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On generalized Fenchel-Moreau theorem and second-order characterization for convex vector functions

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Dedicated to Professor Wataru Takahashi on the occasion of his 70th birthday

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Abstract
Based on the concept of conjugate and biconjugate maps introduced in (Tan and Tinh in Acta Math. Viet. 25:315-345, 2000) we establish a full generalization of the Fenchel-Moreau theorem for the vector case. Besides this, by using the Clarke generalized first-order derivative for locally Lipschitz vector functions, we establish a first-order characterization for monotone operators. Consequently, a second-order characterization for convex vector functions is obtained.
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1 Introduction

Convex functions play an important role in nonlinear analysis, especially in optimization theory since they guarantee several useful properties concerning extremum points. Consequently, characterizations of the class of these functions, first-order as well as second-order, have been studied intensively. We also know that in convex analysis the theory of Fenchel conjugation plays a central role and the Fenchel-Moreau theorem concerning biconjugate functions plays a key role in the duality theory.

In the vector case, there are also many efforts focussing on these topics (see [1–17]). However, the results are still far from the repletion. The main difficulty for ones working on the vector setting is the non-completion of the order under consideration. Hence several generalizations are not complete.

The first purpose of the paper is to generalize the Fenchel-Moreau theorem to the vector case. Based on the concepts of supremum and conjugate and biconjugate maps introduced in [16], we obtain a full generalization of the theorem. Secondly, by using the Clarke generalized first-order derivative for locally Lipschitz vector functions, we establish a first-order characterization for monotone operators. Consequently, a second-order characterization for convex vector functions is obtained.

The paper is organized as follows. In the next section, we present some preliminaries on a cone order in finitely dimensional spaces and on convex vector functions. Section 3 is devoted to a generalization of the Fenchel-Moreau theorem. The last section deals with a second-order characterization of convex vector functions.

2 Preliminaries

Let $C \subseteq \mathbb{R}^m$ be a nonempty set. We recall that C is said to be a cone if $tx \in C, \forall x \in C, t \geq 0$. A cone C is said to be pointed if $C \cap (-C) = \{0\}$. A convex cone $C \subseteq \mathbb{R}^m$ specifies on $\mathbb{R}^m$ a partial order defined by

$$\forall x, y \in \mathbb{R}^m, \quad x \preceq_C y \quad \Leftrightarrow \quad y - x \in C.$$

When $\operatorname{int} C \neq \emptyset$, we shall write $x \ll_C y$ if $y - x \in \operatorname{int} C$. From now on we assume that $\mathbb{R}^m$ is ordered by a convex cone C.

Definition 2.1 [5, Definition 2.1] Let $A \subseteq \mathbb{R}^m$ be a nonempty set, and let $a \in A$. We say that

(i) a is an ideal efficient (or ideal minimum) point of A with respect to C if

$$a \preceq x, \quad \forall x \in A.$$

The set of ideal efficient points of A is denoted by $\operatorname{IMin}(A|C)$.

(ii) a is an efficient (or Pareto minimum) point of A with respect to C if

$$\forall x \in A, \quad x \preceq a \quad \Rightarrow \quad a \preceq x.$$

The set of efficient points of A is denoted by $\operatorname{Min}(A|C)$.

Remark 2.2 When C is pointed and $\operatorname{IMin}(A|C)$ is nonempty, then $\operatorname{IMin}(A|C)$ is a singleton and $\operatorname{Min}(A|C) = \operatorname{IMin}(A|C)$. The concepts of Max and IMax are defined analogously. It is clear that $-\operatorname{Min} A = \operatorname{Max}(-A)$.

Definition 2.3 [16] Let $A \subseteq \mathbb{R}^m$ be a nonempty set, and let $b \in \mathbb{R}^m$. We say that b is an upper bound of A with respect to C if

$$x \preceq b, \quad \forall x \in A.$$

The set of upper bounds of A is denoted by $\operatorname{Ub}(A|C)$.

When $\operatorname{Ub}(A|C) \neq \emptyset$, we say that A is bounded from above. The concept of lower bound is defined analogously. The set of lower bounds of A is denoted by $\operatorname{Lb}(A|C)$.

Definition 2.4 [16, Definition 2.3] Let $A \subseteq \mathbb{R}^m$ be a nonempty set, and let $b \in \mathbb{R}^m$. We say that

(i) b is an ideal supremal point of A with respect to C if $b \in \operatorname{IMin}(\operatorname{Ub} A|C)$, *i.e.*,

$$\begin{cases} x \preceq b, & \forall x \in A, \\ b \preceq y, & \forall y \in \operatorname{Ub}(A|C). \end{cases}$$

The set of ideal supremal points of A is denoted by $\operatorname{ISup}(A|C)$.

(ii) b is a supremal point of A with respect to C if $b \in \operatorname{Min}(\operatorname{Ub} A|C)$, *i.e.*,

$$\begin{cases} x \preceq b, \quad \forall x \in A, \\ \forall y \in \operatorname{Ub}(A|C), \quad y \preceq b \Rightarrow b \preceq y. \end{cases}$$

The set of supremal points of A is denoted by $\operatorname{Sup}(A|C)$.

Remark 2.5 If $\operatorname{ISup} A \neq \emptyset$, then $\operatorname{ISup} A = \operatorname{Sup} A$. In addition, if the ordering cone C is pointed, then $\operatorname{ISup} A$ is a singleton.

In the sequel, when there is no risk of confusion, we omit the phrase 'with respect to C' and the symbol '$|_C$' in the definitions above. We list here some properties of supremum which will be needed in the sequel.

Lemma 2.6 *Assume that the ordering cone $C \subseteq \mathbb{R}^m$ is closed, convex and pointed.*

(i) [16, Corollary 2.21] *Let $A \subseteq \mathbb{R}^m$ be nonempty. If* $\operatorname{Ub} A \cap \overline{\operatorname{co} A} \neq \emptyset$, *then*

$$\operatorname{Ub} A \cap \overline{\operatorname{co} A} = \operatorname{ISup} A$$

(*where* $\overline{\operatorname{co} A}$ *denotes the closure of the convex hull of A*).

(ii) [16, Corollary 2.14] *Let $S \subseteq \mathbb{R}$ be nonempty and bounded from above. Then, for every $c \in C$, we have*

$$\operatorname{ISup}(Sc) = (\sup S)c$$

(*where* $Sc := \{tc : t \in S\}$).

(iii) [16, Theorem 2.16, Remark 2.18] *Let $A \subseteq \mathbb{R}^m$ be nonempty. Then* $\operatorname{Sup} A \neq \emptyset$ *if and only if A is bounded from above. In this case, we have*

$$\operatorname{Ub} A = \operatorname{Sup} A + C.$$

(iv) [16, Proposition 2.22] *Let $A, B \subseteq \mathbb{R}^m$ be nonempty. Then*

(a) *If $A \subseteq B$, then* $\operatorname{Sup} B \subseteq \operatorname{Sup} A + C$;

(b) $\operatorname{Sup} A + \operatorname{Sup} B \subseteq \operatorname{Sup}(A + B) + C$. *If, in addition*, $\operatorname{ISup} A \cup \operatorname{ISup} B \neq \emptyset$, *then*

$$\operatorname{Sup} A + \operatorname{Sup} B = \operatorname{Sup}(A + B).$$

Now let f be a vector function from a nonempty set $D \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$, and let $S \subseteq D$, $x \in S$. We say that f is continuous relative to S at x if for every neighborhood W of $f(x)$, there exists a neighborhood V of x such that

$$x' \in V \cap S \quad \Rightarrow \quad f(x') \in W.$$

f is called continuous relative to S if it is continuous relative to S at every $x \in S$. The epigraph of f (with respect to the ordering cone C) is defined as the set

$$\operatorname{epi} f := \{(x, y) \in D \times \mathbb{R}^m : f(x) \preceq y\}.$$

f is called closed (with respect to C) if $\operatorname{epi} f$ is closed in $\mathbb{R}^n \times \mathbb{R}^m$. Now assume that $D \subseteq \mathbb{R}^n$ is nonempty and convex. We recall that $f : D \to \mathbb{R}^m$ is said to be convex (with respect to C) if for every $x, y \in D$, $\lambda \in [0,1]$,

$$f\bigl(\lambda x + (1-\lambda)y\bigr) \preceq \lambda f(x) + (1-\lambda)f(y).$$

Subdifferential of f at $x \in D$ is defined as the set

$$\partial f(x) := \bigl\{A \in \mathcal{L}\bigl(\mathbb{R}^n, \mathbb{R}^m\bigr) : A(y-x) \preceq f(y) - f(x)\ (\forall y \in D)\bigr\}.$$

Convex vector functions have several nice properties as scalar convex functions (see, [6, 16, 17]). We recall some results which will be used in the sequel.

Lemma 2.7 [6, Theorem 4.12] *Assume that the ordering cone $C \subseteq \mathbb{R}^m$ is closed, convex and pointed. Let f be a convex vector function from a nonempty convex set $D \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$. Then $\partial f(x) \neq \emptyset$ for every $x \in \operatorname{ri} D$.*

From [17, Theorem 3.6] we immediately have the following lemma.

Lemma 2.8 *Assume that the ordering cone $C \subseteq \mathbb{R}^m$ is closed, convex and pointed with $\operatorname{int} C \neq \emptyset$. Let f be a closed convex vector function from a nonempty convex set $D \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$, and let $x, y \in D$ be arbitrary. Then f is continuous relative to $[x,y]$ (where $[x,y] := \{tx + (1-t)y : t \in [0,1]\}$).*

3 Generalized Fenchel-Moreau theorem

Let F be a set-valued map from a finitely dimensional normed space X to $\mathbb{R}^m$. We recall that the epigraph of F with respect to C is defined as the set

$$\operatorname{epi} F := \bigl\{(x,y) \in X \times \mathbb{R}^m : y \in F(x) + C\bigr\}.$$

The effective domain of F is the set

$$\operatorname{dom} F := \bigl\{x \in X : F(x) \neq \emptyset\bigr\}.$$

F is called convex (resp., closed) with respect to C if $\operatorname{epi} F$ is convex (resp., closed) in $X \times \mathbb{R}^m$. Sometimes a vector function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is identified with the set-valued map

$$F(x) := \begin{cases} \{f(x)\}, & x \in D, \\ \emptyset, & x \notin D. \end{cases}$$

Definition 3.1 [16, Definition 3.1] Assume that $\operatorname{dom} F \neq \emptyset$. The conjugate map of F, denoted by F^*, is a set-valued map from $\mathcal{L}(X, \mathbb{R}^m)$ to $\mathbb{R}^m$ defined as follows.

$$F^*(A) := \operatorname{Sup} \bigcup_{x \in X} \bigl[A(x) - F(x)\bigr], \quad \forall A \in \mathcal{L}\bigl(X, \mathbb{R}^m\bigr),$$

where $\mathcal{L}(X, \mathbb{R}^m)$ denotes the space of continuous linear maps from X to $\mathbb{R}^m$.

Definition 3.2 [16, Definition 3.2] Let F be a set-valued map from $\mathbb{R}^n$ to $\mathbb{R}^m$. Assume that $\operatorname{dom} F^* \neq \emptyset$. The biconjugate map of F, denoted by F^{**}, is a set-valued map from $\mathbb{R}^n$ to $\mathbb{R}^m$ defined as follows.

$$F^{**}(x) := \operatorname{Sup} \bigcup_{A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)} \bigl[A(x) - F^*(A)\bigr], \quad \forall x \in \mathbb{R}^n.$$

Remark 3.3 Let F be a set-valued map from $\mathbb{R}^n$ to $\mathbb{R}^m$ with $\operatorname{dom} F^* \neq \emptyset$. By identifying $x \in \mathbb{R}^n$ with the linear map $\bar{x} : \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}^m$ defined as follows:

$$\bar{x}(A) := A(x), \quad \forall A \in \mathcal{L}\bigl(\mathbb{R}^n, \mathbb{R}^m\bigr),$$

we see that F^{**} is the restriction of $(F^*)^*$ on $\mathbb{R}^n$, *i.e.*,

$$F^{**} = \bigl(F^*\bigr)^*|_{\mathbb{R}^n}.$$

In the rest of this section, we assume that the ordering cone $C \subseteq \mathbb{R}^m$ is closed, convex, pointed and $\operatorname{int} C \neq \emptyset$.

Lemma 3.4 [16, Proposition 3.5] *Let F be a set-valued map from $\mathbb{R}^n$ to $\mathbb{R}^m$ with* $\operatorname{dom} F \neq \emptyset$. *Then*

(i) *F^* is closed and convex.*
(ii) *If* $\operatorname{dom} F^* \neq \emptyset$, *then $F(x) \subseteq F^{**}(x) + C$, $\forall x \in \mathbb{R}^n$.*

Lemma 3.5 *Let F be a set-valued map from $\mathbb{R}^n$ to $\mathbb{R}^m$ with* $\operatorname{dom} F^* \neq \emptyset$. *Then F^{**} is closed and convex.*

Proof It is immediate from Remark 3.3 and Lemma 3.4. □

Lemma 3.6 [16, Proposition 3.6] *Let f be a convex vector function from a nonempty convex set $D \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$, and let $x \in D$, $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. Then $A \in \partial f(x)$ if and only if*

$$f^*(A) = A(x) - f(x).$$

Lemma 3.7 *Let f be a convex vector function from a nonempty convex set $D \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$. Then*

$$D \subseteq \operatorname{dom} f^{**} \subseteq \overline{D}.$$

Proof Let $x \in \operatorname{ri} D$ be arbitrary. By Lemma 2.7, $\partial f(x) \neq \emptyset$. Then, by Lemma 3.6, $\partial f(x) \subseteq \operatorname{dom} f^*$. Consequently, $\operatorname{dom} f^* \neq \emptyset$. Then, by Lemma 3.4, $D \subseteq \operatorname{dom} f^{**}$. Now, suppose on the contrary that $\operatorname{dom} f^{**} \nsubseteq \overline{D}$. Then there is $x_0 \in \operatorname{dom} f^{**}$ such that $x_0 \notin \overline{D}$. Using the strong separation theorem, one can find $\xi \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}) \setminus \{0\}$ so that

$$\xi(x_0) > \sup_{x \in \overline{D}} \xi(x). \tag{1}$$

Pick any $y_0 \in \operatorname{ri} D$ and $A_0 \in \partial f(y_0)$. By Lemma 3.6, $f^*(A_0)$ is a singleton. For each $c \in C$, we define a linear map $\beta_c : \mathbb{R} \to \mathbb{R}^m$ as follows:

$$\beta_c(t) = tc \quad (\forall t \in \mathbb{R}).$$

By (1) and by Lemma 2.6(ii),

$$\operatorname{ISup} \bigcup_{x\in D} \{(\beta_c \xi)(x)\} = \Bigl(\sup_{x\in D} \xi(x)\Bigr) c.$$

Then we have

$$\begin{aligned}
f^*(A_0) + \Bigl(\sup_{x\in D} \xi(x)\Bigr) c &= f^*(A_0) + \operatorname{ISup} \bigcup_{x\in D} \{(\beta_c \xi)(x)\} \\
&= \operatorname{Sup} \bigcup_{x\in D} \{A_0(x) - f(x)\} + \operatorname{ISup} \bigcup_{x\in D} (\beta_c \xi)(x) \\
&= \operatorname{Sup} \Bigl(\bigcup_{x\in D} \{A_0(x) - f(x)\} + \bigcup_{x\in D} \{(\beta_c \xi)(x)\} \Bigr) \\
&\qquad \text{(by Lemma 2.6(iv))} \\
&\subseteq \operatorname{Sup} \bigcup_{x\in D} \{A_0(x) - f(x) + (\beta_c \xi)(x)\} + C \\
&\qquad \text{(by Lemma 2.6(iv))} \\
&= f^*(A_0 + \beta_c \xi) + C.
\end{aligned}$$

Then there exists $y_c \in f^*(A_0 + \beta_c \xi)$ such that

$$f^*(A_0) + \Bigl(\sup_{x\in D} \xi(x)\Bigr) c \succeq y_c.$$

Let $z \in f^{**}(x_0)$ be arbitrary. From the definition of f^{**}, one has

$$\begin{aligned}
z &\succeq (A_0 + \beta_c \xi)(x_0) - y_c \\
&\succeq \bigl[A_0(x_0) - f^*(A_0)\bigr] + \Bigl[\xi(x_0) - \sup_{x\in D} \xi(x)\Bigr].c \quad (\forall c \in C).
\end{aligned}$$

By (1), this is impossible since $C \neq \{0\}$ and pointed. Thus, $\operatorname{dom} f^{**} \subseteq \overline{D}$. The proof is complete.

□

Let $x_0, x \in \mathbb{R}^n$, $\{x_k\}_k \subseteq [x_0, x]$. Then we write '$x_k \uparrow x$' if

$$\begin{cases} x_k \to x, \\ \|x_{k+1} - x_0\| \geq \|x_k - x_0\| \quad (\forall k). \end{cases}$$

Lemma 3.8 [16, Lemma 3.16] *Let f be a convex function from a nonempty convex set $D \subseteq \mathbb{R}^n$ to $\mathbb{R}^m$, and let $x \in D$. If there exists $x_0 \in \operatorname{ri} D$ such that*

$$f(x) = \lim_{t\uparrow 1} f\bigl(tx + (1-t)x_0\bigr),$$

then for every sequence $\{(A_k, x_k)\}_k \subseteq \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \times [x_0, x]$ *such that* $x_k \uparrow x$ *and* $A_k \in \partial f(x_k)$, *we have*

$$\lim_{k\to\infty} A_k(x - x_k) = 0.$$

Although biconjugate maps of vector functions have a set-valued structure, under certain conditions, they reduce to single-valued maps. Such conditions are the convexity and closedness of the functions. Moreover, we have the following theorem.

Theorem 3.9 (*Generalized Fenchel-Moreau theorem*) *Let* f *be a vector function from a nonempty convex set* $D \subseteq \mathbb{R}^n$ *to* $\mathbb{R}^m$. *Then* f *is closed and convex if and only if*

$$f = f^{**}.$$

Proof $\Rightarrow$: Let $x \in D$ be arbitrary. Pick a point $x_0 \in \operatorname{ri} D$. By Lemma 2.8, f is continuous relative to $[x_0, x]$. Hence

$$f(x) = \lim_{t\uparrow 1} f\bigl(tx + (1-t)x_0\bigr). \tag{2}$$

Let $\{\lambda_k\}_k \subseteq (0,1)$ be an increasing sequence that converges to 1. Put $x_k = \lambda_k x + (1 - \lambda_k)x_0$. Then $\{x_k\}_k \subseteq \operatorname{ri} D \cap [x_0, x]$ and $x_k \uparrow x$. By Lemma 2.7, $\partial f(x_k) \neq \emptyset$. For each k, pick $A_k \in \partial f(x_k)$. By Lemma 3.6, $f(x_k) = A_k(x_k) - f^*(A_k)$. Hence,

$$\begin{aligned}\bigl\|f(x) - \bigl[A_k(x) - f^*(A_k)\bigr]\bigr\| &= \bigl\|f(x) - \bigl[A_k(x_k) - f^*(A_k)\bigr] + \bigl[A_k(x_k) - A_k(x)\bigr]\bigr\| \\ &\leq \bigl\|f(x) - f(x_k)\bigr\| + \bigl\|A_k(x_k - x)\bigr\|. \end{aligned} \tag{3}$$

Take $k \to \infty$ in (3), by (2) and by Lemma 3.8, we have

$$\bigl\|f(x) - \bigl[A_k(x) - f^*(A_k)\bigr]\bigr\| \to 0,$$

which together with Lemma 3.4(ii) implies

$$f(x) \in \operatorname{Ub}\Bigl(\bigcup_{A\in\mathcal{L}(\mathbb{R}^n,\mathbb{R}^m)} \bigl[A(x) - f^*(A)\bigr]\Bigr) \cap \operatorname{cl}\Bigl(\operatorname{co}\Bigl(\bigcup_{A\in\mathcal{L}(\mathbb{R}^n,\mathbb{R}^m)} \bigl[A(x) - f^*(A)\bigr]\Bigr)\Bigr).$$

Hence, by Lemma 2.6(i), Remark 2.5 and by the definition of biconjugate maps, we have

$$f(x) = \operatorname{ISup} \bigcup_{A\in\mathcal{L}(\mathbb{R}^n,\mathbb{R}^m)} \bigl[A(x) - f^*(A)\bigr] = f^{**}(x). \tag{4}$$

Finally, we shall show that

$$\operatorname{dom} f^{**} = D.$$

Indeed, by the proof above, we have $\operatorname{dom} f^{**} \supseteq D$. Let $x_0 \in \operatorname{dom} f^{**}$ be arbitrary. By Lemma 3.7, $x_0 \in \overline{D}$. Let $y_0 \in f^{**}(x_0)$ and $x \in \operatorname{ri} D$. Then $(x, f(x)), (x_0, y_0) \in \operatorname{epi} f^{**}$. For every natural number $k \geq 1$, put

$$x_k = \frac{1}{k}x + \left(1 - \frac{1}{k}\right)x_0,$$
$$y_k = \frac{1}{k}f(x) + \left(1 - \frac{1}{k}\right)y_0.$$

Obviously, $(x_k, y_k) \to (x_0, y_0)$ and $(x_k, y_k) \in \operatorname{epi} f^{**}$, $\forall k$, since f^{**} is convex. By (4), $f(x_k) = f^{**}(x_k)$ since $x_k \in D$. Hence, $(x_k, y_k) \in \operatorname{epi} f$ $(\forall k)$. This fact together with closedness of f implies

$$(x_0, y_0) \in \operatorname{epi} f.$$

Hence $x_0 \in D$. Thus, $\operatorname{dom} f^{**} = D$ and then $f = f^{**}$.

$\Leftarrow$: It is immediate from Lemma 3.5. The theorem is proved. □

When $m = 1$ and $C = \mathbb{R}_+$, Theorem 3.9 is the famous Fenchel-Moreau theorem in convex analysis.

4 Second-order characterization of convex vector functions

Let X, Y be real finitely dimensional normed spaces. We denote by $\mathcal{L}(X, Y)$ the space of continuous linear maps from X to Y. In $\mathcal{L}(X, Y)$ we equip the norm defined by

$$\|A\| := \sup\{\|A(x)\| : \|x\| \leq 1\}, \quad \forall A \in \mathcal{L}(X, Y).$$

Let $D \subseteq X$ be a nonempty open set, $x_0 \in D$, and let $f : D \to Y$ be a vector function.

Definition 4.1 [18] Assume that f is locally Lipschitz. The *Clarke generalized derivative* of f at x_0 is defined as

$$\partial f(x_0) := \operatorname{co}\Big\{\lim_{k\to\infty} Df(x_k) : x_k \in D, x_k \to x_0, Df(x_k) \text{ exists}\Big\},$$

where $Df(x_k)$ denotes the derivative of f at x_k.

The following definition is suggested by [19, Definition 2.1].

Definition 4.2 Assume that f is a vector function of class $C^{1,1}$. The *Clarke generalized second-order derivative* of f at x_0 is defined as

$$\partial^2 f(x_0) := \operatorname{co}\Big\{\lim_{k\to\infty} D^2 f(x_k) : x_k \in D, x_k \to x_0, D^2 f(x_k) \text{ exists}\Big\},$$

where $D^2 f(x_k)$ denotes the second-order derivative of f at x_0.

In the remainder of this section, we assume that the ordering cone $C \subseteq \mathbb{R}^m$ is closed and convex.

Definition 4.3 Let $D \subseteq \mathbb{R}^n$ be a nonempty set, and let a map $F : D \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. We say that F is monotone with respect to C if

$$F(x)(y - x) + F(y)(x - y) \preceq 0, \quad \forall x, y \in D.$$

When $m = 1$ and $C = \mathbb{R}_+$, Definition 4.3 collapses to the classical concept of monotonicity.

Now assume that $D \subseteq \mathbb{R}^n$ is a nonempty, convex and open set. Let $F : D \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ be a locally Lipschitz map, $x \in D$, $y \in \mathbb{R}^n$. We denote by I the largest open line segment satisfying $x + ty \in D$, $\forall t \in I$. Define

$$\Phi(t) := F(x + ty)(y), \quad t \in I.$$

Set

$$\begin{aligned} &\partial\Phi(t)(\epsilon) := \{l(\epsilon) : l \in \partial\Phi(t)\}, \\ &M(y,y) := [M(y)](y), \quad \forall M \in \mathcal{L}(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)), y \in \mathbb{R}^n, \\ &\partial F(x+ty)(y,y) := \{M(y,y) : M \in \partial F(x+ty)\}. \end{aligned}$$

We have the following lemma.

Lemma 4.4 $\partial\Phi(t)(\epsilon) \subseteq \epsilon\partial F(x + ty)(y, y)$, $\forall t \in I$, $\epsilon \in \mathbb{R}$.

Proof Observe that $\Phi = \varphi \circ F \circ \psi$, where

$$\psi : t \mapsto x + ty, \qquad \varphi : A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \mapsto A(y).$$

Since φ is linear and ψ is affine, we have

$$\begin{aligned} &\partial\psi(t)(\epsilon) = D\psi(t)(\epsilon) = \epsilon y, \quad \forall t \in I, \epsilon \in \mathbb{R} \\ &\partial\varphi(A)(M) = D\varphi(A)(M) = \varphi(M) = M(y), \quad \forall A, M \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m). \end{aligned}$$

Then, applying a chain rule in [18, Corollary 2.6.6], one obtains

$$\begin{aligned} \partial\Phi(t)(\epsilon) &= \partial((\varphi \circ F) \circ \psi)(t)(\epsilon) \\ &\subseteq \operatorname{co}\{\partial(\varphi \circ F)(\psi(t))\partial\psi(t)\}(\epsilon) \\ &= \operatorname{co}\{\partial(\varphi \circ F)(\psi(t))\}(\epsilon y) \\ &= \operatorname{co}\{D\varphi(F(x+ty))\partial F(x+ty)\}(\epsilon y) \\ &= D\varphi(F(x+ty))\partial F(x+ty)(\epsilon y) \quad (\text{since } \partial F(x+ty) \text{ is convex}) \\ &= \epsilon\partial F(x+ty)(y,y). \end{aligned}$$

□

Theorem 4.5 *Let $D \subseteq \mathbb{R}^n$ be a nonempty, convex and open set, and let $F : D \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ be a locally Lipschitz map. Then the following statements are equivalent*:

(i) *F is monotone with respect to C.*

(ii) *For every $x \in D$ at which F is differentiable,*

$$DF(x)(u,u) \in C, \quad \forall u \in \mathbb{R}^n.$$

(iii) *For every* $x \in D$, $A \in \partial F(x)$,

$$A(u,u) \in C, \quad \forall u \in \mathbb{R}^n.$$

Proof (i) ⇒ (ii) Let $x \in D$ at which F is differentiable, and let $u \in \mathbb{R}^n$ be arbitrary. Let $\{t_k\}_k$ be a positive sequence converging to 0. Since F is monotone with respect to C, we have

$$\frac{(F(x+t_k u) - F(x))}{t_k}(u) = \frac{1}{t_k^2}\bigl(F(x+t_k u) - F(x)\bigr)(x + t_k u - x) \in C, \quad \forall k.$$

Taking $k \to \infty$, since C is closed, we obtain $DF(x)(u,u) \in C$.

(ii) ⇒ (iii) Let $x \in D$, $A \in \partial F(x)$ and $u \in \mathbb{R}^n$ be arbitrary. By the definition of the Clark generalized derivative, we can represent A in the form

$$A = \sum_{i=1}^{k} \lambda_i A_i, \tag{5}$$

where $\lambda_i \geq 0$, $\sum_{i=1}^k \lambda_i = 1$ and $A_i = \lim_{j\to\infty} DF(x_{ij})$ with $x_{ij} \to x$ $(j \to \infty)$, and there exists $DF(x_{ij})$ for every $i = 1,\ldots,k$; $j = 1,2,\ldots$. Since $DF(x_{ij})(u,u) \in C$ and C is closed, passing to the limit, we have $A_i(u,u) \in C$, $\forall i = 1,\ldots,k$. By (5) and by the convexity of C, we obtain $A(u,u) \in C$.

(iii) ⇒ (i) Let $x, y \in D$ be arbitrary. Consider the function

$$\Phi(t) = F\bigl(x + t(y-x)\bigr)(y-x).$$

Then Φ is locally Lipschitz on an open line segment I which contains $[0,1]$. Hence Φ is Lipschitz on any compact line segment $[a,b]$ with

$$[0,1] \subseteq (a,b) \subseteq [a,b] \subseteq I.$$

By the mean value theorem, for a vector function [18, Proposition 2.6.5], there exist $\tau_1,\ldots,\tau_k \in [0,1]$, $\lambda_1,\ldots,\lambda_k \geq 0$, $\lambda_1 + \cdots + \lambda_k = 1$ such that

$$\Phi(1) - \Phi(0) \in \sum_{i=1}^{k} \lambda_i \partial\Phi(\tau_i)(1).$$

Hence

$$\begin{aligned}\bigl(F(y) - F(x)\bigr)(y-x) &= \Phi(1) - \Phi(0)\\ &\in \sum_{i=1}^{k} \lambda_i \partial\Phi(\tau_i)(1)\\ &\subseteq \sum_{i=1}^{k} \lambda_i \partial F\bigl(x + \tau_i(y-x)\bigr)(y-x, y-x)\\ &\quad \text{(by Lemma 4.4)}\\ &\subseteq C.\end{aligned}$$

Thus F is monotone. The proof is complete. □

We note that Theorem 4.5 generalizes the corresponding result of Luc and Schaible in [7] in which $m = 1$ and $C = \mathbb{R}_+$.

Theorem 4.6 *Let $D \subseteq \mathbb{R}^n$ be a nonempty convex and open set, and let $f : D \to \mathbb{R}^m$ be a $C^{1,1}$ vector function. Then f is convex with respect to C if and only if for every $x \in D$, $A \in \partial^2 f(x)$, $u \in \mathbb{R}^n$,*

$$A(u,u) \in C.$$

Proof We have

$$\begin{aligned} f \text{ is convex with respect to } C \quad &\Leftrightarrow \quad Df \text{ is monotone with respect to } C \\ &\qquad \text{(by [17, Theorem 4.4])} \\ &\Leftrightarrow \quad A(u,u) \in C, \quad \forall x \in D, A \in \partial^2 f(x), u \in \mathbb{R}^n \\ &\qquad \text{(by Theorem 4.5).} \end{aligned}$$

□

Specially, we have the following.

Corollary 4.7 [17, Theorem 4.9] *Let $D \subseteq \mathbb{R}^n$ be a nonempty convex and open set, and let $f : D \to \mathbb{R}^m$ be a twice continuously differentiable function. Then f is convex with respect to C if and only if*

$$D^2 f(x)(u,u) \in C, \quad \forall x \in D, u \in \mathbb{R}^n.$$

Proof Since continuously differentiable functions are locally Lipschitz, repeating arguments in the proof of the above theorem, we obtain the result. □

We note that when $m = 1$, $C = \mathbb{R}_+$, Corollary 4.7 collapses to the classical result on the second-order characterization of convex functions.

Example 4.8 Let $\mathbb{R}^3$ be ordered by the cone $C = \operatorname{con}(\operatorname{co}\{(1,0,1),(0,-1,-1),(0,0,1)\})$. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $f(x_1,x_2) := (\frac{1}{2}x_1^2 + 2x_1 - x_2, -\frac{1}{2}x_2^2 - x_1 + 2x_2, \frac{1}{2}x_1^2 + x_1 - \frac{1}{2}x_2^2)$. By computing we have

$$D^2 f(x) = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right), \quad \forall x \in \mathbb{R}^2.$$

Then

$$\begin{aligned} D^2 f(x)(y,y) &= \left(y_1^2, -y_2^2, y_1^2 - y_2^2\right) \\ &= y_1^2(1,0,1) + y_2^2(0,-1,-1) \\ &\in C, \quad \forall x, y \in \mathbb{R}^2. \end{aligned}$$

Hence f is convex with respect to C by Corollary 4.7.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in this research work. All authors read and approved the final manuscript.

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