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A new hybrid iteration method for a finite family of asymptotically nonexpansive mappings in Banach spaces

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Abstract

In this paper, we first introduce a new hybrid iteration method for a finite family of asymptotically nonexpansive mappings and nonexpansive mappings in Banach spaces, and then we discuss the strong and weak convergence for the iterative processes. The results presented in this paper extend and improve the corresponding results of Wang and Osilike.

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1 Introduction and preliminaries

Throughout this paper we assume that E is a real Banach space and $T : E \to E$ is a mapping. We denote by $F(T)$ and $D(T)$ the set of fixed points and the domain of T, respectively.

Recently, the convergence problems of an implicit (or non-implicit) iterative process to a common fixed point for a finite family of asymptotically nonexpansive mappings (or nonexpansive mappings) in Hilbert spaces or uniformly convex Banach spaces have been considered by several authors (see, *e.g.*, [1–24]).

Recall that E is said to satisfy *Opial's condition* [11] if for each sequence $\{x_n\}$ in E, the condition that the sequence $x_n \to x$ weakly implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$.

Definition 1.1 Let D be a closed subset of E and $T : D \to D$ be a mapping.

(1) T is said to be *demi-closed* at the origin if for each sequence $\{x_n\}$ in D, the conditions $x_n \to x_0$ weakly and $Tx_n \to 0$ strongly imply $Tx_0 = 0$.

(2) T is said to be *semi-compact* if for any bounded sequence x_n in D such that $\|x_n - Tx_n\| \to 0$ $(n \to \infty)$, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \to x^* \in D$.

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(3) T is said to be *asymptotically nonexpansive* [3] if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\left\|T^n x - T^n y\right\| \leq k_n\|x-y\|, \quad \forall x,y \in D, n \geq 1;$$

when $k_n \equiv 1$, T is known as a nonexpansive mapping.

(4) T is said to be *L-Lipschitzian* if there exists a constant $L > 0$ such that $\|Tx - Ty\| \leq L\|x-y\|$ for all $x,y \in D$.

Proposition 1.1 *Let K be a nonempty subset of E, and let $\{T_i\}_{i=1}^m : K \to K$ be m asymptotically nonexpansive mappings. Then there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ such that*

$$\left\|T_i^n x - T_i^n y\right\| \leq k_n\|x-y\|, \quad \forall n \geq 1, x,y \in K, i = 1,2,\ldots,m. \tag{1.1}$$

Proof Since for each $i = 1,2,\ldots,m$, $T_i : K \to K$ is an asymptotically nonexpansive mapping, there exists a sequence $\{k_n^{(i)}\} \subset [1,\infty)$ with $k_n^{(i)} \to 1$ ($n \to \infty$) such that

$$\left\|T_i^n x - T_i^n y\right\| \leq k_n^{(i)}\|x-y\|, \quad \forall x,y \in K, \forall n \geq 1, i = 1,2,\ldots,m.$$

Letting

$$k_n = \max\left\{k_n^{(1)}, k_n^{(2)}, \ldots, k_n^{(m)}\right\},$$

we have that $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ ($n \to \infty$) and

$$\left\|T_i^n x - T_i^n y\right\| \leq k_n^{(i)}\|x-y\| \leq k_n\|x-y\|, \quad \forall n \geq 1$$

for all $x,y \in K$ and for each $i = 1,2,\ldots,m$. $\square$

In 2007, for studying the strong and weak convergence of fixed points of nonexpansive mappings in a Hilbert space H, Wang [19] introduced the following hybrid iteration scheme:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T^{\lambda_{n+1}} x_n, \quad \forall n \geq 0, \tag{1.2}$$

where $T^{\lambda_{n+1}} x_n = Tx_n - \lambda_{n+1}\mu F(Tx_n)$ for all $x_n \in H$, $x_0 \in H$ is an initial point, $F : H \to H$ is an η-strongly monotone and k-Lipschitzian mapping, μ is a positive fixed constant.

In the same year, Osilike *et al.* [13] extended the results of Wang from Hilbert spaces to arbitrary Banach spaces and proved those theorems by Wang without the strong monotonicity condition.

In this paper, we introduce the following new hybrid iteration method in Banach spaces:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\left[\sum_{i=1}^m \tau_i T_i^n x_n - \lambda_{n+1}\mu f\left(\sum_{i=1}^m \tau_i T_i^n x_n\right)\right], \quad \forall n \geq 0 \tag{1.3}$$

for a finite family of asymptotically nonexpansive mappings $\{T_i\}_{i=1}^m : K \to K$, where $f : K \to K$ is an L-Lipschitzian mapping, μ is a positive fixed constant, $\{\alpha_n\}$ is a sequence in $(0,1)$, $\{\lambda_n\} \subset [0,1)$ and $\{\tau_i\}_{i=1}^m \subset (0,1)$ such that $\sum_{i=1}^m \tau_i = 1$.

Especially, if $\{T_i\}_{i=1}^m : K \to K$ are m nonexpansive mappings, $f : K \to K$ is an L-Lipschitzian mapping, μ is a positive fixed constant, $\{\alpha_n\}$ is a sequence in $(0,1)$, $\{\lambda_n\} \subset [0,1)$ and $\{\tau_i\}_{i=1}^m \subset (0,1)$ such that $\sum_{i=1}^m \tau_i = 1$, then the sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\left[\sum_{i=1}^m \tau_i T_i x_n - \lambda_{n+1}\mu f\left(\sum_{i=1}^m \tau_i T_i x_n\right)\right], \quad \forall n \geq 0, \tag{1.4}$$

is called the hybrid iteration scheme for a finite family of nonexpansive mappings $\{T_i\}_{i=1}^N$.

The purpose of this paper is to study the weak and strong convergence of an iterative sequence $\{x_n\}$ defined by (1.3) and (1.4) to a common fixed point for a finite family of asymptotically nonexpansive mappings and nonexpansive mappings in Banach spaces. The results presented in this paper extend and improve the main results in [13] and [19].

In order to prove the main results of this paper, we need the following lemmas.

Lemma 1.1 [17] *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be three nonnegative real sequences satisfying the following condition*:

$$a_{n+1} \leq (1+b_n)a_n + c_n, \quad \forall n \geq 0.$$

If $\sum_{n=1}^\infty b_n < \infty$ and $\sum_{n=1}^\infty c_n < \infty$, then the limit $\lim_{n\to\infty} a_n$ exists.

Lemma 1.2 [15] *Let E be a uniformly convex Banach space, and let b, c be two constants with $0 < b < c < 1$. Suppose that $\{t_n\}$ is a sequence in $[b,c]$ and $\{x_n\}$, $\{y_n\}$ are two sequences in E. Then the conditions*

$$\begin{cases} \lim_{n\to\infty} \|(1-t_n)x_n + t_n y_n\| = d, \\ \limsup_{n\to\infty} \|x_n\| \leq d, \\ \limsup_{n\to\infty} \|y_n\| \leq d, \end{cases}$$

imply that $\lim_{n\to\infty} \|x_n - y_n\| = 0$, where $d \geq 0$ is some constant.

Lemma 1.3 [4] *Let E be a uniformly convex Banach space, let K be a nonempty closed convex subset of E, and let $T : K \to K$ be an asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. Then $I - T$ is semi-closed at zero, where I is the identity mapping of E, that is, for each sequence $\{x_n\}$ in K, if $\{x_n\}$ converges weakly to $q \in K$ and $\{(I-T)x_n\}$ converges strongly to 0, then $(I-T)q = 0$.*

2 Main results

We are now in a position to prove our main results in this paper.

Theorem 2.1 *Let E be a real uniformly convex Banach space, let K be a nonempty closed convex subset of E, and let $\{T_1, T_2, \ldots, T_m\} : K \to K$ be m asymptotically nonexpansive mappings with $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$ (the set of common fixed points of $\{T_1, T_2, \ldots, T_m\}$); $f : K \to K$ is an L-Lipschitzian mapping. Let the hybrid iteration $\{x_n\}$ be defined by* (1.3), *where* $\{\alpha_n\}$ *and* $\{\lambda_n\}$ *are real sequences in* $[0,1)$, *let* $\{k_n\}$ *be the sequence defined by* (1.1) *satisfying the following conditions*:

(i) $\alpha \le \alpha_n \le \beta$ *for some* $\alpha, \beta \in (0,1)$;
(ii) $\sum_{n=1}^{\infty}(k_n - 1) < \infty$;
(iii) $\sum_{n=1}^{\infty} \lambda_n < \infty$.

Then

(1) $\lim_{n\to\infty} \|x_n - p\|$ *exists for each* $p \in F$,
(2) $\lim_{n\to\infty} \|x_n - T_l x_n\| = 0$, $\forall l = 1, 2, 3, \ldots, m$,
(3) $\{x_n\}$ *converges strongly to a common fixed point of* $\{T_1, T_2, T_3, \ldots, T_m\}$ *if and only if* $\lim_{n\to\infty} d(x_n, F) = 0$.

Proof (1) Since $F = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$, for each $p \in F$, it follows from Proposition 1.1 that

$$\begin{aligned}
\|x_{n+1} - p\| &= \left\| \alpha_n x_n + (1-\alpha_n)\left[\sum_{i=1}^{m} \tau_i T_i^n x_n - \lambda_{n+1}\mu f\left(\sum_{i=1}^{m} \tau_i T_i^n x_n\right)\right] - p \right\| \\
&\le \alpha_n \|x_n - p\| + (1-\alpha_n)\left\| \sum_{i=1}^{m} \tau_i \left(T_i^n x_n - T_i^n p\right) \right\| \\
&\quad + (1-\alpha_n)\lambda_{n+1}\mu \left\| f\left(\sum_{i=1}^{m} \tau_i T_i^n x_n\right) \right\| \\
&\le \alpha_n \|x_n - p\| + (1-\alpha_n) \sum_{i=1}^{m} \tau_i \left\| T_i^n x_n - T_i^n p \right\| \\
&\quad + (1-\alpha_n)\lambda_{n+1}\mu \left\| f\left(\sum_{i=1}^{m} \tau_i T_i^n x_n\right) - f(p) \right\| + (1-\alpha_n)\lambda_{n+1}\mu \left\| f(p) \right\| \\
&\le \alpha_n \|x_n - p\| + (1-\alpha_n) k_n \|x_n - p\| \\
&\quad + (1-\alpha_n)\lambda_{n+1}\mu k_n L \|x_n - p\| + (1-\alpha_n)\lambda_{n+1}\mu \left\| f(p) \right\|.
\end{aligned} \tag{2.1}$$

Since $k_n \to 1$ ($n \to \infty$), we know that $\{k_n\}$ is bounded, and there exists $M_1 \ge 1$ such that $k_n \le M_1$. Let $u_n = k_n - 1$, $\forall n \ge 1$, by condition (ii) we have $\sum_{n=1}^{\infty} u_n < \infty$. Therefore we have

$$\begin{aligned}
\|x_{n+1} - p\| &\le \alpha_n \|x_n - p\| + (1-\alpha_n)(1+u_n)\|x_n - p\| \\
&\quad + (1-\alpha_n)\lambda_{n+1}\mu M_1 L \|x_n - p\| + (1-\alpha_n)\lambda_{n+1}\mu \left\| f(p) \right\| \\
&\le \alpha_n \|x_n - p\| + (1-\alpha_n+u_n)\|x_n - p\| \\
&\quad + \lambda_{n+1}\mu M_1 L \|x_n - p\| + \lambda_{n+1}\mu \left\| f(p) \right\| \\
&\le (1 + u_n + \lambda_{n+1}\mu M_1 L)\|x_n - p\| + \lambda_{n+1}\mu \left\| f(p) \right\|.
\end{aligned} \tag{2.2}$$

Taking $a_n = \|x_n - p\|$, $b_n = u_n + \lambda_{n+1}\mu M_1 L$, $c_n = \lambda_{n+1}\mu\|f(p)\|$ and by using condition (iii) and $\sum_{n=1}^{\infty} u_n < \infty$, it is easy to see that

$$\sum_{n=1}^{\infty} b_n < \infty; \qquad \sum_{n=1}^{\infty} c_n < \infty.$$

It follows from Lemma 1.1 that $\lim_{n\to\infty} \|x_n - p\|$ exists.

(2) Since $\{\|x_n - p\|\}$ is bounded, there exists $M_2 > 0$ such that

$$\|x_n - p\| \le M_2, \quad \forall n \ge 1. \tag{2.3}$$

We can assume that

$$\lim_{n\to\infty} \|x_n - p\| = d, \tag{2.4}$$

where $d \ge 0$ is some number. Since $\{\|x_n - p\|\}$ is a convergent sequence, so $\{x_n\}$ is a bounded sequence in K. Let

$$\sigma_n = \sum_{i=1}^{m} \tau_i T_i^n x_n - \lambda_{n+1}\mu f\left(\sum_{i=1}^{m} \tau_i T_i^n x_n\right),$$

then

$$\|x_{n+1} - p\| = \left\|\alpha_n(x_n - p) + (1-\alpha_n)(\sigma_n - p)\right\|. \tag{2.5}$$

By (2.4) we have that

$$\limsup_{n\to\infty} \|x_n - p\| = d. \tag{2.6}$$

From (2.1) and (2.3) we have

$$\begin{aligned}\left\|f\left(\sum_{i=1}^{m} \tau_i T_i^n x_n\right)\right\| &\le Lk_n\|x_n - p\| + \left\|f(p)\right\| \\ &\le LM_1M_2 + \left\|f(p)\right\|.\end{aligned} \tag{2.7}$$

By condition (iii), $k_n \le M_1$, $\sum_{n=1}^{\infty} u_n < \infty$ and (2.3), (2.4), (2.6), we have that

$$\begin{aligned}\limsup_{n\to\infty} \|\sigma_n - p\| &\le \limsup_{n\to\infty}\left\{\left\|\sum_{i=1}^{m} \tau_i T_i^n x_n - p\right\| + \lambda_{n+1}\mu\left\|f\left(\sum_{i=1}^{m} \tau_i T_i^n x_n\right)\right\|\right\} \\ &\le \limsup_{n\to\infty}\left\{k_n\|x_n - p\| + \lambda_{n+1}\mu\left(LM_1M_2 + \left\|f(p)\right\|\right)\right\} \\ &= \limsup_{n\to\infty}\left\{(1+u_n)\|x_n - p\| + \lambda_{n+1}\mu\left(LM_1M_2 + \left\|f(p)\right\|\right)\right\} \\ &\le \limsup_{n\to\infty}\left\{\|x_n - p\| + u_nM_2 + \lambda_{n+1}\mu\left(LM_1M_2 + \left\|f(p)\right\|\right)\right\} \\ &\le d.\end{aligned} \tag{2.8}$$

Thus from (2.4), (2.5), (2.6), (2.8) and Lemma 1.2 we know that

$$\lim_{n\to\infty} \|\sigma_n - x_n\| = 0. \tag{2.9}$$

By (2.9), we have that

$$\begin{aligned}\|x_{n+1} - x_n\| &= \left\|(\alpha_n - 1)x_n + (1-\alpha_n)\sigma_n\right\| \\ &\le (1-\alpha_n)\|\sigma_n - x_n\| \to 0 \quad (n\to\infty).\end{aligned} \tag{2.10}$$

From (2.10) we obtain that

$$\lim_{n\to\infty} \|x_{n+j} - x_n\| = 0, \quad \forall j = 1, 2, 3, \ldots, m. \tag{2.11}$$

It follows from (2.7) and (2.9) that

$$\begin{aligned}
\lim_{n\to\infty} \left\| x_n - \sum_{i=1}^{m} \tau_i T_i^n x_n \right\| &\le \lim_{n\to\infty} \left(\|x_n - \sigma_n\| + \left\| \sigma_n - \sum_{i=1}^{m} \tau_i T_i^n x_n \right\| \right) \\
&\le \lim_{n\to\infty} \left(\|x_n - \sigma_n\| + \lambda_{n+1}\mu \left\| f\left(\sum_{i=1}^{m} \tau_i T_i^n x_n \right) \right\| \right) \\
&\le \lim_{n\to\infty} \left(\|x_n - \sigma_n\| + \lambda_{n+1}\mu \left(LM_1M_2 + \|f(p)\| \right) \right) \\
&= 0.
\end{aligned} \tag{2.12}$$

Let $\xi_{l,n} = \|x_n - T_l^n x_n\|$, $l \in \{1, 2, 3, \ldots, m\}$, then from (2.12) we have

$$\begin{aligned}
\xi_{l,n} &= \left\| x_n - T_l^n x_n \right\| = \frac{1}{\tau_l} \cdot \tau_l \left\| x_n - T_l^n x_n \right\| \\
&\le \frac{1}{\tau_l} \sum_{i=1}^{m} \tau_i \left\| x_n - T_i^n x_n \right\| = \frac{1}{\tau_l} \left\| x_n - \sum_{i=1}^{m} \tau_i T_i^n x_n \right\| \\
&\to 0 \quad (n \to \infty).
\end{aligned} \tag{2.13}$$

It follows from (2.10) and (2.13) that

$$\begin{aligned}
\|x_n - T_l x_n\| &\le \left\| x_n - T_l^n x_n \right\| + \left\| T_l^n x_n - T_l x_n \right\| \\
&\le \xi_{l,n} + k_1 \left\| T_l^{n-1} x_n - x_n \right\| \\
&\le \xi_{l,n} + k_1 \left(\left\| T_l^{n-1} x_n - T_l^{n-1} x_{n-1} \right\| + \left\| T_l^{n-1} x_{n-1} - x_{n-1} \right\| + \|x_{n-1} - x_n\| \right) \\
&\le \xi_{l,n} + k_1 \left(k_{n-1} \|x_n - x_{n-1}\| + \xi_{l,n-1} + \|x_{n-1} - x_n\| \right) \\
&= \xi_{l,n} + k_1 (1 + k_{n-1}) \|x_n - x_{n-1}\| + \xi_{l,n-1} \\
&\to 0 \quad (n \to \infty).
\end{aligned} \tag{2.14}$$

(3) From (2.2) and (2.3), we have that

$$\|x_{n+1} - p\| \le (1 + b_n)\|x_n - p\| + c_n, \quad \forall n \ge 1, \tag{2.15}$$

where $b_n = u_n + \lambda_{n+1}\mu M_1 L$ and $c_n = \lambda_{n+1}\mu\|f(p)\|$ with $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Hence, we have

$$d(x_n, F) \le (1 + b_n) d(x_{n-1}, F) + c_n, \quad \forall n \ge 1. \tag{2.16}$$

It follows from (2.16) and Lemma 1.1 that the limit $\lim_{n\to\infty} d(x_n, F)$ exists.

If $\{x_n\}$ converges strongly to a common fixed point p of $\{T_1, T_2, T_3, \ldots, T_m\}$, then it follows from (2.3) and Lemma 1.2 that the limit $\lim_{n\to\infty} \|x_n - p\| = 0$. Since $0 \le d(x_n, F) \le \|x_n - p\|$, we know that $\lim_{n\to\infty} d(x_n, F) = 0$, and so $\limsup_{n\to\infty} d(x_n, F) = 0$.

Conversely, suppose $\limsup_{n\to\infty} d(x_n, F) = 0$, then $\lim_{n\to\infty} d(x_n, F) = 0$.

Next we prove that the sequence $\{x_n\}$ is a Cauchy sequence in K. In fact, since $\sum_{n=1}^{\infty} b_n < \infty$, $1 + t \le \exp\{t\}$ for all $t > 0$, from (2.15) we have

$$\|x_n - p\| \le \exp\{b_n\}\|x_{n-1} - p\| + c_n. \tag{2.17}$$

Hence, for any positive integers n, m, from (2.17) it follows that

$$\begin{aligned}
\|x_{n+m} - p\| &\le \exp\{b_{n+m}\}\|x_{n+m-1} - p\| + c_{n+m} \\
&\le \exp\{b_{n+m}\}\bigl[\exp\{b_{n+m-1}\}\|x_{n+m-2} - p\| + c_{n+m-1}\bigr] + c_{n+m} \\
&= \exp\{b_{n+m} + b_{n+m-1}\}\|x_{n+m-2} - p\| + \exp\{b_{n+m}\}c_{n+m-1} + c_{n+m} \\
&\le \cdots \\
&\le \exp\left\{\sum_{i=n}^{n+m} b_i\right\}\|x_n - p\| + \exp\left\{\sum_{i=n+1}^{n+m} b_i\right\}\sum_{i=n+1}^{n+m} c_i \\
&\le W\|x_n - p\| + W\sum_{i=n+1}^{\infty} c_i,
\end{aligned}$$

where $W = \exp\{\sum_{n=1}^{\infty} b_n\} < \infty$.

Since $\lim_{n\to\infty} d(x_n, F) = 0$ and $\sum_{n=1}^{\infty} c_n < \infty$, for any given $\epsilon > 0$, there exists a positive integer n_0 such that

$$d(x_n, F) < \frac{\epsilon}{4(W+1)}, \qquad \sum_{i=n+1}^{\infty} c_i < \frac{\epsilon}{2W}, \quad \forall n \ge n_0.$$

Therefore there exists $p_1 \in F$ such that

$$d(x_n, p_1) < \frac{\epsilon}{2(W+1)}, \quad \forall n \ge n_0.$$

Consequently, for any $n \ge n_0$ and for all $m \ge 1$, we have

$$\begin{aligned}
\|x_{n+m} - x_n\| &\le \|x_{n+m} - p_1\| + \|x_n - p_1\| \\
&< (1 + W)\|x_n - p_1\| + W\sum_{i=n+1}^{\infty} c_i \\
&\le \frac{\epsilon}{2(W+1)}(1 + W) + W \cdot \frac{\epsilon}{2W} \\
&= \epsilon.
\end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in K. By the completeness of K, we can assume that $x_n \to x^* \in K$. Then from (2) and Lemma 1.3 we have $x^* \in F$, and so x^* is a common fixed point of $T_1, T_2, T_3, \ldots, T_m$. This completes the proof of Theorem 2.1. □

Theorem 2.2 *Let E be a real uniformly convex Banach space, let K be a nonempty closed convex subset of E, and let $\{T_1, T_2, T_3, \ldots, T_m\} : K \to K$ be m asymptotically nonexpansive* *mappings with* $F = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$, *and at least there exists* T_l, $1 \leq l \leq m$, *which is semi-compact.* $f : K \to K$ *is an L-Lipschitzian mapping. Let* $\{\alpha_n\}$ *and* $\{\lambda_n\}$ *be real sequences in* $[0,1)$, $\{k_n\}$ *be the sequence defined by* (1.1) *satisfying the following conditions*:

(i) $\alpha \leq \alpha_n \leq \beta$ *for some* $\alpha, \beta \in (0,1)$;

(ii) $\sum_{n=1}^{\infty}(k_n - 1) < \infty$;

(iii) $\sum_{n=1}^{\infty} \lambda_n < \infty$.

Then the hybrid iterative process $\{x_n\}$ *defined by* (1.3) *converges strongly to a common fixed point of* $\{T_1, T_2, T_3, \ldots, T_m\}$ *in* K.

Proof From the proof of Theorem 2.1, $\{x_n\}$ is bounded, and $\lim_{n\to\infty} \|x_n - T_l x_n\| = 0$, $\forall l = 1, 2, 3, \ldots, m$. Especially, we have

$$\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0. \tag{2.18}$$

By the assumption of Theorem 2.2, we may assume that T_1 is semi-compact, without loss of generality. Then it follows from (2.18) that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to $p \in K$, and we have

$$\|p - T_l p\| = \lim_{n_k\to\infty} \|x_{n_k} - T_l x_{n_k}\| = \lim_{n\to\infty} \|x_n - T_l x_n\| = 0, \quad \forall l = 1, 2, 3, \ldots, m.$$

This implies that $p \in F$. In addition, since $\lim_{n\to\infty} \|x_n - p\|$ exists, therefore $\lim_{n\to\infty} \|x_n - p\| = 0$, that is, $\{x_n\}$ converges strongly to a fixed point of $\{T_1, T_2, T_3, \ldots, T_m\}$ in K. This completes the proof of Theorem 2.2. □

Theorem 2.3 *Under the conditions of Theorem* 2.1, *if* E *satisfies Opial's condition*, *then the hybrid iterative process* $\{x_n\}$ *defined by* (1.3) *converges weakly to a common fixed point of* $\{T_1, T_2, T_3, \ldots, T_m\}$ *in* K.

Proof From the proof of Theorem 2.1, we know that $\{x_n\}$ is a bounded sequence in K. Since E is uniformly convex, it must be reflexive, so every bounded subset of E is weakly compact. Therefore, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $x^* \in K$. From (2.14) we have

$$\lim_{j\to\infty} \|x_{n_j} - T_l x_{n_j}\| = 0, \quad \forall l = 1, 2, 3, \ldots, m. \tag{2.19}$$

By Lemma 1.3, we know that $x^* \in F(T_l)$. By the arbitrariness of $l \in \{1, 2, 3, \ldots, m\}$, we have that $x^* \in \bigcap_{l=1}^{m} F(T_i)$.

Suppose that there exists some subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to y^* \in K$ weakly and $y^* \neq x^*$. From Lemma 1.3, $y^* \in F$. By (2.2) we know that $\lim_{n\to\infty} \|x_n - x^*\|$ and $\lim_{n\to\infty} \|x_n - y^*\|$ exist. By the virtue of *Opial's* condition of E, we have

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - x^*\| &= \lim_{j\to\infty} \|x_{n_j} - x^*\| < \lim_{j\to\infty} \|x_{n_j} - y^*\| \\
&= \lim_{n\to\infty} \|x_n - y^*\| = \lim_{k\to\infty} \|x_{n_k} - y^*\| \\
&< \lim_{k\to\infty} \|x_k - x^*\| = \lim_{n\to\infty} \|x_n - x^*\|,
\end{aligned}$$

which is a contraction. Hence $x^* = y^*$. This implies that $\{x_n\}$ converges weakly to a common fixed point of $\{T_1, T_2, T_3, \dots, T_m\}$ in K. This completes the proof of Theorem 2.3. □

Remark 2.1 Theorems 2.1, 2.2 and 2.3 extend the results of [13] and [19] from a nonexpansive mapping to a finite family of asymptotically nonexpansive mappings.

Theorem 2.4 *Let E be a real uniformly convex Banach space, let K be a nonempty closed convex subset of E, and let $\{T_1, T_2, T_3, \dots, T_m\} : K \to K$ be m nonexpansive mappings with $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$; $f : K \to K$ is an L-Lipschitzian mapping. Let a hybrid iterative sequence $\{x_n\}$ be defined by* (1.4), *where $\{\alpha_n\}$ and $\{\lambda_n\}$ are real sequences in* $[0,1)$ *satisfying the following conditions*:

(i) $\alpha \le \alpha_n \le \beta$ *for some* $\alpha, \beta \in (0,1)$;
(ii) $\sum_{n=1}^{\infty} \lambda_n < \infty$.

Then

(1) $\lim_{n\to\infty} \|x_n - p\|$ *exists for each* $p \in F$,
(2) $\lim_{n\to\infty} \|x_n - T_l x_n\| = 0$, $\forall l = 1, 2, 3, \dots, m$,
(3) $\{x_n\}$ *converges strongly to a common fixed point of* $\{T_1, T_2, T_3, \dots, T_m\}$ *if and only if* $\lim_{n\to\infty} d(x_n, F) = 0$.

Theorem 2.5 *Let E be a real uniformly convex Banach space, let K be a nonempty closed convex subset of E, and let $\{T_1, T_2, T_3, \dots, T_m\} : K \to K$ be m nonexpansive mappings with $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$, and at least there exists T_l, $1 \le l \le m$, which is semi-compact. $f : K \to K$ is an L-Lipschitzian mapping. Let $\{\alpha_n\}$ and $\{\lambda_n\}$ be real sequences in* $[0,1)$ *satisfying the following conditions*:

(i) $\alpha \le \alpha_n \le \beta$ *for some* $\alpha, \beta \in (0,1)$;
(ii) $\sum_{n=1}^{\infty} \lambda_n < \infty$.

Then the hybrid iterative process $\{x_n\}$ defined by (1.4) *converges strongly to a common fixed point of $\{T_1, T_2, T_3, \dots, T_m\}$ in K.*

Theorem 2.6 *Under the conditions of Theorem* 2.4, *if E satisfies Opial's condition, then the hybrid iterative process $\{x_n\}$ defined by* (1.4) *converges weakly to a common fixed point of $\{T_1, T_2, T_3, \dots, T_m\}$ in K.*

The proofs of Theorems 2.4, 2.5 and 2.6 can be obtained from those of Theorems 2.1, 2.2 and 2.3 with the condition that $\{T_1, T_2, T_3, \dots, T_m\} : K \to K$ are m nonexpansive mappings.

Remark 2.2 Theorems 2.4, 2.5 and 2.6 extend the results of [13] and [19] from a nonexpansive mapping to a finite family of nonexpansive mappings.

Competing interests
The author declares that they have no competing interests.

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