb{R}^+$,

then φ is said to be a (c)-comparison function.

Several results regarding comparison functions can be found in [14] and [15]. Referring to [14] and [15], we have:

1. Any (c)-comparison function is a comparison function;
2. Any comparison function satisfies $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$;
3. Any subadditive comparison function is continuous;
4. Condition (iii_φ) is equivalent to the following one:
   There exist $k_0 \in \mathbb{N}$, $\alpha \in (0,1)$ and a convergent series of nonnegative terms $\sum v_n$ such that
   $$\varphi^{k+1}(t) \le \alpha\varphi^k(t) + v_k$$
   holds for all $k \ge k_0$ and any $t \in \mathbb{R}^+$.

Berinde [16] expanded the concept of (c)-comparison functions in b-metric spaces to s-comparison functions as follows.

Definition 2.1 Let $s \ge 1$ be a real number. A mapping $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an s-comparison function if it satisfies (i_φ) and

(iv_φ) There exist $k_0 \in \mathbb{N}$, $\alpha \in (0,1)$, and a convergent series of nonnegative terms $\sum v_n$ such that
$$s^{k+1}\varphi^{k+1}(t) \le \alpha s^k\varphi^k(t) + v_k$$
holds for all $k \ge k_0$ and any $t \in \mathbb{R}^+$.

Applying results 4 and 1 regarding comparison functions, it is easy to conclude that every s-comparison function is a comparison function.

In the sequel, some lemmas which are useful to obtain our main results are stated.

Lemma 2.2 ([17]) *Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a comparison function, and let ε_n be a sequence of positive numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$. Then*
$$\lim_{n\to\infty} \sum_{k=0}^{n} \varphi^{n-k}(\varepsilon_k) = 0.$$

Lemma 2.3 ([18]) *Let $\{u_n\}$, $\{\alpha_n\}$, and $\{\varepsilon_n\}$ be sequences of nonnegative real numbers satisfying the inequality*
$$u_{n+1} \le \alpha_n u_n + \varepsilon_n, \quad n \in \mathbb{N}.$$
If $\alpha_n \ge 1$, $\sum_{n=1}^{\infty}(\alpha_n - 1) < \infty$ and $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, then $\lim_{n\to\infty} u_n$ exists.

Lemma 2.4 *Suppose that $\{u_n\}$ and $\{\varepsilon_n\}$ are two sequences of nonnegative numbers such that*
$$u_{n+1} \le \varphi(u_n) + \varepsilon_n, \quad n = 0, 1, 2, \ldots, \tag{2.1}$$
where φ is a subadditive comparison function. If $\lim_{n\to\infty} \varepsilon_n = 0$, then $\lim_{n\to\infty} u_n = 0$.

Proof The monotone increasing and the subadditivity of φ together with inequality (2.1) imply that

$$\begin{aligned} u_{n+1} &\le \varphi(u_n) + \varepsilon_n \\ &\le \varphi\bigl(\varphi(u_{n-1}) + \varepsilon_{n-1}\bigr) + \varepsilon_n \\ &\le \varphi^2(u_{n-1}) + \varphi(\varepsilon_{n-1}) + \varepsilon_n \\ &\vdots \\ &\le \varphi^{n+1}(u_0) + \sum_{i=0}^{n} \varphi^{n-i}(\varepsilon_i), \end{aligned} \tag{2.2}$$

where $\varphi^0 = I$ (identity mapping). Moreover, since any comparison function satisfies (ii_φ), hence $\lim_{n\to\infty} \varphi^{n+1}(u_0) = 0$. Also, we have $\lim_{n\to\infty} \sum_{i=0}^{n} \varphi^{n-i}(\varepsilon_i) = 0$ from Lemma 2.2. Thus, inequality (2.2) implies that $\lim_{n\to\infty} u_n = 0$. □

Lemma 2.5 *Let $\{\alpha_n\}$ be a real sequence in $[0,1]$, let $\{\varepsilon_n\}$ be a sequence of positive numbers such that $\sum_{n=0}^{\infty} \varepsilon_n$ converges, and let $\{u_n\}$ be a sequence of nonnegative numbers such that*

$$u_{n+1} \le (1-\alpha_n)u_n + \alpha_n\varphi(u_n) + \varepsilon_n, \quad n = 0,1,2,\ldots, \tag{2.3}$$

where φ is a convex subadditive comparison function. If $\sum_{n=0}^{\infty} \alpha_n = \infty$, then $\lim_{n\to\infty} u_n = 0$.

Proof Since $\varphi(t) \le t$ for all $t \ge 0$, using a straightforward induction and (2.3), one can obtain

$$\begin{aligned} u_{n+p+1} &\le (1-\alpha_{n+p})u_{n+p} + \alpha_{n+p}\varphi(u_{n+p}) + \varepsilon_{n+p} \\ &\le (1-\alpha_{n+p})\bigl[(1-\alpha_{n+p-1})u_{n+p-1} + \alpha_{n+p-1}\varphi(u_{n+p-1}) + \varepsilon_{n+p-1}\bigr] \\ &\quad + \alpha_{n+p}\bigl[(1-\alpha_{n+p-1})\varphi(u_{n+p-1}) + \alpha_{n+p-1}\varphi^2(u_{n+p-1}) + \varphi(\varepsilon_{n+p-1})\bigr] + \varepsilon_{n+p} \\ &\le (1-\alpha_{n+p})(1-\alpha_{n+p-1})u_{n+p-1} + \bigl[1-(1-\alpha_{n+p})(1-\alpha_{n+p-1})\bigr]\varphi(u_{n+p-1}) \\ &\quad + \varepsilon_{n+p-1} + \varepsilon_{n+p} \\ &\vdots \\ &\le \left(\prod_{i=n}^{n+p}(1-\alpha_i)\right)u_n + \left(1-\prod_{i=n}^{n+p}(1-\alpha_i)\right)\varphi(u_n) + \sum_{i=n}^{n+p}\varepsilon_i \\ &\le \left(\prod_{i=n}^{n+p}(1-\alpha_i)\right)u_n + \varphi(u_n) + \sum_{i=n}^{n+p}\varepsilon_i \\ &\le \exp\left(-\sum_{i=n}^{n+p}\alpha_i\right)u_n + \varphi(u_n) + \sum_{i=n}^{n+p}\varepsilon_i \end{aligned}$$

for all $n,p \in \mathbb{N}$. Now, $\sum_{n=0}^{\infty} \alpha_n = \infty$ yields that $\lim_{p\to\infty} \exp(-\sum_{i=n}^{n+p} \alpha_i) = 0$. Then

$$\limsup_{p\to\infty} u_p = \limsup_{p\to\infty} u_{n+p+1} \le \varphi(u_n) + \sum_{i=n}^{\infty}\varepsilon_i, \quad n = 0,1,2,\ldots, \tag{2.4}$$

which implies that

$$\limsup_{p\to\infty} u_p \le \liminf_{n\to\infty} \varphi(u_n) \le \liminf_{n\to\infty} u_n.$$

Therefore, there exists $u \in \mathbb{R}^+$ such that $\lim_{n\to\infty} u_n = u$. Assume that $u > 0$. Since φ is continuous and $\sum_{n=0}^{\infty} \varepsilon_n$ converges, letting $n \to \infty$ in (2.4), we get that $u \le \varphi(u) < u$, which is a contradiction. Hence $u = 0$ and the desired conclusion follows. □

3 Weak Jungck (φ, ψ)-contractive mappings

In this section, the class of weak Jungck (φ, ψ)-contractive mappings which contains the class of Jungck φ-quasinonexpansive mappings is studied. Furthermore, it is showed that this class includes the various classes of contractive mappings which is introduced in Section 1.

Definition 3.1 Let Y be an arbitrary subset of a b-metric space (X, d), and let $S, T : Y \to X$ be such that z is a coincidence point of S and T, *i.e.*, $Sz = Tz = p$. We say that T is a Jungck φ-quasinonexpansive mapping with respect to S if there exists a function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$d(Tx, p) \le \varphi\bigl(d(Sx, p)\bigr)$$

for all $x \in Y$.

The above definition was used in [19] when S is the identity mapping on $Y = X$.

Definition 3.2 Let Y be an arbitrary subset of a b-metric space (X, d) and $S, T : Y \to X$. A mapping T is said to be a weak Jungck (φ, ψ)-contractive mapping with respect to S if there exist an s-comparison function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ and a monotone increasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ with upper semicontinuity from the right at $\psi(0) = 0$ such that for all $x, y \in Y$,

$$d(Tx, Ty) \le \varphi\bigl(d(Sx, Sy)\bigr) + \psi\bigl(\min\bigl\{d(Sx, Tx), d(Sx, Ty)\bigr\}\bigr). \tag{3.1}$$

It is obvious that any weak Jungck (φ, ψ)-contraction is also Jungck φ-quasinonexpansive, but the reverse is not true. The next example illustrates this matter.

Example 3.1 Let $S, T : [0, 1] \to [0, 1]$ be given by $Sx = x$ and

$$Tx = \begin{cases} 0, & 0 \le x \le \frac{1}{2}, \\ \frac{1}{2}, & \frac{1}{2} < x \le 1, \end{cases}$$

where $[0, 1]$ is endowed with the usual metric. It is easy to see that T satisfies the following property:

$$d(Tx, p) \le \varphi\bigl(d(x, p)\bigr)$$

for all $x \in [0, 1]$, $p \in F(T) = \{0\}$, and $\varphi(x) = x$. But T is not a weak Jungck (φ, ψ)-contractive mapping. Indeed, if there exist a 1-comparison function φ and a monotone increasing function ψ with upper semicontinuity from the right at $\psi(0) = 0$ such that for all $x, y \in [0,1]$,

$$d(Tx, Ty) \le \varphi\big(d(x,y)\big) + \psi\big(\min\big\{d(x, Tx), d(x, Ty)\big\}\big),$$

then, taking $x = \frac{1}{2}$, $y = 1$, we have $\frac{1}{2} \le \varphi(\frac{1}{2}) + \psi(0)$. This shows that the class of φ-quasinonexpansive mappings properly includes the class of weak Jungck (φ, ψ)-contractive mappings.

In what follows, we prove that all the mappings introduced in Section 1 are in the class of weak Jungck (φ, ψ)-contractive mappings. It is clear that every Jungck contractive mapping is a weak Jungck (φ, ψ)-contractive mapping with $\varphi(t) = \alpha t$ and $\psi(t) = 0$, where $0 \le \alpha < \frac{1}{s}$.

Proposition 3.3 *Let (X, d) be a b-metric space with parameter s, let Y be an arbitrary subset of X, and let $S, T : Y \to X$. If T is a Jungck-Zamfirescu contraction (JZ), then T is a weak Jungck (φ, ψ)-contractive mapping if $\alpha < \frac{1}{s}$ and $\beta, \gamma < \frac{1}{s(1+s^2)}$. Moreover, it is a weak Jungck (φ, ψ)-contraction with $\varphi(t) = \max\{\alpha, \frac{\beta s^2}{1-\beta s}, \frac{\gamma s^2}{1-\gamma s}\}t$ and $\psi(t) = \max\{\frac{\beta(1+s^2)}{1-\beta s}, \frac{\gamma(1+s^2)}{1-\gamma s}\}t$ for all $t \in \mathbb{R}^+$.*

Proof If $\min\{d(Sx, Tx), d(Sx, Ty)\} = d(Sx, Tx)$, then for all $x, y \in Y$,

$$\begin{aligned} d(Tx, Ty) &\le \beta\big[d(Sx, Tx) + d(Sy, Ty)\big] \\ &\le \beta d(Sx, Tx) + \beta s\big[d(Sy, Tx) + d(Tx, Ty)\big] \\ &\le \beta d(Sx, Tx) + \beta s^2\big[d(Sy, Sx) + d(Sx, Tx)\big] + \beta s d(Tx, Ty), \end{aligned}$$

which implies that

$$d(Tx, Ty) \le \frac{\beta s^2}{1 - \beta s} d(Sx, Sy) + \frac{\beta(1 + s^2)}{1 - \beta s} d(Sx, Tx).$$

Also

$$\begin{aligned} d(Tx, Ty) &\le \gamma\big[d(Sx, Ty) + d(Sy, Tx)\big] \\ &\le \gamma s\big[d(Sx, Tx) + d(Tx, Ty)\big] + \gamma s\big[d(Sy, Sx) + d(Sx, Tx)\big] \end{aligned}$$

yields that

$$d(Tx, Ty) \le \frac{\gamma s}{1 - \gamma s} d(Sx, Sy) + \frac{2\gamma s}{1 - \gamma s} d(Sx, Tx).$$

Similarly, if $\min\{d(Sx, Tx), d(Sx, Ty)\} = d(Sx, Ty)$, then for all $x, y \in Y$,

$$\begin{aligned} d(Tx, Ty) &\le \beta\big[d(Sx, Tx) + d(Sy, Ty)\big] \\ &\le \beta s\big[d(Sx, Ty) + d(Ty, Tx)\big] + \beta s\big[d(Sy, Sx) + d(Sx, Ty)\big], \end{aligned}$$

thus

$$d(Tx,Ty) \leq \frac{\beta s}{1-\beta s}d(Sx,Sy) + \frac{2\beta s}{1-\beta s}d(Sx,Ty).$$

In addition,

$$\begin{aligned} d(Tx,Ty) &\leq \gamma\bigl[d(Sx,Ty) + d(Sy,Tx)\bigr] \\ &\leq \gamma d(Sx,Ty) + \gamma s\bigl[d(Sy,Ty) + d(Ty,Tx)\bigr] \\ &\leq \gamma d(Sx,Ty) + \gamma s^2\bigl[d(Sy,Sx) + d(Sx,Ty)\bigr] + \gamma s d(Tx,Ty) \end{aligned}$$

implies that

$$d(Tx,Ty) \leq \frac{\gamma s^2}{1-\gamma s}d(Sx,Sy) + \frac{\gamma(1+s^2)}{1-\gamma s}d(Sx,Ty).$$

Now, let

$$\varphi(t) := \max\left\{\alpha, \frac{\beta s}{1-\beta s}, \frac{\beta s^2}{1-\beta s}, \frac{\gamma s}{1-\gamma s}, \frac{\gamma s^2}{1-\gamma s}\right\}t = \max\left\{\alpha, \frac{\beta s^2}{1-\beta s}, \frac{\gamma s^2}{1-\gamma s}\right\}t$$

and

$$\begin{aligned} \psi(t) &:= \max\left\{0, \frac{2\beta s}{1-\beta s}, \frac{\beta(1+s^2)}{1-\beta s}, \frac{2\gamma s}{1-\gamma s}, \frac{\gamma(1+s^2)}{1-\gamma s}\right\}t \\ &= \max\left\{\frac{\beta(1+s^2)}{1-\beta s}, \frac{\gamma(1+s^2)}{1-\gamma s}\right\}t \end{aligned}$$

for all $t \in \mathbb{R}^+$. It is clear that φ is an s-comparison function, where $\alpha < \frac{1}{s}$ and $\beta, \gamma < \frac{1}{s(1+s^2)}$ and ψ is a monotone increasing function which is continuous from the right at $\psi(0) = 0$. □

The following result shows that this fact is still true for a more general class of mappings.

Proposition 3.4 *Let X, Y and $S, T : Y \to X$ be as in the above proposition. If T satisfies* (JS), *then T is a weak Jungck (φ,ψ)-contractive mapping, provided that $q < \frac{1}{s(1+s^2)}$. Furthermore, it is a weak Jungck (φ,ψ)-contraction with $\varphi(t) = \frac{qs^2}{1-qs}t$ and $\psi(t) = \frac{qs^2}{1-qs}t$ for all $t \in \mathbb{R}^+$.*

Proof If $\min\{d(Sx,Tx), d(Sx,Ty)\} = d(Sx,Tx)$, then according to the inequality

$$\begin{aligned} d(Tx,Ty) &\leq qd(Sy,Ty) \leq qs\bigl[d(Sy,Tx) + d(Tx,Ty)\bigr] \\ &= qs^2\bigl[d(Sy,Sx) + d(Sx,Tx)\bigr] + qsd(Tx,Ty), \end{aligned}$$

we have

$$d(Tx,Ty) \leq \frac{qs^2}{1-qs}d(Sx,Sy) + \frac{qs^2}{1-qs}d(Sx,Tx)$$

for all $x, y \in Y$. Moreover,

$$\begin{aligned} d(Tx,Ty) &\le \frac{q}{2}\big[d(Sx,Ty)+d(Tx,Sy)\big] \\ &\le \frac{qs}{2}\big[d(Sx,Tx)+d(Tx,Ty)\big]+\frac{qs}{2}\big[d(Tx,Sx)+d(Sx,Sy)\big] \end{aligned}$$

implies that

$$d(Tx,Ty) \le \frac{qs}{2-qs}d(Sx,Sy)+\frac{2qs}{2-qs}d(Sx,Tx).$$

On the other hand, if $\min\{d(Sx,Tx), d(Sx,Ty)\} = d(Sx,Ty)$, then

$$d(Tx,Ty) \le qd(Sx,Tx) \le qs\big[d(Sx,Ty)+d(Ty,Tx)\big]$$

yields that

$$d(Tx,Ty) \le \frac{qs}{1-qs}d(Sx,Ty)$$

for all $x, y \in Y$. Also

$$d(Tx,Ty) \le qd(Sy,Ty) \le qs\big[d(Sy,Sx)+d(Sx,Ty)\big].$$

Moreover,

$$\begin{aligned} d(Tx,Ty) &\le \frac{q}{2}\big[d(Sx,Ty)+d(Tx,Sy)\big] \\ &\le \frac{q}{2}d(Sx,Ty)+\frac{qs}{2}\big[d(Tx,Ty)+d(Ty,Sy)\big] \\ &\le \frac{q}{2}d(Sx,Ty)+\frac{qs}{2}d(Tx,Ty)+\frac{qs^2}{2}\big[d(Ty,Sx)+d(Sx,Sy)\big] \end{aligned}$$

yields that

$$d(Tx,Ty) \le \frac{qs^2}{2-qs}d(Sx,Sy)+\frac{q(1+s^2)}{2-qs}d(Sx,Ty).$$

Now, we take

$$\varphi(t) := \max\left\{0, q, qs, \frac{qs}{2-qs}, \frac{qs^2}{1-qs}, \frac{qs^2}{2-qs}\right\}t = \frac{qs^2}{1-qs}t$$

and

$$\psi(t) := \max\left\{0, q, qs, \frac{qs}{1-qs}, \frac{2qs}{2-qs}, \frac{qs^2}{1-qs}, \frac{q(1+s^2)}{2-qs}\right\}t = \frac{qs^2}{1-qs}t$$

for all $t \in \mathbb{R}^+$. It shows that φ is an s-comparison function provided that $q < \frac{1}{s(1+s^2)}$ and ψ is a monotone increasing function which is continuous at $\psi(0) = 0$. $\square$

Similar arguments illustrate that every (JR) mapping is a weak Jungck (φ, ψ)-contractive mapping, provided that $q < \frac{1}{s(1+s^2)}$. In fact, it is a weak Jungck (φ, ψ)-contraction with $\varphi(t) = \psi(t) = \frac{qs^2}{1-qs}t$ for all $t \in \mathbb{R}^+$. Also, every (JQC) mapping is a weak Jungck (φ, ψ)-contractive mapping with $\varphi(t) = \psi(t) = \frac{qs^2}{1-qs}t$ for all $t \in \mathbb{R}^+$, provided that $q < \frac{1}{s(1+s^2)}$.

4 Convergence results

In 1970, Takahashi [20] defined a convex structure on metric spaces. In this section a version of the convexity notion in b-metric spaces is stated. Then, using some Jungck-type iterative procedures, we prove the existence of coincidence points as well as the strong convergence theorems for the weak Jungck (φ, ψ)-contractive mappings.

Definition 4.1 Let (X, d) be a b-metric space. A mapping $W : X \times X \times [0,1] \to X$ is said to be a convex structure on X if for each $(x, y, \lambda) \in X \times X \times [0,1]$ and $z \in X$,

$$d\big(z, W(x, y, \lambda)\big) \leq \lambda d(z, x) + (1 - \lambda)d(z, y). \tag{4.1}$$

A b-metric space X equipped with the convex structure W is called a convex b-metric space, which is denoted by (X, d, W).

Example 4.1 The space l^p $(p > 1)$ consisting of all the sequences $\{x_n\}$ of real numbers for which $\sum_{n=1}^{\infty} |x_n|^p$ converges, with the function $d : l^p \times l^p \to \mathbb{R}$ given by

$$d(x, y) = \sum_{n=1}^{\infty} |x_n - y_n|^p,$$

for all $x, y \in l^p$, is a b-metric space with $s = 2^{p-1} > 1$. Also, regarding the convexity of $f(t) = t^p$, we obtain that $d(z, \lambda x + (1 - \lambda)y) \leq \lambda d(z, x) + (1 - \lambda)d(z, y)$ for all $z \in l^p$, that is, l^p $(p > 1)$ is a convex b-metric space with $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$. (In a similar way, the space L^p $(p > 1)$ is a convex b-metric space.)

Now, the iterative procedures in a convex b-metric space are ready to be illustrated. From now on, it is assumed that (X, d) is a b-metric space (resp. (X, d, W) is a convex b-metric space) with parameter s and that $S, T : Y \to X$ are two nonself mappings on a subset Y of X such that $T(Y) \subset S(Y)$, where $S(Y)$ is a complete subspace of X.

Let $\{x_n\}$ be the sequence generated by an iterative procedure involving the mapping T and S, that is,

$$Sx_{n+1} = f(T, x_n), \quad n = 0, 1, 2, \ldots, \tag{4.2}$$

where $x_0 \in Y$ is the initial approximation and f is a function.

In the sequel, we discuss several special cases of (4.2):

1. The Jungck iteration (or Jungck-Picard iteration) is given from (4.2) for $f(T, x_n) = Tx_n$. This process was essentially introduced by Jungck [12] and it reduces to the Picard iterative process, when S is the identity mapping on $Y = X$;

2. The Jungck-Krasnoselskij iteration is defined by (4.2) with

$$f(T,x_n) = W(Sx_n, Tx_n, \lambda), \tag{4.3}$$

where $0 \leq \lambda \leq 1$;

3. The Jungck-Mann iteration is stated by (4.2) with

$$f(T,x_n) = W(Sx_n, Tx_n, \alpha_n), \tag{4.4}$$

where $\{\alpha_n\}$ is a sequence of real numbers such that $0 \leq \alpha_n \leq 1$;

4. The Jungck-Ishikawa iteration is introduced by (4.2) with

$$\begin{aligned} f(T,x_n) &= W(Sx_n, Ty_n, \alpha_n), \\ Sy_n &= W(Sx_n, Tx_n, \beta_n), \end{aligned} \tag{4.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences of real numbers such that $0 \leq \alpha_n, \beta_n \leq 1$.

It is worth noting that Olatinwo and Postolache [21] used the above iterative procedures in the setting of convex metric spaces.

Theorem 4.2 *Suppose that (X,d) is a b-metric space, and let $S,T : Y \to X$ be such that T is a weak Jungck (φ,ψ)-contractive mapping. Then S and T have a coincidence point. Moreover, for any $x_0 \in Y$, the sequence $\{Sx_n\}$ generated by the Jungck-Picard iterative process converges strongly to the coincidence value.*

Proof First, we prove that S and T have at least one coincidence point in Y. To do this, let $\{x_n\}$ be the Jungck-Picard iterative process defined by $Sx_{n+1} = Tx_n$ and $x_0 \in Y$. Taking $x = x_n$ and $y = x_{n-1}$ in (3.1), we obtain

$$d(Tx_n, Tx_{n-1}) \leq \varphi\bigl(d(Sx_n, Sx_{n-1})\bigr) + \psi\bigl(\min\bigl\{d(Sx_n, Tx_n), d(Sx_n, Tx_{n-1})\bigr\}\bigr),$$

which implies that

$$d(Sx_{n+1}, Sx_n) \leq \varphi\bigl(d(Sx_n, Sx_{n-1})\bigr),$$

and, inductively,

$$d(Sx_{n+1}, Sx_n) \leq \varphi^n\bigl(d(Sx_1, Sx_0)\bigr).$$

Therefore

$$\begin{aligned} d(Sx_{n+p}, Sx_n) &\leq s^{p-1}d(Sx_{n+p}, Sx_{n+p-1}) + s^{p-1}d(Sx_{n+p-1}, Sx_{n+p-2}) \\ &\quad + \cdots + s^2 d(Sx_{n+2}, Sx_{n+1}) + sd(Sx_{n+1}, Sx_n) \\ &\leq s^p\varphi^{n+p-1}\bigl(d(Sx_1, Sx_0)\bigr) + s^{p-1}\varphi^{n+p-2}\bigl(d(Sx_1, Sx_0)\bigr) \\ &\quad + \cdots + s^2\varphi^{n+1}\bigl(d(Sx_1, Sx_0)\bigr) + s\varphi^n\bigl(d(Sx_1, Sx_0)\bigr) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{p} s^i \varphi^{n+i-1}\bigl(d(Sx_1, Sx_0)\bigr) \\
&= \frac{1}{s^{n-1}} \sum_{i=n}^{n+p-1} s^i \varphi^i\bigl(d(Sx_1, Sx_0)\bigr), \quad n, p \in \mathbb{N}, p \neq 0.
\end{aligned}
$$

Since $\sum_{i=1}^{\infty} s^i \varphi^i(t) < \infty$ for all $t \in \mathbb{R}^+$, $\{Sx_n\}$ is a Cauchy sequence. Also, $S(Y)$ is complete, so $\{Sx_n\}$ has a limit in $S(Y)$, that is, there exists $z \in S^{-1}p$ such that $p = \lim_{n\to\infty} Sx_n$. Hence, $Sz = p$ and

$$
\begin{aligned}
d(Sz, Tz) &\leq sd(Sz, Sx_{n+1}) + sd(Sx_{n+1}, Tz) = sd(Sx_{n+1}, Sz) + sd(Tz, Tx_n) \\
&\leq sd(Sx_{n+1}, Sz) + s\varphi\bigl(d(Sz, Sx_n)\bigr) + s\psi\bigl(\min\bigl\{d(Sz, Tz), d(Sz, Tx_n)\bigr\}\bigr) \\
&\leq sd(Sx_{n+1}, p) + sd(Sx_n, p) + s\psi\bigl(d(Sx_{n+1}, p)\bigr).
\end{aligned}
$$

Taking the upper limit in the above inequality, we obtain $d(Sz, Tz) = 0$. Hence, $Tz = Sz = p$, *i.e.*, z is a coincidence point.

Now, we show that S and T have a unique coincidence value. Assume that S and T have two coincidence values $p, q \in X$ such that $p \neq q$. Then there exist $z_1, z_2 \in Y$ such that $Sz_1 = Tz_1 = p$ and $Sz_2 = Tz_2 = q$. Thus, we conclude that

$$
d(p, q) = d(Tz_1, Tz_2) \leq \varphi\bigl(d(Sz_1, Sz_2)\bigr) + \psi\bigl(\min\bigl\{d(Sz_1, Tz_1), d(Sz_1, Tz_2)\bigr\}\bigr) = \varphi\bigl(d(p, q)\bigr).
$$

From our assumptions on φ, it is impossible unless $d(p, q) = 0$, that is, $p = q$, which is a contradiction. □

Using Proposition 3.3, one can conclude that the above theorem is a significant extension of [22, Theorem 3.1] and [23, Theorem 3.1].

Theorem 4.3 *Let (X, d, W) be a convex b-metric, and let $S, T : Y \to X$ be such that T is a weak Jungck (φ, ψ)-contractive mapping such that φ is a convex subadditive function. Let $\{\alpha_n\}$ be a real sequence in $[0, 1]$ such that $\sum_{n=0}^{\infty}(1 - \alpha_n) = \infty$. Then, for any $x_0 \in Y$, the sequence $\{Sx_n\}$ defined by the Jungck-Ishikawa iterative process converges strongly to the coincidence value of S and T.*

Proof Theorem 4.3 states the existence of coincidence points in Y and one can obtain the uniqueness of coincidence value in a similar way. We now show that the Jungck-Ishikawa iteration given by $Sx_{n+1} = W(Sx_n, Ty_n, \alpha_n)$, where $Sy_n = W(Sx_n, Tx_n, \beta_n)$ for each $x_0 \in Y$, converges to $p = Sz = Tz$, where z is a coincidence point of S and T. Using (3.1), we have

$$
\begin{aligned}
d(Sx_{n+1}, p) &\leq \alpha_n d(Sx_n, p) + (1 - \alpha_n) d(Ty_n, p) \\
&\leq \alpha_n d(Sx_n, p) + (1 - \alpha_n) \\
&\quad \times \bigl[\varphi\bigl(d(Sz, Sy_n)\bigr) + \psi\bigl(\min\bigl\{d(Sz, Tz), d(Sz, Ty_n)\bigr\}\bigr)\bigr] \\
&= \alpha_n d(Sx_n, p) + (1 - \alpha_n)\varphi\bigl(d(Sy_n, p)\bigr), \qquad (4.6)
\end{aligned}
$$

and

$$\begin{aligned}
d(Sy_n,p) &\le \beta_n d(Sx_n,p) + (1-\beta_n)d(Tx_n,p) \\
&\le \beta_n d(Sx_n,p) + (1-\beta_n) \\
&\quad \times \bigl[\varphi\bigl(d(Sz,Sx_n)\bigr) + \psi\bigl(\min\bigl\{d(Sz,Tz),d(Sz,Tx_n)\bigr\}\bigr)\bigr] \\
&\le \beta_n d(Sx_n,p) + (1-\beta_n)\varphi\bigl(d(Sx_n,p)\bigr) \\
&\le \beta_n d(Sx_n,p) + (1-\beta_n)d(Sx_n,p) \\
&= d(Sx_n,p).
\end{aligned} \tag{4.7}$$

Substituting (4.7) in (4.6), it follows that

$$d(Sx_{n+1},p) \le \alpha_n d(Sx_n,p) + (1-\alpha_n)\varphi\bigl(d(Sx_n,p)\bigr), \quad n = 0,1,2,\ldots.$$

Since φ is a convex subadditive comparison function, we have the desired result from Lemma 2.5. □

Remark 4.1

(1) Based on Theorem 4.3, it is clear that the Jungck-Mann iterative process as well as the Jungck-Krasnoselskij iterative process converge;
(2) In normed linear spaces, the generalization of this theorem is stated by Olatinwo [9, 24];
(3) In Hilbert spaces, assuming that $q < \frac{1}{s(1+s^2)}$ in (JQC), Theorem 4.3 is an extension of the results in [25].

The following example shows that condition (3.1) in Theorem 4.3 is necessary.

Example 4.2 Let $S, T : [0,1] \to [0,1]$ be given by $Sx = x$ and

$$Tx = \begin{cases} 0, & 0 \le x \le \frac{1}{2}, \\ \frac{1}{2}, & \frac{1}{2} < x \le 1, \end{cases}$$

where $[0,1]$ is endowed with the usual metric. Let $x_0 \in (\frac{1}{2},1]$ and $x_{n+1} = \lambda x_n + (1-\lambda)Tx_n$ for $n = 0,1,2,\ldots$. Then $x_{n+1} = \lambda^{n+1}x_0 + \frac{1-\lambda^{n+1}}{2}$, which implies that $\lim_{n\to\infty} x_n = \frac{1}{2}$ if $0 \le \lambda < 1$ and $\lim_{n\to\infty} x_n = x_0 \ne 0$ if $\lambda = 1$. Therefore, the Krasnoselskij iteration associated to T does not converge strongly to the coincidence value.

5 Stability results

This section is devoted entirely to the stability of some various iterative procedures in b-metric spaces. This concept was first proposed by Ostrowski [2] in metric spaces. Then, Czerwik *et al.* [26, 27] extended Ostrowski's classical theorem in the setting of b-metric spaces. In addition, Singh *et al.* [13] introduced the stability and almost stability of Jungck-type iterative procedures in metric spaces. Below, we state these concepts in convex b-metric spaces.

Definition 5.1 Let (X,d,W) be a convex b-metric space, let Y be a subset of X, and let $S,T:Y\to Y$ be such that $T(Y)\subset S(Y)$. For any $x_0\in Y$, let the sequence $\{Sx_n\}$, generated by iterative procedure (4.2), converges to p. Also, let $\{Sy_n\}\subset X$ be an arbitrary sequence and let $\varepsilon_n = d(Sy_{n+1}, f(T,y_n))$, $n=0,1,2,\ldots$. Then

(i) Iterative procedure (4.2) will be called (S,T)-stable if $\lim_{n\to\infty}\varepsilon_n = 0$ implies that $\lim_{n\to\infty} Sy_n = p$.
(ii) Iterative procedure (4.2) will be called almost (S,T)-stable if $\sum_{n=0}^{\infty}\varepsilon_n<\infty$ implies that $\lim_{n\to\infty} Sy_n = p$.

The above definition reduces to the concept of the stability of iterative procedure due to Harder and Hicks [3] when S is the identity mapping on $Y=X$.

Example 5.1 Let $S,T:[0,1]\to[0,\frac{3}{2}]$ be given by $Sx = x^2+\frac{x}{2}$ and

$$Tx = \begin{cases} 0, & 0\le x\le \frac{1}{2},\\ \frac{1}{2}, & \frac{1}{2}<x\le 1,\end{cases}$$

where $[0,\frac{3}{2}]$ is endowed with the usual metric. Let $x_0\in[0,1]$ and $Sx_{n+1}=Tx_n$ for $n=0,1,2,\ldots$. If $0\le x_0\le\frac{1}{2}$, then $Sx_{n+1}=Tx_n=0$, and if $\frac{1}{2}<x_0\le 1$, we have $Sx_1=Tx_0=\frac{1}{2}$ and $Sx_{n+1}=Tx_n=0$ for all $n\in\mathbb{N}$. Thus $\lim_{n\to\infty}Sx_n = 0 = S(0)=T(0)$; *i.e.*, the Picard iteration converges strongly to the coincidence value. But the Picard iteration is not (S,T)-stable. Indeed, take the sequence $\{y_n\}$ given by $y_n=\frac{n+2}{2n}$, $n\in\mathbb{N}$. One can see easily that the sequence $\{Sy_n\}$ does not converge to the coincidence value, while $\varepsilon_n = d(Sy_{n+1},Ty_n) = \frac{1}{(n+1)^2}+\frac{3}{2(n+1)}\to 0$ as $n\to\infty$.

Our next theorem is presented for a pair of mappings on a nonempty subset with values in b-metric spaces under a condition more general than the condition stated by Singh and Prasad [23, Theorem 4.2]. Further, this theorem reduces the condition $s^2q<1$ to the condition $sq<1$.

Theorem 5.2 *Let (X,d) be a b-metric space and T be a weak Jungck (φ,ψ)-contractive mapping such that φ is subadditive. For $x_0\in Y$, let $\{Sx_n\}$ be the Picard iterative process defined by $Sx_{n+1}=Tx_n$. Then the Jungck-Picard iteration is (S,T)-stable.*

Proof Note that, by Theorem 4.2, there exists a coincidence point $z\in Y$ such that $\{Sx_n\}$ converges to $p=Sz=Tz$. Suppose that $\{Sy_n\}\subset X$ and define $\varepsilon_n = d(Sy_{n+1}, f(T,y_n))$, where $f(T,y_n)=Ty_n$. Assume that $\lim_{n\to\infty}\varepsilon_n=0$. Then we have

$$\begin{aligned} d(Sy_{n+1},p) &\le s\bigl[d(Sy_{n+1},Ty_n)+d(Ty_n,p)\bigr]\\ &\le s\varepsilon_n + s\bigl[\varphi\bigl(d(Sz,Sy_n)\bigr)+\psi\bigl(\min\bigl\{d(Sz,Tz),d(Sz,Ty_n)\bigr\}\bigr)\bigr]\\ &= s\varepsilon_n + s\varphi\bigl(d(Sy_n,p)\bigr). \end{aligned}$$

Since φ is a subadditive s-comparison function, we get that $s\varphi$ is a subadditive comparison function. Therefore, Lemma 2.4 yields that $\lim_{n\to\infty}d(Sy_n,p)=0$, that is, $\lim_{n\to\infty}Sy_n=p$. □

Remark 5.1 Theorem 5.2 is a generalization of Theorem 3.2 of Singh and Alam [22], Theorem 3.4 of Singh *et al.* [13], Theorems 4.1 and 4.2 of Singh and Prasad [23], Theorem 1 of Osilike [6], Theorem 2 of Berinde [28], Theorem 2.1 of Bosede and Rhoades [29] as well as Corollary 2 of Qing and Rhoades [30].

The following example shows that the Ishikawa iterative process is not (S,T)-stable.

Example 5.2 Let $S,T:[0,1]\to\mathbb{R}$ be given by $Sx=x$ and $Tx=\frac{-x}{2}$, where $\mathbb{R}$ is again endowed with the usual metric. Then T is a weak Jungck $(\frac{I}{2},0)$-contraction. Let $\{x_n\}$ be a sequence generated by the Ishikawa iterative process with $\alpha_n=\beta_n=1-\frac{1}{n+1}$ and $x_0\in[0,1]$. Then

$$\begin{cases} z_n = Sz_n = \beta_n Sx_n + (1-\beta_n)Tx_n = (1-\frac{1}{n+1})x_n + \frac{1}{n+1}Tx_n = (1-\frac{3}{2(n+1)})x_n, \\ x_{n+1} = Sx_{n+1} = \alpha_n Sx_n + (1-\alpha_n)Tz_n = (1-\frac{1}{n+1})x_n + \frac{1}{n+1}Tz_n = (1-\frac{3}{2(n+1)} + \frac{3}{4(n+1)^2})x_n. \end{cases}$$

Suppose that $t_n=\frac{3}{2(n+1)}-\frac{3}{4(n+1)^2}$. As $t_n\in(0,1)$ and $\sum_{n=0}^{\infty} t_n=\infty$, Lemma 2 of [31] implies that $\lim_{n\to\infty} x_n = 0 = S(0) = T(0)$ (the unique coincidence value of S and T).

To prove the fact that the Ishikawa iteration is not (S,T)-stable, we use the sequence $\{y_n\}$ given by $y_n=\frac{n+1}{n+2}$. Then

$$\begin{aligned} \varepsilon_n &= \left|y_{n+1} - f(T,y_n)\right| \\ &= \left|y_{n+1} - \left(1-\frac{3}{2(n+1)}+\frac{3}{4(n+1)^2}\right)y_n\right| \\ &= \left|\frac{n+2}{n+3} - \left(1-\frac{3}{2(n+1)}+\frac{3}{4(n+1)^2}\right)\frac{n+1}{n+2}\right| \\ &= \frac{6n^2+25n+13}{4(n+1)(n+2)(n+3)}. \end{aligned}$$

It is clear that $\lim_{n\to\infty}\varepsilon_n=0$ and $\sum_{n=0}^{\infty}\varepsilon_n=\infty$, while $\lim_{n\to\infty} y_n=1$. Therefore, the Ishikawa iterative procedure is not (S,T)-stable, but it is almost (S,T)-stable. (The almost (S,T)-stability is shown in the following.)

The following theorem states that Jungck-Mann iterative and Jungck-Ishikawa iterative process are almost (S,T)-stable provided that $\sum_{n=0}^{\infty}\alpha_n<\infty$.

Theorem 5.3 *Let (X,d,W) be a convex b-metric space and let T be a weak Jungck (φ,ψ)-contractive mapping such that φ is a convex subadditive function. Let $\{\alpha_n\}$ be a real sequence in* $[0,1]$ *such that $\sum_{n=0}^{\infty}\alpha_n<\infty$. For $x_0\in Y$, let $\{Sx_n\}$ be the Ishikawa iterative process given by* (4.5). *Then the Jungck-Ishikawa iteration is almost* (S,T)*-stable.*

Proof In view of Theorem 4.3, there exists a coincidence point $z\in Y$ such that $\{Sx_n\}$ converges to $p=Sz=Tz$. Suppose that $\{Sy_n\}\subset X$, $\varepsilon_n=d(Sy_{n+1},W(Sy_n,Tu_n,\alpha_n))$, $n=0,1,2,\dots$, where $Su_n=W(Sy_n,Ty_n,\beta_n)$. Assume that $\sum_{n=0}^{\infty}\varepsilon_n<\infty$. Then

$$\begin{aligned} d(Sy_{n+1},p) &\le s\bigl[d\bigl(Sy_{n+1},W(Sy_n,Tu_n,\alpha_n)\bigr)+d\bigl(W(Sy_n,Tu_n,\alpha_n),p\bigr)\bigr] \\ &\le s\varepsilon_n + s\bigl[\alpha_n d(Sy_n,p)+(1-\alpha_n)d(Tu_n,p)\bigr] \end{aligned}$$

$$\begin{aligned}
&\leq s\varepsilon_n + s\alpha_n d(Sy_n, p) + s(1-\alpha_n) \\
&\quad \times \bigl[\varphi\bigl(d(Sz, Su_n)\bigr) + \psi\bigl(\min\bigl\{d(Sz, Tz), d(Sz, Tu_n)\bigr\}\bigr)\bigr] \\
&\leq s\varepsilon_n + s\alpha_n d(Sy_n, p) + s(1-\alpha_n)\varphi\bigl(d(Su_n, p)\bigr),
\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}
d(Su_n, p) &\leq \beta_n d(Sy_n, p) + (1-\beta_n) d(Ty_n, p) \\
&\leq \beta_n d(Sy_n, p) + (1-\beta_n) \\
&\quad \times \bigl[\varphi\bigl(d(Sz, Sy_n)\bigr) + \psi\bigl(\min\bigl\{d(Sz, Tz), d(Sz, Ty_n)\bigr\}\bigr)\bigr] \\
&\leq \beta_n d(Sy_n, p) + (1-\beta_n)\varphi\bigl(d(Sy_n, p)\bigr) \\
&\leq \beta_n d(Sy_n, p) + (1-\beta_n) d(Sy_n, p) \\
&= d(Sy_n, p).
\end{aligned} \tag{5.2}$$

From (5.1) and (5.2), we conclude that

$$d(Sy_{n+1}, p) \leq s\varepsilon_n + s\alpha_n d(Sy_n, p) + s(1-\alpha_n)\varphi\bigl(d(Sy_n, p)\bigr). \tag{5.3}$$

Since φ is an s-comparison function, $s\varphi$ is a comparison function. Thus, inequality (5.3) implies that

$$d(Sy_{n+1}, p) \leq s\varepsilon_n + s\alpha_n d(Sy_n, p) + (1-\alpha_n) d(Sy_n, p) = \bigl(1 + (s-1)\alpha_n\bigr) d(Sy_n, p) + s\varepsilon_n.$$

Now, according to Lemma 2.3, $\lim_{n\to\infty} d(Sy_n, p)$ exists. Therefore, there exists $u \in \mathbb{R}^+$ such that $\lim_{n\to\infty} d(Sy_n, p) = u$. Assume that $u > 0$. Since $s\varphi$ is a subadditive comparison function, φ is continuous and $s\varphi(t) < t$ for all $t > 0$. Then, letting $n \to \infty$ in (5.3), we get $u \leq s\varphi(u) < u$, which is a contradiction. Hence, $u = 0$ and this completes the proof. □

In a similar way, using Lemma 1 of [32] in place of Lemma 2.3 in the previous proof, by omitting the condition $\sum \alpha_n < \infty$, one can prove that Theorem 5.3 holds in convex metric spaces. This indicates that the Ishikawa iterative process given Example 5.2 is almost (S, T)-stable.

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