Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Some fixed point results in ordered G_p-metric spaces

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Abstract

In this paper, first we present some coincidence point results for six mappings satisfying the generalized (ψ, φ)-weakly contractive condition in the framework of partially ordered G_p-metric spaces. Secondly, we consider α-admissible mappings in the setup of G_p-metric spaces. An example is also provided to support our results.

MSC: Primary 47H10; secondary 54H25

Keywords: coincidence point; common fixed point; generalized weak contraction; generalized metric space; partially weakly increasing mapping; altering distance function

1 Introduction and mathematical preliminaries

Recently, Zand and Nezhad [1] have introduced a new generalized metric space, a G_p-metric space, as a generalization of both partial metric spaces [2] and G-metric spaces [3].

We will use the following definition of a G_p-metric space.

Definition 1.1 [4] Let X be a nonempty set. Suppose that a mapping $G_p : X \times X \times X \to \mathbb{R}^+$ satisfies:

(G_p1) $x = y = z$ if $G_p(x,y,z) = G_p(z,z,z) = G_p(y,y,y) = G_p(x,x,x)$;
(G_p2) $G_p(x,x,x) \le G_p(x,x,y) \le G_p(x,y,z)$ for all $x,y,z \in X$ with $z \ne y$;
(G_p3) $G_p(x,y,z) = G_p(p\{x,y,z\})$, where p is any permutation of x, y, z (symmetry in all three variables);
(G_p4) $G_p(x,y,z) \le G_p(x,a,a) + G_p(a,y,z) - G_p(a,a,a)$ for all $x,y,z,a \in X$ (rectangle inequality).

Then G_p is called a G_p-metric and (X, G_p) is called a G_p-metric space.

The G_p-metric G_p is called symmetric if

$$G_p(x,x,y) = G_p(x,y,y) \tag{1}$$

holds for all $x,y \in X$. Otherwise, G_p is an asymmetric G_p-metric.

Remark 1 In [1] (see also [5]), instead of (G_p2), the following condition was used:

(G_p2') $G_p(x,x,x) \le G_p(x,x,y) \le G_p(x,y,z)$ for all $x,y,z \in X$.

However, with this assumption, it is very easy to obtain that (1) holds for all $x, y \in X$, *i.e.*, the respective space is symmetric. On the other hand, there are a lot of examples of non-symmetric G-metric spaces. Hence, the conclusion stated in [1, 5] that each G-metric space is a G_p-metric space (satisfying (G_p2')) does not hold. With our assumption (G_p2), this conclusion holds true.

The following are some easy examples of G_p-metric spaces.

Example 1.1 Let $X = [0, +\infty)$, and let $G_p : X^3 \to \mathbb{R}^+$ be given by $G_p(x, y, z) = \max\{x, y, z\}$. Obviously, (X, G_p) is a symmetric G_p-metric space which is not a G-metric space.

Example 1.2 Let $X = \{0, 1, 2, 3, \ldots\}$. Define $G_p : X^3 \to X$ by

$$G_p(x, y, z) = \begin{cases} x + y + z + 1, & x \neq y \neq z, \\ x + z + 1, & y = z \neq x, \\ y + z + 1, & x = z \neq y, \\ x + z + 1, & x = y \neq z, \\ 1, & x = y = z. \end{cases}$$

It is easy to see that (X, G_p) is a symmetric G_p-metric space.

Example 1.3 [4] Let $X = \{0, 1, 2, 3\}$. Let

$$\begin{aligned} A = \bigl\{&(1,0,0),(0,1,0),(0,0,1),(2,0,0),(0,2,0),(0,0,2),(3,0,0),(0,3,0),(0,0,3), \\ &(1,2,2),(2,1,2),(2,2,1),(2,3,3),(3,2,3),(3,3,2)\bigr\}, \\ B = \bigl\{&(0,1,1),(1,0,1),(1,1,0),(0,2,2),(2,0,2),(2,2,0),(0,3,3),(3,0,3),(3,3,0), \\ &(2,1,1),(1,2,1),(1,1,2),(3,2,2),(2,3,2),(2,2,3)\bigr\}. \end{aligned}$$

Define $G_p : X^3 \to \mathbb{R}^+$ by

$$G(x, y, z) = \begin{cases} 1 & \text{if } x = y = z \neq 2, \\ 0 & \text{if } x = y = z = 2, \\ 2 & \text{if } (x, y, z) \in A, \\ \frac{5}{2} & \text{if } (x, y, z) \in B, \\ 3 & \text{if } x \neq y \neq z. \end{cases}$$

It is easy to see that (X, G_p) is an asymmetric G_p-metric space.

Proposition 1.1 [1] *Every G_p-metric space (X, G_p) defines a metric space (X, d_{G_p}) where*

$$d_{G_p}(x, y) = G_p(x, y, y) + G_p(y, x, x) - G_p(x, x, x) - G_p(y, y, y)$$

for all $x, y \in X$.

Proposition 1.2 [1] *Let X be a G_p-metric space. Then, for each $x, y, z, a \in X$, it follows that*:

(1) $G_p(x,y,z) \le G_p(x,a,a) + G_p(y,a,a) + G_p(z,a,a) - 2G_p(a,a,a)$;

(2) $G_p(x,y,z) \le G_p(x,x,y) + G_p(x,x,z) - G_p(x,x,x)$;

(3) $G_p(x,y,y) \le 2G_p(x,x,y) - G_p(x,x,x)$;

(4) $G_p(x,y,z) \le G_p(x,a,z) + G_p(a,y,z) - G_p(a,a,a)$, $a \neq z$.

Definition 1.2 [1] Let (X, G_p) be a G_p-metric space. Let $\{x_n\}$ be a sequence of points of X.

1. A point $x \in X$ is said to be a limit of the sequence $\{x_n\}$, denoted by $x_n \to x$, if $\lim_{n,m\to\infty} G_p(x, x_n, x_m) = G_p(x,x,x)$.
2. $\{x_n\}$ is said to be a G_p-Cauchy sequence if $\lim_{n,m\to\infty} G_p(x_n, x_m, x_m)$ exists (and is finite).
3. (X, G_p) is said to be G_p-complete if every G_p-Cauchy sequence in X is G_p-convergent to $x \in X$.

Using the above definitions, one can easily prove the following proposition.

Proposition 1.3 [1] *Let (X, G_p) be a G_p-metric space. Then, for any sequence $\{x_n\}$ in X and a point $x \in X$, the following are equivalent*:

(1) *$\{x_n\}$ is G_p-convergent to x.*

(2) *$G_p(x_n, x_n, x) \to G_p(x,x,x)$ as $n \to \infty$.*

(3) *$G_p(x_n, x, x) \to G_p(x,x,x)$ as $n \to \infty$.*

Lemma 1.1 [4] *If G_p is a G_p-metric on X, then the mappings $d_{G_p}, d'_{G_p} : X \times X \to R^+$, given by*

$$d_{G_p}(x,y) = G_p(x,y,y) + G_p(y,x,x) - G_p(x,x,x) - G_p(y,y,y)$$

and

$$d'_{G_p}(x,y) = \max\{G_p(x,y,y) - G_p(x,x,x), G_p(y,x,x) - G_p(y,y,y)\},$$

define equivalent metrics on X.

Proof $\frac{a+b}{2} \le \max\{a,b\} \le a+b$ for all nonnegative real numbers a, b. □

Based on Lemma 2.2 of [6], Parvaneh *et al.* have proved the following essential lemma.

Lemma 1.2 [4] (1) *A sequence $\{x_n\}$ is a G_p-Cauchy sequence in a G_p-metric space (X, G_p) if and only if it is a Cauchy sequence in the metric space (X, d_{G_p}).*

(2) *A G_p-metric space (X, G_p) is G_p-complete if and only if the metric space (X, d_{G_p}) is complete. Moreover, $\lim_{n\to\infty} d_{G_p}(x, x_n) = 0$ if and only if*

$$\begin{aligned}\lim_{n\to\infty} G_p(x, x_n, x_n) &= \lim_{n\to\infty} G_p(x_n, x, x) = \lim_{n,m\to\infty} G_p(x_n, x_n, x_m)\\ &= \lim_{n,m\to\infty} G_p(x_n, x_m, x_m) = G_p(x,x,x).\end{aligned}$$

Lemma 1.3 [4] *Assume that $x_n \to x$ as $n \to \infty$ in a G_p-metric space (X, G_p) such that $G_p(x,x,x) = 0$. Then, for every $y \in X$,*

(i) $\lim_{n\to\infty} G_p(x_n, y, y) = G_p(x, y, y)$,
(ii) $\lim_{n\to\infty} G_p(x_n, x_n, y) = G_p(x, x, y)$.

Lemma 1.4 [4] *Assume that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are three sequences in a G_p-metric space X such that*

$$\lim_{n\to\infty} G_p(x_n, x, x) = \lim_{n\to\infty} G_p(x_n, x_n, x_n) = G_p(x, x, x),$$
$$\lim_{n\to\infty} G_p(y_n, y, y) = \lim_{n\to\infty} G_p(y_n, y_n, y_n) = G_p(y, y, y)$$

and

$$\lim_{n\to\infty} G_p(z_n, z, z) = \lim_{n\to\infty} G_p(z_n, z_n, z_n) = G_p(z, z, z).$$

Then
(i) $\lim_{n\to\infty} G_p(x_n, y_n, z_n) = G_p(x, y, z)$ *and*
(ii) $\lim_{n\to\infty} G_p(x_n, x_n, y) = G_p(x, x, y)$
for every $y, z \in X$.

Lemma 1.5 [5] *Let (X, G_p) be a G_p-metric space. Then*
(A) *If $G_p(x, y, z) = 0$, then $x = y = z$.*
(B) *If $x \neq y$, then $G_p(x, y, y) > 0$.*

Definition 1.3 [1] Let (X_1, G_1) and (X_2, G_2) be two G_p-metric spaces, and let $f : (X_1, G_1) \to (X_2, G_2)$ be a mapping. Then f is said to be G_p-continuous at a point $a \in X_1$ if for a given $\varepsilon > 0$, there exists $\delta > 0$ such that $x, y \in X_1$ and $G_1(a, x, y) < \delta + G_1(a, a, a)$ imply that $G_2(f(a), f(x), f(y)) < \varepsilon + G_2(f(a), f(a), f(a))$. The mapping f is G_p-continuous on X_1 if it is G_p-continuous at all $a \in X_1$.

Proposition 1.4 [1] *Let (X_1, G_1) and (X_2, G_2) be two G_p-metric spaces. Then a mapping $f : X_1 \to X_2$ is G_p-continuous at a point $x \in X_1$ if and only if it is G_p-sequentially continuous at x; that is, whenever $\{x_n\}$ is G_p-convergent to x, $\{f(x_n)\}$ is G_p-convergent to $f(x)$.*

The concept of an altering distance function was introduced by Khan *et al.* [7] as follows.

Definition 1.4 The function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:
1. ψ is continuous and nondecreasing.
2. $\psi(t) = 0$ if and only if $t = 0$.

A self-mapping f on X is called a weak contraction if the following contractive condition is satisfied:

$$d(fx, fy) \leq d(x, y) - \varphi\big(d(x, y)\big),$$

for all $x, y \in X$, where φ is an altering distance function.

The concept of a weakly contractive mapping was introduced by Alber and Guerre-Delabrere [8] in the setup of Hilbert spaces. Rhoades [9] considered this class of mappings in the setup of metric spaces and proved that a weakly contractive mapping defined on a complete metric space has a unique fixed point.

Zhang and Song [10] introduced the concept of a generalized φ-weakly contractive mapping as follows.

Definition 1.5 Self-mappings f and g on a metric space X are called generalized φ-weak contractions if there exists a lower semicontinuous function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that for all $x, y \in X$,

$$d(fx, gy) \leq N(x,y) - \varphi\bigl(N(x,y)\bigr),$$

where

$$N(x,y) = \max\left\{d(x,y), d(x,fx), d(y,gy), \frac{1}{2}\bigl[d(x,gy) + d(y,fx)\bigr]\right\}.$$

Based on the above definition, they proved the following common fixed point result.

Theorem 1.1 [10] *Let (X,d) be a complete metric space. If $f, g : X \to X$ are generalized φ-weakly contractive mappings, then there exists a unique point $u \in X$ such that $u = fu = gu$.*

So far, many authors extended Theorem 1.1 (see [11–13] and [14]). Moreover, Đorić [12] generalized it by the definition of generalized (ψ,φ)-weak contractions.

Definition 1.6 Two mappings $f, g : X \to X$ are called generalized (ψ,φ)-weakly contractive if there exist two maps $\psi, \varphi : [0,\infty) \to [0,\infty)$ such that

$$\psi\bigl(d(fx,gy)\bigr) \leq \psi\bigl(N(x,y)\bigr) - \varphi\bigl(N(x,y)\bigr),$$

for all $x, y \in X$, where N and φ are as in Definition 1.5 and $\psi : [0,\infty) \to [0,\infty)$ is an altering distance function.

Theorem 1.2 [12] *Let (X,d) be a complete metric space, and let $f, g : X \to X$ be generalized (ψ,φ)-weakly contractive self-mappings. Then there exists a unique point $u \in X$ such that $u = fu = gu$.*

Recently, many researchers have focused on different contractive conditions in various metric spaces endowed with a partial order and studied fixed point theory in the so-called bi-structured spaces. For more details on fixed point results, their applications, comparison of different contractive conditions and related results in ordered various metric spaces, we refer the reader to [15–29] and the references mentioned therein.

Let X be a nonempty set and $f : X \to X$ be a given mapping. For every $x \in X$, let $f^{-1}(x) = \{u \in X : fu = x\}$.

Definition 1.7 [24] Let $(X,\preceq)$ be a partially ordered set, and let $f, g, h : X \to X$ be given mappings such that $fX \subseteq hX$ and $gX \subseteq hX$. We say that f and g are weakly increasing with respect to h if for all $x \in X$, we have

$$fx \preceq gy \quad \text{for all } y \in h^{-1}(fx)$$

and

$$gx \preceq fy \quad \text{for all } y \in h^{-1}(gx).$$

If $f = g$, we say that f is weakly increasing with respect to h.

If $h = I$ (the identity mapping on X), then the above definition reduces to that of a weakly increasing mapping [30] (see also [24, 31]).

Definition 1.8 A partially ordered G_p-metric space $(X, \preceq, G_p)$ is said to have the sequential limit comparison property if for every nondecreasing sequence (nonincreasing sequence) $\{x_n\}$ in X, $x_n \to x$ implies that $x_n \preceq x$ ($x \preceq x_n$).

The aim of this paper is to prove some coincidence and common fixed point theorems for weakly (ψ, φ)-contractive mappings in partially ordered G_p-metric spaces.

2 Main results

Let $(X, \preceq, G_p)$ be an ordered G_p-metric space and $f, g, h, R, S, T : X \to X$ be six self-mappings. Throughout this paper, unless otherwise stated, for all $x, y, z \in X$, let

$$\begin{aligned} M(x,y,z) = \max\Big\{ & G_p(Tx, Ry, Sz), \\ & G_p(Tx, fx, fx), G_p(Ry, gy, gy), G_p(Sz, hz, hz), \\ & \frac{G_p(Tx, Tx, fx) + G_p(Ry, Ry, gy) + G_p(Sz, Sz, hz)}{3} \Big\}. \end{aligned}$$

Theorem 2.1 *Let $(X, \preceq, G_p)$ be a partially ordered G_p-metric space with the sequential limit comparison property. Let $f, g, h, R, S, T : X \to X$ be six mappings such that $f(X) \subseteq R(X)$, $g(X) \subseteq S(X)$ and $h(X) \subseteq T(X)$, and RX, SX and TX are G_p-complete subsets of X. Suppose that for comparable elements $Tx, Ry, Sz \in X$, we have*

$$\psi\big(2G_p(fx, gy, hz)\big) \le \psi\big(M(x,y,z)\big) - \varphi\big(M(x,y,z)\big), \tag{2}$$

where $\psi, \varphi : [0, \infty) \to [0, \infty)$ are altering distance functions. Then the pairs (f, T), (g, R) and (h, S) have a coincidence point z^* in X provided that the pairs (f, T), (g, R) and (h, S) are weakly compatible and the pairs (f, g), (g, h) and (h, f) are partially weakly increasing with respect to R, S and T, respectively. Moreover, if Rz^*, Sz^* and Tz^* are comparable, then $z^* \in X$ is a coincidence point of f, g, h, R, S and T.

Proof Let x_0 be an arbitrary point of X. Choose $x_1 \in X$ such that $fx_0 = Rx_1$, $x_2 \in X$ such that $gx_1 = Sx_2$ and $x_3 \in X$ such that $hx_2 = Tx_3$. This can be done as $f(X) \subseteq R(X)$, $g(X) \subseteq S(X)$ and $h(X) \subseteq T(X)$.

Continuing this way, construct a sequence $\{z_n\}$ defined by $z_{3n+1} = Rx_{3n+1} = fx_{3n}$, $z_{3n+2} = Sx_{3n+2} = gx_{3n+1}$ and $z_{3n+3} = Tx_{3n+3} = hx_{3n+2}$ for all $n \ge 0$. The sequence $\{z_n\}$ in X is said to be a Jungck-type iterative sequence with initial guess x_0.

As $x_1 \in R^{-1}(fx_0)$, $x_2 \in S^{-1}(gx_1)$ and $x_3 \in T^{-1}(hx_2)$ and the pairs (f,g), (g,h) and (h,f) are partially weakly increasing with respect to R, S and T, respectively, we have

$$Rx_1 = fx_0 \preceq gx_1 = Sx_2 \preceq hx_2 = Tx_3 \preceq fx_3 = Rx_4.$$

Continuing this process, we obtain $Rx_{3n+1} \preceq Sx_{3n+2} \preceq Tx_{3n+3}$ for all $n \geq 0$.

We will complete the proof in three steps.

Step I. We will prove that $\{z_n\}$ is a G_p-Cauchy sequence. First, we show that $\lim_{k\to\infty} G_p(z_k, z_{k+1}, z_{k+2}) = 0$.

Define $G_{p_k} = G_p(z_k, z_{k+1}, z_{k+2})$. Suppose $G_{p_{k_0}} = 0$ for some k_0. Then $z_{k_0} = z_{k_0+1} = z_{k_0+2}$. In the case that $k_0 = 3n$, then $z_{3n} = z_{3n+1} = z_{3n+2}$ gives $z_{3n+1} = z_{3n+2} = z_{3n+3}$. Indeed,

$$\begin{aligned}\psi\bigl(2G_p(z_{3n+1}, z_{3n+2}, z_{3n+3})\bigr) &= \psi\bigl(2G_p(fx_{3n}, gx_{3n+1}, hx_{3n+2})\bigr)\\ &\leq \psi\bigl(M(x_{3n}, x_{3n+1}, x_{3n+2})\bigr) - \varphi\bigl(M(x_{3n}, x_{3n+1}, x_{3n+2})\bigr),\end{aligned}$$

where

$$\begin{aligned}&M(x_{3n}, x_{3n+1}, x_{3n+2})\\ &\quad= \max\Biggl\{G_p(Tx_{3n}, Rx_{3n+1}, Sx_{3n+2}), G_p(Tx_{3n}, fx_{3n}, fx_{3n}),\\ &\qquad G_p(Rx_{3n+1}, gx_{3n+1}, gx_{3n+1}), G_p(Sx_{3n+2}, hx_{3n+2}, hx_{3n+2}),\\ &\qquad \frac{G_p(Tx_{3n}, Tx_{3n}, fx_{3n}) + G_p(Rx_{3n+1}, Rx_{3n+1}, gx_{3n+1}) + G_p(Sx_{3n+2}, Sx_{3n+2}, hx_{3n+2})}{3}\Biggr\}\\ &\quad= \max\Biggl\{G_p(z_{3n}, z_{3n+1}, z_{3n+2}), G_p(z_{3n}, z_{3n+1}, z_{3n+1}),\\ &\qquad G_p(z_{3n+1}, z_{3n+2}, z_{3n+2}), G_p(z_{3n+2}, z_{3n+3}, z_{3n+3}),\\ &\qquad \frac{G_p(z_{3n}, z_{3n}, z_{3n+1}) + G_p(z_{3n+1}, z_{3n+1}, z_{3n+2}) + G_p(z_{3n+2}, z_{3n+2}, z_{3n+3})}{3}\Biggr\}\\ &\quad= \max\Biggl\{0, 0, 0, G_p(z_{3n+2}, z_{3n+3}, z_{3n+3}), \frac{0 + 0 + G_p(z_{3n+2}, z_{3n+2}, z_{3n+3})}{3}\Biggr\}\\ &\quad= G_p(z_{3n+2}, z_{3n+3}, z_{3n+3})\\ &\quad\leq 2G_p(z_{3n+2}, z_{3n+2}, z_{3n+3})\\ &\quad= 2G_p(z_{3n+1}, z_{3n+2}, z_{3n+3}).\end{aligned}$$

Thus

$$\psi\bigl(2G_p(z_{3n+1}, z_{3n+2}, z_{3n+3})\bigr) \leq \psi\bigl(2G_p(z_{3n+1}, z_{3n+2}, z_{3n+3})\bigr) - \varphi\bigl(G_p(z_{3n+2}, z_{3n+3}, z_{3n+3})\bigr)$$

implies that $\varphi(G_p(z_{3n+2}, z_{3n+3}, z_{3n+3})) = 0$, that is, $z_{3n+1} = z_{3n+2} = z_{3n+3}$. Similarly, if $k_0 = 3n + 1$, then $z_{3n+1} = z_{3n+2} = z_{3n+3}$ gives $z_{3n+2} = z_{3n+3} = z_{3n+4}$. Also, if $k_0 = 3n + 2$, then $z_{3n+2} = z_{3n+3} = z_{3n+4}$ implies that $z_{3n+3} = z_{3n+4} = z_{3n+5}$. Consequently, the sequence $\{z_k\}$ becomes constant for $k \geq k_0$, hence $\{z_k\}$ is G_p-Cauchy.

Suppose that

$$z_k \neq z_{k+1} \neq z_{k+2} \tag{3}$$

for each k. We now claim that the following inequality holds:

$$G_p(z_{k+1}, z_{k+2}, z_{k+3}) \leq G_p(z_k, z_{k+1}, z_{k+2}) = M(x_k, x_{k+1}, x_{k+2}) \tag{4}$$

for each $k = 1, 2, 3, \ldots$.

Let $k = 3n$ and for $n \geq 0$, $G_p(z_{3n+1}, z_{3n+2}, z_{3n+3}) > G_p(z_{3n}, z_{3n+1}, z_{3n+2}) > 0$. Then, as $Tx_{3n} \preceq Rx_{3n+1} \preceq Sx_{3n+2}$, using (2) we obtain that

$$\begin{aligned} \psi\bigl(G_p(z_{3n+1}, z_{3n+2}, z_{3n+3})\bigr) &\leq \psi\bigl(2G_p(z_{3n+1}, z_{3n+2}, z_{3n+3})\bigr) \\ &= \psi\bigl(2G_p(fx_{3n}, gx_{3n+1}, hx_{3n+2})\bigr) \\ &\leq \psi\bigl(M(x_{3n}, x_{3n+1}, x_{3n+2})\bigr) - \varphi\bigl(M(x_{3n}, x_{3n+1}, x_{3n+2})\bigr), \end{aligned} \tag{5}$$

where

$$\begin{aligned} &M(x_{3n}, x_{3n+1}, x_{3n+2}) \\ &\quad = \max\biggl\{G_p(Tx_{3n}, Rx_{3n+1}, Sx_{3n+2}), \\ &\qquad G_p(Tx_{3n}, fx_{3n}, fx_{3n}), G_p(Rx_{3n+1}, gx_{3n+1}, gx_{3n+1}), G_p(Sx_{3n+2}, hx_{3n+2}, hx_{3n+2}), \\ &\qquad \frac{G_p(Tx_{3n}, Tx_{3n}, fx_{3n}) + G_p(Rx_{3n+1}, Rx_{3n+1}, gx_{3n+1}) + G_p(Sx_{3n+2}, Sx_{3n+2}, hx_{3n+2})}{3}\biggr\} \\ &\quad = \max\biggl\{G_p(z_{3n}, z_{3n+1}, z_{3n+2}), \\ &\qquad G_p(z_{3n}, z_{3n+1}, z_{3n+1}), G_p(z_{3n+1}, z_{3n+2}, z_{3n+2}), G_p(z_{3n+2}, z_{3n+3}, z_{3n+3}), \\ &\qquad \frac{G_p(z_{3n}, z_{3n}, z_{3n+1}) + G_p(z_{3n+1}, z_{3n+1}, z_{3n+2}) + G_p(z_{3n+2}, z_{3n+2}, z_{3n+3})}{3}\biggr\} \\ &\quad \leq \max\biggl\{G_p(z_{3n}, z_{3n+1}, z_{3n+2}), G_p(z_{3n+1}, z_{3n+2}, z_{3n+3}), \\ &\qquad \frac{2G_p(z_{3n}, z_{3n+1}, z_{3n+2}) + G_p(z_{3n+1}, z_{3n+2}, z_{3n+3})}{3}\biggr\} \\ &\quad = G_p(z_{3n+1}, z_{3n+2}, z_{3n+3}). \end{aligned}$$

Hence (5) implies that

$$\psi\bigl(G_p(z_{3n+1}, z_{3n+2}, z_{3n+3})\bigr) \leq \psi\bigl(G_p(z_{3n+1}, z_{3n+2}, z_{3n+3})\bigr) - \varphi\bigl(M(x_{3n}, x_{3n+1}, x_{3n+2})\bigr),$$

which is possible only if $M(x_{3n}, x_{3n+1}, x_{3n+2}) = 0$, that is, $G_p(z_{3n}, z_{3n+1}, z_{3n+2}) = 0$. A contradiction to (3). Hence, $G_p(z_{3n+1}, z_{3n+2}, z_{3n+3}) \leq G_p(z_{3n}, z_{3n+1}, z_{3n+2})$ and

$$M(x_{3n}, x_{3n+1}, x_{3n+2}) = G_p(z_{3n}, z_{3n+1}, z_{3n+2}).$$

Therefore, (4) is proved for $k = 3n$.

Similarly, it can be shown that

$$G_p(z_{3n+2}, z_{3n+3}, z_{3n+4}) \leq G_p(z_{3n+1}, z_{3n+2}, z_{3n+3}) = M(x_{3n+1}, x_{3n+2}, x_{3n+3}) \tag{6}$$

and

$$G_p(z_{3n+3}, z_{3n+4}, z_{3n+5}) \leq G_p(z_{3n+2}, z_{3n+3}, z_{3n+4}) = M(x_{3n+2}, x_{3n+3}, x_{3n+4}). \tag{7}$$

Hence, $\{G_p(z_k, z_{k+1}, z_{k+2})\}$ is a nonincreasing sequence of nonnegative real numbers. Therefore, there is $r \geq 0$ such that

$$\lim_{k\to\infty} G_p(z_k, z_{k+1}, z_{k+2}) = r. \tag{8}$$

Since

$$G_p(z_{k+1}, z_{k+2}, z_{k+3}) \leq M(x_k, x_{k+1}, x_{k+2}) \leq G_p(z_k, z_{k+1}, z_{k+2}), \tag{9}$$

taking the limit as $k \to \infty$ in (9), we obtain

$$\lim_{k\to\infty} M(x_k, x_{k+1}, x_{k+2}) = r. \tag{10}$$

Taking the limit as $n \to \infty$ in (5), using (8), (10) and the continuity of ψ and φ, we have $\psi(r) \leq \psi(r) - \varphi(r)$. Therefore, $\varphi(r) = 0$. Hence

$$\lim_{k\to\infty} G_p(z_k, z_{k+1}, z_{k+2}) = 0 \tag{11}$$

from our assumptions about φ. Also, from Definition 1.1, part (G_p2), we have

$$\lim_{k\to\infty} G_p(z_k, z_{k+1}, z_{k+1}) = 0, \tag{12}$$

and since $G_p(x, y, y) \leq 2G_p(x, x, y)$ for all $x, y \in X$, we have

$$\lim_{k\to\infty} G_p(z_k, z_k, z_{k+1}) = 0. \tag{13}$$

Step II. We now show that $\{z_n\}$ is a G_p-Cauchy sequence in X. Therefore, we will show that

$$\lim_{m,n\to\infty} G_p(z_m, z_n, z_n) = 0.$$

Because of (11), (12) and (13), it is sufficient to show that

$$\lim_{m,n\to\infty} G_p(z_{3m}, z_{3n}, z_{3n}) = 0,$$

i.e., we prove that $\{z_{3n}\}$ is G_p-Cauchy.

Suppose the opposite. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{z_{3m(k)}\}$ and $\{z_{3n(k)}\}$ of $\{z_{3n}\}$ such that $n(k) > m(k) \geq k$ and

$$G_p(z_{3m(k)}, z_{3n(k)}, z_{3n(k)}) \geq \varepsilon, \tag{14}$$

and $n(k)$ is the smallest number such that the above statement holds; *i.e.*,

$$G_p(z_{3m(k)}, z_{3n(k)-3}, z_{3n(k)-3}) < \varepsilon. \tag{15}$$

From the rectangle inequality and (15), we have

$$\begin{aligned} &G_p(z_{3m(k)}, z_{3n(k)}, z_{3n(k)}) \\ &\quad \leq G_p(z_{3m(k)}, z_{3n(k)-3}, z_{3n(k)-3}) + G_p(z_{3n(k)-3}, z_{3n(k)}, z_{3n(k)}) \\ &\quad < \varepsilon + G_p(z_{3n(k)-3}, z_{3n(k)}, z_{3n(k)}) \\ &\quad < \varepsilon + G_p(z_{3n(k)-3}, z_{3n(k)-2}, z_{3n(k)-2}) + G_p(z_{3n(k)-2}, z_{3n(k)-1}, z_{3n(k)-1}) \\ &\qquad + G_p(z_{3n(k)-1}, z_{3n(k)}, z_{3n(k)}). \end{aligned} \tag{16}$$

Taking limit as $k \to \infty$ in (16), from (12) and (14) we obtain that

$$\lim_{k\to\infty} G_p(z_{3m(k)}, z_{3n(k)}, z_{3n(k)}) = \varepsilon. \tag{17}$$

Using the rectangle inequality and (G_p2), we have

$$\begin{aligned} &G_p(z_{3m(k)}, z_{3n(k)+1}, z_{3n(k)+2}) \\ &\quad \leq G_p(z_{3m(k)}, z_{3n(k)}, z_{3n(k)}) + G_p(z_{3n(k)}, z_{3n(k)+1}, z_{3n(k)+2}) \\ &\quad \leq G_p(z_{3m(k)}, z_{3n(k)+1}, z_{3n(k)+1}) + G_p(z_{3n(k)+1}, z_{3n(k)}, z_{3n(k)}) \\ &\qquad + G_p(z_{3n(k)}, z_{3n(k)+1}, z_{3n(k)+2}) \\ &\quad \leq G_p(z_{3m(k)}, z_{3n(k)+1}, z_{3n(k)+2}) + G_p(z_{3n(k)+1}, z_{3n(k)+2}, z_{3n(k)+2}) \\ &\qquad + G_p(z_{3n(k)+1}, z_{3n(k)}, z_{3n(k)}) + G_p(z_{3n(k)}, z_{3n(k)+1}, z_{3n(k)+2}). \end{aligned} \tag{18}$$

Taking limit as $k \to \infty$, we have

$$\lim_{k\to\infty} G_p(z_{3m(k)}, z_{3n(k)+1}, z_{3n(k)+2}) \leq \varepsilon \leq \lim_{k\to\infty} G_p(z_{3m(k)}, z_{3n(k)+1}, z_{3n(k)+2}).$$

Finally,

$$\begin{aligned} &G_p(z_{3m(k)+1}, z_{3n(k)+2}, z_{3n(k)+3}) \\ &\quad \leq G_p(z_{3m(k)+1}, z_{3m(k)}, z_{3m(k)}) + G_p(z_{3m(k)}, z_{3n(k)+2}, z_{3n(k)+3}) \\ &\quad \leq G_p(z_{3m(k)+1}, z_{3m(k)}, z_{3m(k)}) + G_p(z_{3m(k)}, z_{3n(k)}, z_{3n(k)}) \\ &\qquad + G_p(z_{3n(k)}, z_{3n(k)+2}, z_{3n(k)+3}). \end{aligned} \tag{19}$$

Taking limit as $k \to \infty$ and using (17), we have

$$\lim_{k\to\infty} G_p(z_{3m(k)+1}, z_{3n(k)+2}, z_{3n(k)+3}) \le \varepsilon.$$

Consider,

$$\begin{aligned}
&G_p(z_{3m(k)}, z_{3n(k)+1}, z_{3n(k)+2}) \\
&\quad \le G_p(z_{3m(k)}, z_{3m(k)+1}, z_{3m(k)+1}) + G_p(z_{3m(k)+1}, z_{3n(k)+1}, z_{3n(k)+2}) \\
&\quad \le G_p(z_{3m(k)}, z_{3m(k)+1}, z_{3m(k)+1}) + G_p(z_{3m(k)+1}, z_{3n(k)+3}, z_{3n(k)+3}) \\
&\qquad + G_p(z_{3n(k)+3}, z_{3n(k)+1}, z_{3n(k)+2}) \\
&\quad \le G_p(z_{3m(k)}, z_{3m(k)+1}, z_{3m(k)+1}) + G_p(z_{3m(k)+1}, z_{3n(k)+2}, z_{3n(k)+3}) \\
&\qquad + G_p(z_{3n(k)+1}, z_{3n(k)+2}, z_{3n(k)+3}).
\end{aligned} \tag{20}$$

Taking limit as $k \to \infty$ and using (11), (12) and (13), we have

$$\varepsilon \le \lim_{k\to\infty} G_p(z_{3m(k)+1}, z_{3n(k)+2}, z_{3n(k)+3}).$$

Therefore,

$$\lim_{k\to\infty} G_p(z_{3m(k)+1}, z_{3n(k)+2}, z_{3n(k)+3}) = \varepsilon. \tag{21}$$

As $Tx_{m(k)} \preceq Rx_{n(k)+1} \preceq Sx_{n(k)+2}$, so from (2) we have

$$\begin{aligned}
&\psi\bigl(G_p(z_{3m(k)+1}, z_{3n(k)+2}, z_{3n(k)+3})\bigr) \\
&\quad = \psi\bigl(G_p(fx_{3m(k)}, gx_{3n(k)+1}, hx_{3n(k)+2})\bigr) \\
&\quad \le \psi\bigl(M(x_{3m(k)}, x_{3n(k)+1}, x_{3n(k)+2})\bigr) - \varphi\bigl(M(x_{3m(k)}, x_{3n(k)+1}, x_{3n(k)+2})\bigr),
\end{aligned} \tag{22}$$

where

$$\begin{aligned}
&M(x_{3m(k)}, x_{3n(k)+1}, x_{3n(k)+2}) \\
&\quad = \max\Biggl\{ G_p(Tx_{3m(k)}, Rx_{3n(k)+1}, Sx_{3n(k)+2}), G_p(Tx_{3m(k)}, fx_{3m(k)}, fx_{3m(k)}), \\
&\qquad G_p(Rx_{3n(k)+1}, gx_{3n(k)+1}, gx_{3n(k)+1}), G_p(Sx_{3n(k)+2}, hx_{3n(k)+2}, hx_{3n(k)+2}), \\
&\qquad \frac{\begin{array}{c} G_p(Tx_{3m(k)}, Tx_{3m(k)}, fx_{3m(k)}) + G_p(Rx_{3n(k)+1}, Rx_{3n(k)+1}, gx_{3n(k)+1}) \\ + G_p(Sx_{3n(k)+2}, Sx_{3n(k)+2}, hx_{3n(k)+2}) \end{array}}{3} \Biggr\} \\
&\quad = \max\Biggl\{ G_p(z_{3m(k)}, z_{3n(k)+1}, z_{3n(k)+2}), G_p(z_{3m(k)}, z_{3m(k)+1}, z_{3m(k)+1}), \\
&\qquad G_p(z_{3n(k)+1}, z_{3n(k)+2}, z_{3n(k)+2}), G_p(z_{3n(k)+2}, z_{3n(k)+3}, z_{3n(k)+3}), \\
&\qquad \frac{\begin{array}{c} G_p(z_{3m(k)}, z_{3m(k)}, z_{3m(k)+1}) + G_p(z_{3n(k)+1}, z_{3n(k)+1}, z_{3n(k)+2}) \\ + G_p(z_{3n(k)+2}, z_{3n(k)+2}, z_{3n(k)+3}) \end{array}}{3} \Biggr\}.
\end{aligned}$$

Taking limit as $k \to \infty$ and using (12), (13), (17), (21) in (22), we have

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \varphi(\varepsilon) < \psi(\varepsilon),$$

a contradiction. Hence, $\{z_n\}$ is a G_p-Cauchy sequence.

Step III. We will show that f, g, h, R, S and T have a coincidence point.

Since $\{z_n\}$ is a G_p-Cauchy sequence in the complete G_p-metric space X, from Lemma 1.2, $\{z_n\}$ is a Cauchy sequence in the metric space (X, d_{G_p}). Completeness of (X, G_p) yields that (X, d_{G_p}) is also complete. Then there exists $z^* \in X$ such that

$$\lim_{n\to\infty} d_{G_p}\bigl(z_n, z^*\bigr) = 0. \tag{23}$$

Now, since $\lim_{m,n\to\infty} G_p(z_m, z_n, z_n) = 0$, (23) and part (2) of Lemma 1.2 yield that $G_p(z^*, z^*, z^*) = 0$.

Since $R(X)$ is G_p-complete and $\{z_{3n+1}\} \subseteq R(X)$, there exists $u \in X$ such that $z^* = Ru$ and

$$\begin{aligned} &\lim_{n\to\infty} G_p(z_{3n+1}, z_{3n+1}, Ru) \\ &\quad = \lim_{n\to\infty} G_p(Rx_{3n+1}, Rx_{3n+1}, Ru) = \lim_{n\to\infty} G_p(fx_{3n}, fx_{3n}, Ru) = G(Ru, Ru, Ru) = 0. \end{aligned} \tag{24}$$

By similar arguments, there exist $v, w \in X$ such that $z^* = Sv = Tw$ and

$$\begin{aligned} &\lim_{n\to\infty} G_p\bigl(z_{3n+2}, z_{3n+2}, z^*\bigr) \\ &\quad = \lim_{n\to\infty} G_p\bigl(Sx_{3n+2}, Sx_{3n+2}, z^*\bigr) = \lim_{n\to\infty} G_p\bigl(gx_{3n+1}, gx_{3n+1}, z^*\bigr) = G\bigl(z^*, z^*, z^*\bigr) = 0 \end{aligned} \tag{25}$$

and

$$\begin{aligned} &\lim_{n\to\infty} G_p\bigl(z_{3n+3}, z_{3n+3}, z^*\bigr) \\ &\quad = \lim_{n\to\infty} G_p\bigl(Tx_{3n+3}, Tx_{3n+3}, z^*\bigr) = \lim_{n\to\infty} G_p\bigl(hx_{3n+2}, hx_{3n+2}, z^*\bigr) = G\bigl(z^*, z^*, z^*\bigr) = 0. \end{aligned} \tag{26}$$

Now, we prove that w is a coincidence point of f and T.

Since $Sx_{3n+2} \to z^* = Tw = Ru$ as $n \to \infty$, so $Sx_{3n+2} \preceq Tw = Ru$. Therefore, from (2), we have

$$\psi\bigl(G_p(fw, gu, hx_{3n+2})\bigr) \leq \psi\bigl(M(w, u, x_{3n+2})\bigr) - \varphi\bigl(M(w, u, x_{3n+2})\bigr), \tag{27}$$

where

$$\begin{aligned} &M(w, u, x_{3n+2}) \\ &\quad = \max\biggl\{G_p(Tw, Ru, Sx_{3n+2}), G(Tw, fw, fw), \\ &\qquad G_p(Ru, gu, gu), G(Sx_{3n+2}, hx_{3n+2}, hx_{3n+2}), \\ &\qquad \frac{G_p(Tw, Tw, fw) + G(Ru, Ru, gu) + G_p(Sx_{3n+2}, Sx_{3n+2}, hx_{3n+2})}{3}\biggr\}. \end{aligned}$$

Taking limit as $n \to \infty$ in (27), as $G(z^*, z^*, z^*) = 0$, from Lemma 1.3, we obtain that

$$\begin{aligned}
&\psi\bigl(G_p\bigl(fw, gu, z^*\bigr)\bigr) \\
&\quad \le \psi\bigl(G_p\bigl(fw, gu, z^*\bigr)\bigr) \\
&\qquad - \varphi\left(\max\left\{G_p\bigl(z^*, fw, fw\bigr), G_p\bigl(z^*, gu, gu\bigr), \frac{G_p(z^*, z^*, fw) + G_p(z^*, z^*, gu)}{3}\right\}\right),
\end{aligned}$$

which implies that $gu = fw = z^* = Tw = Ru$.

As f and T are weakly compatible, we have $fz^* = fTw = Tfw = Tz^*$. Thus z^* is a coincidence point of f and T.

Similarly it can be shown that z^* is a coincidence point of the pairs (g, R) and (h, S).

Now, let Rz^*, Sz^* and Tz^* be comparable. By (2) we have

$$\psi\bigl(G_p\bigl(fz^*, gz^*, hz^*\bigr)\bigr) \le \psi\bigl(M\bigl(z^*, z^*, z^*\bigr)\bigr) - \varphi\bigl(M\bigl(z^*, z^*, z^*\bigr)\bigr), \tag{28}$$

where

$$\begin{aligned}
M\bigl(z^*, z^*, z^*\bigr) = \max\Bigl\{&G_p\bigl(Tz^*, Rz^*, Sz^*\bigr), \\
&G_p\bigl(Tz^*, fz^*, fz^*\bigr), G_p\bigl(Rz^*, gz^*, gz^*\bigr), G_p\bigl(Sz^*, hz^*, hz^*\bigr), \\
&\frac{G_p(Tz^*, Tz^*, fz^*) + G_p(Rz^*, Rz^*, gz^*) + G_p(Sz^*, Sz^*, hz^*)}{3}\Bigr\} \\
= {}&G_p\bigl(Tz^*, Rz^*, Sz^*\bigr) = G_p\bigl(fz^*, gz^*, hz^*\bigr).
\end{aligned}$$

Hence (28) gives

$$\psi\bigl(G_p\bigl(fz^*, gz^*, hz^*\bigr)\bigr) \le \psi\bigl(G_p\bigl(fz^*, gz^*, hz^*\bigr)\bigr) - \varphi\bigl(G_p\bigl(fz^*, gz^*, hz^*\bigr)\bigr) = 0.$$

Therefore $fz^* = gz^* = hz^* = Tz^* = Rz^* = Sz^*$. □

Theorem 2.2 *Let $(X, \preceq, G_p)$ be a partially ordered complete G_p-metric space. Let $f, g, h : X \to X$ be three mappings. Suppose that for every three comparable elements $x, y, z \in X$, we have*

$$\psi\bigl(2G_p(fx, gy, hz)\bigr) \le \psi\bigl(M(x, y, z)\bigr) - \varphi\bigl(M(x, y, z)\bigr), \tag{29}$$

where

$$\begin{aligned}
M(x, y, z) = \max\Bigl\{&G_p(x, y, z), \\
&G_p(x, fx, fx), G_p(y, gy, gy), G_p(z, hz, hz), \\
&\frac{G_p(x, x, fx) + G_p(y, y, gy) + G_p(z, z, hz)}{3}\Bigr\}
\end{aligned}$$

and $\psi, \varphi : [0, \infty) \to [0, \infty)$ are altering distance functions. Let f, g, h be continuous and the pairs (f, g), (g, h) and (h, f) be partially weakly increasing. Then f, g and h have a common fixed point z^* in X.

Proof Let x_0 be an arbitrary point and $x_{3n+1} = fx_{3n}$, $x_{3n+2} = gx_{3n+1}$ and $x_{3n+3} = hx_{3n+2}$ for all $n \geq 0$.

Following the proof of the previous theorem, we can show that there exists $z^* \in X$ such that

$$G_p(z^*, z^*, z^*) = 0 \tag{30}$$

and

$$\lim_{n\to\infty} G_p(x_{3n}, x_{3n}, z^*) = 0. \tag{31}$$

Continuity of f yields that

$$\lim_{n\to\infty} G_p(fx_{3n}, fx_{3n}, fz^*) = G_p(fz^*, fz^*, fz^*). \tag{32}$$

By the rectangle inequality, we have

$$G_p(fz^*, z^*, z^*) \leq G_p(fz^*, fx_{3n}, fx_{3n}) + G_p(x_{3n+1}, z^*, z^*) \tag{33}$$

and

$$G_p(fz^*, fz^*, z^*) \leq G_p(z^*, fx_{3n}, fx_{3n}) + G_p(fx_{3n}, fz^*, fz^*). \tag{34}$$

Taking limit as $n \to \infty$ in (33) and (34), from (30) we obtain

$$G_p(fz^*, z^*, z^*) \leq G_p(fz^*, fz^*, fz^*)$$

and

$$G_p(fz^*, fz^*, z^*) \leq G_p(fz^*, fz^*, fz^*).$$

Similar inequalities are obtained for g and h.

On the other hand, as $z^* \preceq z^* \preceq z^*$, using (29) we obtain that

$$\begin{aligned} \psi\big(G_p(fz^*, gz^*, hz^*)\big) &\leq \psi\big(2G_p(fz^*, gz^*, hz^*)\big) \\ &\leq \psi\big(M(z^*, z^*, z^*)\big) - \varphi\big(M(z^*, z^*, z^*)\big), \end{aligned} \tag{35}$$

where

$$\begin{aligned} M(z^*, z^*, z^*) = \max\bigg\{ & G_p(z^*, z^*, z^*), \\ & G_p(z^*, fz^*, fz^*), G_p(z^*, gz^*, gz^*), G_p(z^*, hz^*, hz^*), \\ & \frac{G_p(z^*, z^*, fz^*) + G_p(z^*, z^*, gz^*) + G_p(z^*, z^*, hz^*)}{3} \bigg\} \\ \leq \max\big\{ & G_p(fz^*, fz^*, fz^*), G_p(gz^*, gz^*, gz^*), G_p(hz^*, hz^*, hz^*)\big\}. \end{aligned} \tag{36}$$

We consider three cases as follows:

1. $fz^* = gz^* = hz^*$.
2. $fz^* \neq gz^* \neq hz^*$.
3. a. $fz^* = gz^* \neq hz^*$, or b. $fz^* \neq gz^* = hz^*$.

For case 1, by (36), $M(z^*,z^*,z^*) \leq G_p(fz^*,gz^*,hz^*)$.

For case 2, by (G_p2), $M(z^*,z^*,z^*) \leq G_p(fz^*,gz^*,hz^*)$.

Now, from (35),

$$\psi\bigl(G_p\bigl(fz^*,gz^*,hz^*\bigr)\bigr) \leq \psi\bigl(G_p\bigl(fz^*,gz^*,hz^*\bigr)\bigr) - \varphi\bigl(M\bigl(z^*,z^*,z^*\bigr)\bigr), \tag{37}$$

hence $M(z^*,z^*,z^*) = 0$. Therefore, $z^* = fz^* = gz^* = hz^*$.

On the other hand, for case 3, part a, by (G_p2), $M(z^*,z^*,z^*) \leq 2G_p(fz^*,gz^*,hz^*)$ and hence from (35), we have

$$\psi\bigl(2G_p\bigl(fz^*,gz^*,hz^*\bigr)\bigr) \leq \psi\bigl(2G_p\bigl(fz^*,gz^*,hz^*\bigr)\bigr) - \varphi\bigl(M\bigl(z^*,z^*,z^*\bigr)\bigr), \tag{38}$$

hence $M(z^*,z^*,z^*) = 0$. Therefore, $z^* = fz^* = gz^* = hz^*$.

Now, let x^* and y^* as two fixed points of f, g and h be comparable. So, from (29) we have

$$\begin{aligned}\psi\bigl(2G_p\bigl(x^*,x^*,y^*\bigr)\bigr) &= \psi\bigl(2G_p\bigl(fx^*,gx^*,hy^*\bigr)\bigr)\\ &\leq \psi\bigl(M\bigl(x^*,x^*,y^*\bigr)\bigr) - \varphi\bigl(M\bigl(x^*,x^*,y^*\bigr)\bigr),\end{aligned} \tag{39}$$

where

$$\begin{aligned}M\bigl(x^*,x^*,y^*\bigr) = \max\Bigl\{&G_p\bigl(x^*,x^*,y^*\bigr),\\ &G_p\bigl(x^*,fx^*,fx^*\bigr), G_p\bigl(x^*,gx^*,gx^*\bigr), G_p\bigl(y^*,hy^*,hy^*\bigr),\\ &\frac{G_p(x^*,x^*,fx^*) + G_p(x^*,x^*,gx^*) + G_p(y^*,y^*,hy^*)}{3}\Bigr\}\\ \leq{}& 2G_p\bigl(x^*,x^*,y^*\bigr).\end{aligned}$$

Hence (39) gives

$$\psi\bigl(2G_p\bigl(x^*,x^*,y^*\bigr)\bigr) \leq \psi\bigl(2G_p\bigl(x^*,x^*,y^*\bigr)\bigr) - \varphi\bigl(M\bigl(x^*,x^*,y^*\bigr)\bigr).$$

Therefore, $\varphi(M(x^*,x^*,y^*)) = 0$ and hence $x^* = y^*$. □

The following corollaries are special cases of the above results.

Corollary 2.1 *Let $(X,\preceq,G_p)$ be a partially ordered complete G_p-metric space. Let $f : X \to X$ be a mapping such that for every three comparable elements $x,y,z \in X$, we have*

$$\psi\bigl(2G_p(fx,fy,fz)\bigr) \leq \psi\bigl(M(x,y,z)\bigr) - \varphi\bigl(M(x,y,z)\bigr), \tag{40}$$

where

$$M(x,y,z) = \max\Big\{G_p(x,y,z), \\ G_p(x,fx,fx), G_p(y,fy,fy), G_p(z,fz,fz), \\ \frac{G_p(x,x,fx) + G_p(y,y,fy) + G_p(z,z,fz)}{3}\Big\}$$

and $\psi, \varphi : [0,\infty) \to [0,\infty)$ *are altering distance functions. Then* f *has a fixed point in* X *provided that* $fx \preceq f(fx)$ *for all* $x \in X$ *and either*

a. f *is continuous, or*
b. X *has the sequential limit comparison property.*

Moreover, f *has a unique fixed point provided that the fixed points of* f *are comparable.*

Taking $y = z$ in Corollary 2.1, we obtain the following common fixed point result.

Corollary 2.2 *Let* $(X, \preceq, G_p)$ *be a partially ordered complete* G_p*-metric space, and let* f *be a self-mapping on* X *such that for every comparable elements* $x, y \in X$,

$$\psi\big(2G_p(fx,fy,fy)\big) \le \psi\big(M(x,y,y)\big) - \varphi\big(M(x,y,y)\big), \tag{41}$$

where

$$M(x,y,y) = \max\Big\{G_p(x,y,y), G(x,fx,fx), G_p(y,fy,fy), \frac{G_p(x,x,fx) + 2G_p(y,y,fy)}{3}\Big\},$$

and $\psi, \varphi : [0,\infty) \to [0,\infty)$ *are altering distance functions. Then* f *has a fixed point in* X *provided that* $fx \preceq f(fx)$ *for all* $x \in X$ *and either*

a. f *is continuous, or*
b. X *has the sequential limit comparison property.*

3 Fixed point results via an α-admissible mapping with respect to η in G_p-metric spaces

Samet *et al.* [32] defined the notion of α-admissible mappings and proved the following result.

Definition 3.1 Let T be a self-mapping on X and $\alpha : X \times X \to [0,+\infty)$ be a function. We say that T is an α-admissible mapping if

$$x, y \in X, \quad \alpha(x,y) \ge 1 \quad \Longrightarrow \quad \alpha(Tx,Ty) \ge 1.$$

Denote with Ψ the family of all nondecreasing functions $\psi : [0,+\infty) \to [0,+\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for all $t > 0$, where ψ^n is the nth iterate of ψ.

Theorem 3.1 *Let* (X,d) *be a complete metric space and* T *be an* α*-admissible mapping. Assume that*

$$\alpha(x,y)d(Tx,Ty) \le \psi\big(d(x,y)\big), \tag{42}$$

where $\psi \in \Psi$. *Also suppose that the following assertions hold*:

(i) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$;
(ii) *either* T *is continuous or for any sequence* $\{x_n\}$ *in* X *with* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $x_n \to x$ *as* $n \to +\infty$, *we have* $\alpha(x_n, x) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$.

Then T *has a fixed point.*

For more details on α-admissible mappings, we refer the reader to [33–37].

Very recently, Salimi *et al.* [38] modified and generalized the notions of α-ψ-contractive mappings and α-admissible mappings as follows.

Definition 3.2 [38] Let T be a self-mapping on X and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that T is an α-admissible mapping with respect to η if

$$x, y \in X, \quad \alpha(x, y) \geq \eta(x, y) \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq \eta(Tx, Ty).$$

Note that if we take $\eta(x, y) = 1$, then this definition reduces to Definition 3.1. Also, if we take $\alpha(x, y) = 1$, then we say that T is an η-subadmissible mapping.

The following result properly contains Theorem 3.1 and Theorems 2.3 and 2.4 of [37].

Theorem 3.2 [38] *Let* (X, d) *be a complete metric space and* T *be an* α-*admissible mapping with respect to* η. *Assume that*

$$x, y \in X, \quad \alpha(x, y) \geq \eta(x, y) \quad \Longrightarrow \quad d(Tx, Ty) \leq \psi\big(M(x, y)\big), \tag{43}$$

where $\psi \in \Psi$ *and*

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}.$$

Also, suppose that the following assertions hold:
(i) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;
(ii) *either* T *is continuous or for any sequence* $\{x_n\}$ *in* X *with* $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $x_n \to x$ *as* $n \to +\infty$, *we have* $\alpha(x_n, x) \geq \eta(x_n, x)$ *for all* $n \in \mathbb{N} \cup \{0\}$.

Then T *has a fixed point.*

In fact, the Banach contraction principle and Theorem 3.2 hold for the following example, but Theorem 3.1 does not hold.

Example 3.1 [38] Let $X = [0, \infty)$ be endowed with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$, and let $T : X \to X$ be defined by $Tx = \frac{1}{4}x$. Also, define $\alpha : X^2 \to [0, \infty)$ by $\alpha(x, y) = 3$ and $\psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = \frac{1}{2}t$.

Theorem 3.3 *Let* (X, G_p) *be a* G_p-*complete* G_p-*metric space,* f *be a continuous* α-*admissible mapping with respect to* η *on* X, *there exists* $x_0 \in X$ *such that* $\alpha(x_0, fx_0) \geq \eta(x_0, fx_0)$ *and if any sequence* $\{x_n\}$ *in* X *converges to a point* x, *then we have* $\alpha(x, x) \geq \eta(x, x)$. *Assume* *that*

$$\begin{aligned}&\alpha(x,y)\geq\eta(x,y)\\&\quad\Longrightarrow\quad G_p(fx,fy,fy)\leq r\max\bigl\{G_p(x,y,y),G_p(x,fx,fx),G_p(y,fy,fy)\bigr\}\end{aligned}\tag{44}$$

for all $x,y\in X$, *where* $0\leq r<1$. *Then* f *has a fixed point.*

Proof Let $x_0\in X$ and define a sequence $\{x_n\}$ by $x_n=f^nx_0$ for all $n\in\mathbb{N}$. Since f is an α-admissible mapping with respect to η and $\alpha(x_0,x_1)=\alpha(x_0,fx_0)\geq\eta(x_0,fx_0)=\eta(x_0,x_1)$, we deduce that $\alpha(x_1,x_2)=\alpha(fx_0,fx_1)\geq\eta(fx_0,fx_1)=\eta(x_1,x_2)$. Continuing this process, we get $\alpha(x_n,x_{n+1})\geq\eta(x_n,x_{n+1})$ for all $n\in\mathbb{N}\cup\{0\}$. Now, from (44) we have

$$\begin{aligned}&G_p\bigl(ff^nx_0,f^2f^nx_0,f^2f^nx_0\bigr)\\&\quad\leq r\max\bigl\{G_p\bigl(f^nx_0,ff^nx_0,ff^nx_0\bigr),G_p\bigl(ff^nx_0,f^2f^nx_0,f^2f^nx_0\bigr)\bigr\},\end{aligned}$$

which implies

$$G_p\bigl(f^{n+1}x_0,f^{n+2}x_0,f^{n+2}x_0\bigr)\leq rG_p\bigl(f^nx_0,f^{n+1}x_0,f^{n+1}x_0\bigr).\tag{45}$$

Continuing the above process, we can obtain

$$G_p\bigl(f^nx_0,f^{n+1}x_0,f^{n+1}x_0\bigr)\leq rG_p\bigl(f^{n-1}x_0,f^nx_0,f^nx_0\bigr)\leq\cdots\leq r^nG_p(x_0,fx_0,fx_0).\tag{46}$$

Then, for any $m>n$, by (46) we get

$$\begin{aligned}G_p\bigl(f^nx_0,f^mx_0,f^mx_0\bigr)&\leq G_p\bigl(f^nx_0,f^{n+1}x_0,f^{n+1}x_0\bigr)+G_p\bigl(f^{n+1}x_0,f^mx_0,f^mx_0\bigr)\\&\leq G_p\bigl(f^nx_0,f^{n+1}x_0,f^{n+1}x_0\bigr)+G_p\bigl(f^{n+1}x_0,f^{n+2}x_0,f^{n+2}x_0\bigr)\\&\quad+G_p\bigl(f^{n+2}x_0,f^mx_0,f^mx_0\bigr)\\&\leq G\bigl(f^nx_0,f^{n+1}x_0,f^{n+1}x_0\bigr)+G_p\bigl(f^{n+1}x_0,f^{n+2}x_0,f^{n+2}x_0\bigr)\\&\quad+G_p\bigl(f^{n+2}x_0,f^{n+3}x_0,f^{n+3}x_0\bigr)+\cdots+G_p\bigl(f^{m-1}x_0,f^mx_0,f^mx_0\bigr)\\&\leq\frac{r^n}{1-r}G_p(x_0,fx_0,fx_0).\end{aligned}$$

This implies that $\lim_{m,n\to+\infty}G_p(f^nx_0,f^mx_0,f^mx_0)=0$, that is, $\{x_n\}$ is a G_p-Cauchy sequence.

Since $\{x_n\}$ is a G_p-Cauchy sequence in the complete G_p-metric space X, from Lemma 1.2, $\{x_n\}$ is a Cauchy sequence in the metric space (X,d_{G_p}). Completeness of (X,G_p) yields that (X,d_{G_p}) is also complete. Then there exists $z\in X$ such that

$$\lim_{n\to\infty}d_{G_p}(x_n,z)=0.\tag{47}$$

Since $\lim_{m,n\to+\infty}G_p(x_n,x_m,x_m)=0$, from Lemma 1.2 we get

$$\lim_{n\to+\infty}G_p(x_n,z,z)=\lim_{n\to+\infty}G_p(x_n,x_n,z)=G_p(z,z,z)=0.\tag{48}$$

From the continuity of f, we have

$$\lim_{n\to+\infty} G_p(x_{n+1},fz,fz) = G_p(fz,fz,fz),$$

and hence we get

$$G_p(z,fz,fz) \le \lim_{n\to+\infty} G(z,x_{n+1},x_{n+1}) + \lim_{n\to+\infty} G(x_{n+1},fz,fz) = G_p(fz,fz,fz).$$

So, we get that $G_p(z,fz,fz) \le G_p(fz,fz,fz)$. Since the opposite inequality always holds, we get that

$$G_p(z,fz,fz) = G_p(fz,fz,fz).$$

As $\alpha(z,z) \ge \eta(z,z)$ we have

$$G_p(z,fz,fz) = G_p(fz,fz,fz) \le r\max\bigl\{G_p(z,z,z), G_p(z,fz,fz), G_p(z,fz,fz)\bigr\}, \tag{49}$$

where $0 \le r < 1$. Hence, $G_p(z,fz,fz) \le rG_p(z,fz,fz)$. Thus, $G_p(z,fz,fz) = 0$, that is, $z = fz$. □

If in Theorem 3.3 we take $\eta(x,y) = 1$, then we deduce the following corollary.

Corollary 3.1 *Let (X,G_p) be a G_p-complete G_p-metric space, f be a continuous α-admissible mapping on X, and there exists $x_0 \in X$ such that $\alpha(x_0,fx_0) \ge 1$. Assume that*

$$\alpha(x,y) \ge 1 \quad\Longrightarrow\quad G_p(fx,fy,fy) \le r\max\bigl\{G_p(x,y,y), G_p(x,fx,fx), G_p(y,fy,fy)\bigr\}$$

for all $x,y \in X$, where $0 \le r < 1$, and if any sequence $\{x_n\}$ in X converges to a point x, then we have $\alpha(x,x) \ge 1$. Then f has a fixed point.

If in Theorem 3.3 we take $\alpha(x,y) = 1$, then we deduce the following corollary.

Corollary 3.2 *Let (X,G_p) be a G_p-complete G_p-metric space, f be a continuous η-subadmissible mapping on X, and there exists $x_0 \in X$ such that $\eta(x_0,fx_0) \le 1$. Assume that*

$$\eta(x,y) \le 1 \quad\Longrightarrow\quad G_p(fx,fy,fy) \le r\max\bigl\{G_p(x,y,y), G_p(x,fx,fx), G_p(y,fy,fy)\bigr\} \tag{50}$$

for all $x,y \in X$, where $0 \le r < 1$, and if any sequence $\{x_n\}$ in X converges to a point x, then we have $1 \ge \eta(x,x)$. Then f has a fixed point.

In the following theorem, we omit the continuity of the mapping f.

Theorem 3.4 *Let (X,G_p) be a G_p-complete G_p-metric space and f be an α-admissible mapping with respect to η on X such that*

$$\begin{aligned}&\alpha(x,y) \ge \eta(x,y)\\ &\quad\Longrightarrow\quad G_p(fx,fy,fy) \le r\max\bigl\{G_p(x,y,y), G_p(x,fx,fx), G_p(y,fy,fy)\bigr\}\end{aligned} \tag{51}$$

for all $x,y \in X$, where $0 \le r < 1$. Assume that the following conditions hold:

(i) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, fx_0) \geq \eta(x_0, fx_0)$;
(ii) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ *for all* n *and* $x_n \to x$ *as* $n \to +\infty$, *then* $\alpha(x_n, x) \geq \eta(x_n, x)$ *for all* $n \in \mathbb{N} \cup \{0\}$.

Then f *has a fixed point.*

Proof Let $x_0 \in X$ be such that $\alpha(x_0, fx_0) \geq \eta(x_0, fx_0)$ and define a sequence $\{x_n\}$ in X by $x_n = f^n x_0 = fx_{n-1}$ for all $n \in \mathbb{N}$. Following the proof of Theorem 3.1, we have $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and there exists $x \in X$ such that $x_n \to x$ as $n \to +\infty$. Hence, from (ii) we deduce that $\alpha(x_n, x) \geq \eta(x_n, x)$ for all $n \in \mathbb{N} \cup \{0\}$.

Hence, by (51), it follows that for all n,

$$G_p(x_{n+1}, fx, fx) \leq r \max\{G_p(x_n, x, x), G_p(x_n, x_{n+1}, x_{n+1}), G_p(x, fx, fx)\}.$$

Taking the limit as $n \to +\infty$ in the above inequality, from Lemma 1.3 we obtain $(1 - r)G(x, fx, fx) \leq 0$, which implies that $x = fx$. □

Corollary 3.3 *Let* (X, G_p) *be a* G_p*-complete* G_p*-metric space and* f *be an* α*-admissible mapping on* X *such that*

$$\alpha(x, y) \geq 1 \quad \Longrightarrow \quad G_p(fx, fy, fy) \leq r \max\{G_p(x, y, y), G_p(x, fx, fx), G_p(y, fy, fy)\} \tag{52}$$

for all $x, y \in X$, *where* $0 \leq r < 1$. *Assume that the following conditions hold*:
(i) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, fx_0) \geq 1$;
(ii) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and* $x_n \to x$ *as* $n \to +\infty$, *then* $\alpha(x_n, x) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$.

Then f *has a fixed point.*

Example 3.2 Let $X = [0, +\infty)$ and $G_p(x, y, z) = \max\{x, y, z\}$ be a G_p-metric on X. Define $f : X \to X$ by

$$fx = \begin{cases} \frac{x}{24} & \text{if } x \in [0,1] \cup \{2\} = U, \\ 37/12 & \text{if } x = 3, \\ (1+x)^x & \text{if } x \in [0, +\infty) \setminus ([0,1] \cup \{2,3\}) = V, \end{cases}$$

and $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0,1], \\ 1/8 & \text{if } x = 2 \text{ and } y = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Now, we prove that all the hypotheses of Corollary 3.3 are satisfied and hence f has a fixed point.

Let $x, y \in X$, if $\alpha(x, y) \geq 1$, then $x, y \in [0,1]$. On the other hand, for all $x \in [0,1]$, we have $fx \leq 1$ and hence $\alpha(fx, fy) \geq 1$. This implies that f is an α-admissible mapping on X. Obviously, $\alpha(0, f0) \geq 1$.

Now, if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $\{x_n\} \subseteq [0,1]$ and hence $x \in [0,1]$. This implies that $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.

If $\alpha(x,y) \geq 1$, then $x, y \in [0,1]$. Hence,

$$\begin{aligned} G_p(fx,fy,fy) &= \max\{fx,fy\} = \max\left\{\frac{x}{24}, \frac{y}{24}\right\} \\ &\leq \frac{1}{12}\max\{x,y\} \\ &\leq \frac{1}{12}\max\bigl\{G_p(x,y,y), G_p(x,fx,fx), G_p(y,fy,fy)\bigr\}. \end{aligned}$$

Thus, all the conditions of Corollary 3.3 are satisfied and therefore f has a fixed point ($x = 0$).

Corollary 3.4 *Let (X, G_p) be a G_p-complete G_p-metric space and f be an η-subadmissible mapping on X such that*

$$\eta(x,y) \leq 1 \quad \Longrightarrow \quad G_p(fx,fy,fy) \leq r\max\bigl\{G_p(x,y,y), G_p(x,fx,fx), G_p(y,fy,fy)\bigr\}$$

for all $x, y \in X$, where $0 \leq r < 1$. Assume that the following conditions hold:

(i) *there exists $x_0 \in X$ such that $\eta(x_0, fx_0) \leq 1$;*

(ii) *if $\{x_n\}$ is a sequence in X such that $\eta(x_n, x_{n+1}) \leq 1$ for all n and $x_n \to x$ as $n \to +\infty$, then $\eta(x_n, x) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then f has a fixed point.

4 Consequences

Theorem 4.1 *Let (X, G_p) be a G_p-complete G_p-metric space, f be a continuous α-admissible mapping on X, and there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$. Assume that*

$$\alpha(x,y)G_p(fx,fy,fy) \leq r\max\bigl\{G_p(x,y,y), G_p(x,fx,fx), G_p(y,fy,fy)\bigr\} \tag{53}$$

for all $x, y \in X$, where $0 \leq r < 1$ and if any sequence $\{x_n\}$ in X converges to a point x, then we have $\alpha(x,x) \geq \eta(x,x)$. Then f has a fixed point.

Proof Assume that $\alpha(x,y) \geq 1$, then from (53) we get

$$G_p(fx,fy,fy) \leq r\max\bigl\{G_p(x,y,y), G_p(x,fx,fx), G_p(y,fy,fy)\bigr\}.$$

That is,

$$\alpha(x,y) \geq 1 \quad \Longrightarrow \quad G_p(fx,fy,fy) \leq r\max\bigl\{G_p(x,y,y), G_p(x,fx,fx), G_p(y,fy,fy)\bigr\}.$$

Hence all the conditions of Corollary 3.1 hold and f has a fixed point. □

Similarly, we can deduce the following results.

Theorem 4.2 *Let (X, G_p) be a G_p-complete G_p-metric space, f be a continuous α-admissible mapping on X, and there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$. Assume that*

$$\left(G_p(fx, fy, fy) + \ell\right)^{\alpha(x,y)} \leq r \max\left\{G_p(x, y, y), G_p(x, fx, fx), G_p(y, fy, fy)\right\} + \ell$$

for all $x, y \in X$, where $0 \leq r < 1$ and $\ell \geq 1$, and if any sequence $\{x_n\}$ in X converges to a point x, then we have $\alpha(x, x) \geq 1$. Then f has a fixed point.

Theorem 4.3 *Let (X, G_p) be a G_p-complete G_p-metric space, f be a continuous α-admissible mapping on X, and there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$. Assume that*

$$\left(\alpha(x, y) + \ell\right)^{G_p(fx, fy, fy)} \leq (1 + \ell)^{r \max\{G_p(x,y,y), G_p(x,fx,fx), G_p(y,fy,fy)\}} \tag{54}$$

for all $x, y \in X$, where $0 \leq r < 1$ and $\ell > 0$, and if any sequence $\{x_n\}$ in X converges to a point x, then we have $\alpha(x, x) \geq 1$. Then f has a fixed point.

Theorem 4.4 *Let (X, G_p) be a G_p-complete G_p-metric space, f be a continuous η-subadmissible mapping on X, and there exists $x_0 \in X$ such that $\eta(x_0, fx_0) \leq 1$. Assume that*

$$G_p(fx, fy, fy) \leq r\eta(x, y) \max\left\{G_p(x, y, y), G_p(x, fx, fx), G_p(y, fy, fy)\right\} \tag{55}$$

for all $x, y \in X$, where $0 \leq r < 1$, and if any sequence $\{x_n\}$ in X converges to a point x, then we have $1 \geq \eta(x, x)$. Then f has a fixed point.

Theorem 4.5 *Let (X, G_p) be a G_p-complete G_p-metric space, f be a continuous η-subadmissible mapping on X, and there exists $x_0 \in X$ such that $\eta(x_0, fx_0) \leq 1$. Assume that*

$$G_p(fx, fy, fy) + \ell \leq \left(r \max\left\{G_p(x, y, y), G_p(x, fx, fx), G_p(y, fy, fy)\right\} + \ell\right)^{\eta(x,y)}$$

for all $x, y \in X$, where $0 \leq r < 1$ and $\ell \geq 1$, and if any sequence $\{x_n\}$ in X converges to a point x, then we have $1 \geq \eta(x, x)$. Then f has a fixed point.

Theorem 4.6 *Let (X, G_p) be a G_p-complete G_p-metric space, f be a continuous η-subadmissible mapping on X, and there exists $x_0 \in X$ such that $\eta(x_0, fx_0) \leq 1$. Assume that*

$$(1 + \ell)^{G_p(fx, fy, fy)} \leq \left(\eta(x, y) + \ell\right)^{r \max\{G_p(x,y,y), G_p(x,fx,fx), G_p(y,fy,fy)\}} \tag{56}$$

for all $x, y \in X$, where $0 \leq r < 1$ and $\ell > 0$, and if any sequence $\{x_n\}$ in X converges to a point x, then we have $1 \geq \eta(x, x)$. Then f has a fixed point.

Theorem 4.7 *Let (X, G_p) be a G_p-complete G_p-metric space, f be an α-admissible mapping on X, and there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$. Assume that*

$$\alpha(x, y) G_p(fx, fy, fy) \leq r \max\left\{G_p(x, y, y), G_p(x, fx, fx), G_p(y, fy, fy)\right\}$$

for all $x, y \in X$, where $0 \leq r < 1$. If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$ as $n \to +\infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then f has a fixed point.

Theorem 4.8 *Let (X, G_p) be a G_p-complete G_p-metric space, f be an α-admissible mapping on X, and there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$. Assume that*

$$\left(G_p(fx, fy, fy) + \ell\right)^{\alpha(x,y)} \leq r \max\left\{G_p(x, y, y), G_p(x, fx, fx), G_p(y, fy, fy)\right\} + \ell$$

for all $x, y \in X$, where $0 \leq r < 1$ and $\ell \geq 1$. If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$ as $n \to +\infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then f has a fixed point.

Theorem 4.9 *Let (X, G_p) be a G_p-complete G_p-metric space, f be an α-admissible mapping on X, and there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$. Assume that*

$$\left(\alpha(x, y) + \ell\right)^{G_p(fx,fy,fy)} \leq (1 + \ell)^{r \max\{G_p(x,y,y), G_p(x,fx,fx), G_p(y,fy,fy)\}} \tag{57}$$

for all $x, y \in X$, where $0 \leq r < 1$ and $\ell > 0$. If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x$ as $n \to +\infty$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then f has a fixed point.

Theorem 4.10 *Let (X, G_p) be a G_p-complete G_p-metric space, f be an η-subadmissible mapping on X, and there exists $x_0 \in X$ such that $\eta(x_0, fx_0) \leq 1$. Assume that*

$$G_p(fx, fy, fy) \leq r\eta(x, y) \max\left\{G_p(x, y, y), G_p(x, fx, fx), G_p(y, fy, fy)\right\} \tag{58}$$

for all $x, y \in X$, where $0 \leq r < 1$. If $\{x_n\}$ is a sequence in X such that $\eta(x_n, x_{n+1}) \leq 1$ for all n and $x_n \to x$ as $n \to +\infty$, we have $\eta(x_n, x) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then f has a fixed point.

Theorem 4.11 *Let (X, G_p) be a G_p-complete G_p-metric space, f be an η-subadmissible mapping on X and there exists $x_0 \in X$ such that $\eta(x_0, fx_0) \leq 1$. Assume that*

$$G_p(fx, fy, fy) + \ell \leq \left(r \max\left\{G_p(x, y, y), G_p(x, fx, fx), G_p(y, fy, fy)\right\} + \ell\right)^{\eta(x,y)}$$

for all $x, y \in X$, where $0 \leq r < 1$ and $\ell \geq 1$. If $\{x_n\}$ is a sequence in X such that $\eta(x_n, x_{n+1}) \leq 1$ for all n and $x_n \to x$ as $n \to +\infty$, we have $\eta(x_n, x) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then f has a fixed point.

Theorem 4.12 *Let (X, G_p) be a G_p-complete G_p-metric space, f be an η-subadmissible mapping on X, and there exists $x_0 \in X$ such that $\eta(x_0, fx_0) \leq 1$. Assume that*

$$(1 + \ell)^{G_p(fx,fy,fy)} \leq \left(\eta(x, y) + \ell\right)^{r \max\{G_p(x,y,y), G_p(x,fx,fx), G_p(y,fy,fy)\}} \tag{59}$$

for all $x, y \in X$, where $0 \leq r < 1$ and $\ell > 0$. If $\{x_n\}$ is a sequence in X such that $\eta(x_n, x_{n+1}) \leq 1$ for all n and $x_n \to x$ as $n \to +\infty$, then $\eta(x_n, x) \leq 1$ for all $n \in \mathbb{N} \cup \{0\}$, then f has a fixed point.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in this research. All authors read and approved the final manuscript.

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Acknowledgements
This research was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The authors, therefore, acknowledge with thanks the DSR financial support.

Received: 9 August 2013 Accepted: 25 October 2013 Published: 25 Nov 2013

References
1. Zand, MRA, Nezhad, AD: A generalization of partial metric spaces. J. Contemp. Appl. Math. **24**, 86-93 (2011)
2. Matthews, SG: Partial metric topology. In: General Topology and Its Applications. Proc. 8th Summer Conf., Queen's College, 1992. Annals of the New York Academy of Sciences, vol. 728, pp. 183-197 (1994)
3. Mustafa, Z, Sims, B: A new approach to generalized metric spaces. J. Nonlinear Convex Anal. **7**(2), 289-297 (2006)
4. Parvaneh, V, Roshan, JR, Kadelburg, Z: On generalized weakly *GP*-contractive mappings in ordered *GP*-metric spaces. Gulf J. Math. **1**, 78-97 (2013)
5. Aydi, H, Karapınar, E, Salimi, P: Some fixed point results in *GP*-metric spaces. J. Appl. Math. **2012**, Article ID 891713 (2012). doi:10.1155/2012/891713
6. Oltra, S, Valero, O: Banach's fixed point theorem for partial metric spaces. Rend. Ist. Mat. Univ. Trieste **36**(1-2), 17-26 (2004)
7. Khan, MS, Swaleh, M, Sessa, S: Fixed point theorems by altering distances between the points. Bull. Aust. Math. Soc. **30**, 1-9 (1984)
8. Alber, YI, Guerre-Delabriere, S: Principle of weakly contractive maps in Hilbert spaces. In: Gohberg, I, Lyubich, Y (eds.) New Results in Operator Theory. Advances and Appl., vol. 98, pp. 7-22. Birkhäuser, Basel (1997)
9. Rhoades, BE: Some theorems on weakly contractive maps. Nonlinear Anal. **47**, 2683-2693 (2001)
10. Zhang, Q, Song, Y: Fixed point theory for generalized φ-weak contractions. Appl. Math. Lett. **22**, 75-78 (2009)
11. Abbas, M, Đorić, D: Common fixed point theorem for four mappings satisfying generalized weak contractive condition. Filomat **24**(2), 1-10 (2010)
12. Đorić, D: Common fixed point for generalized (ψ,φ)-weak contractions. Appl. Math. Lett. **22**, 1896-1900 (2009)
13. Moradi, S, Fathi, Z, Analouee, E: Common fixed point of single valued generalized φ_f-weak contractive mappings. Appl. Math. Lett. **24**(5), 771-776 (2011)
14. Razani, A, Parvaneh, V, Abbas, M: A common fixed point for generalized $(\psi,\varphi)_{f,g}$-weak contractions. Ukr. Math. J. **63**, 11 (2012)
15. Abbas, M, Nazir, T, Radenović, S: Common fixed points of four maps in partially ordered metric spaces. Appl. Math. Lett. **24**, 1520-1526 (2011)
16. Abbas, M, Parvaneh, V, Razani, A: Periodic points of *T*-Ćirić generalized contraction mappings in ordered metric spaces. Georgian Math. J. **19**(4), 597-610 (2012)
17. Agarwal, RP, El-Gebeily, MA, O'Regan, D: Generalized contractions in partially ordered metric spaces. Appl. Anal. **87**(1), 109-116 (2008)
18. Al-Khaleel, M, Al-Sharif, S, Khandaqji, M: Fixed point for contraction mappings in generalized cone metric spaces. Jordan J. Math. Stat. **5**(4), 291-307 (2012)
19. Aghajani, A, Radenović, S, Roshan, JR: Common fixed point results for four mappings satisfying almost generalized (S,T)-contractive condition in partially ordered metric spaces. Appl. Math. Comput. **218**, 5665-5670 (2012)
20. Khan, MA, Sumitra, Kumar, R: Subcompatible and subsequential continuous maps in non Archimedean Menger PM-spaces. Jordan J. Math. Stat. **5**(2), 137-150 (2012)
21. Khan, MA, Sumitra, Kumar, R: Semi-compatible maps and common fixed point theorems in non-Archimedean Menger PM-spaces. Jordan J. Math. Stat. **5**(3), 185-199 (2012)
22. Esmaily, J, Vaezpour, SM, Rhoades, BE: Coincidence point theorem for generalized weakly contractions in ordered metric spaces. Appl. Math. Comput. **219**, 1536-1548 (2012)
23. Harjani, J, López, B, Sadarangani, K: Fixed point theorems for weakly *C*-contractive mappings in ordered metric spaces. Comput. Math. Appl. **61**, 790-796 (2011)
24. Nashine, HK, Samet, B: Fixed point results for mappings satisfying (ψ,φ)-weakly contractive condition in partially ordered metric spaces. Nonlinear Anal. **74**, 2201-2209 (2011)
25. Nieto, JJ, López, RR: Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. Order **22**, 223-239 (2005)
26. Nieto, JJ, López, RR: Existence and uniqueness of fixed points in partially ordered sets and applications to ordinary differential equations. Acta Math. Sin. Engl. Ser. **23**, 2205-2212 (2007)
27. Radenović, S, Kadelburg, Z: Generalized weak contractions in partially ordered metric spaces. Comput. Math. Appl. **60**, 1776-1783 (2010)
28. Ran, ACM, Reurings, MCB: A fixed point theorem in partially ordered sets and some application to matrix equations. Proc. Am. Math. Soc. **132**, 1435-1443 (2004)
29. Shatanawi, W, Samet, B: On (ψ,ϕ)-weakly contractive condition in partially ordered metric spaces. Comput. Math. Appl. **62**, 3204-3214 (2011)
30. Altun, I, Simsek, H: Some fixed point theorems on ordered metric spaces and application. Fixed Point Theory Appl. **2010**, Article ID 621492 (2010)
31. Lakshmikantham, V, Ćirić, LB: Coupled fixed point theorems for nonlinear contractions in partially ordered metric space. Nonlinear Anal. TMA **70**, 4341-4349 (2009)

32. Samet, B, Vetro, C, Vetro, P: Fixed point theorem for α-ψ-contractive type mappings. Nonlinear Anal. **75**, 2154-2165 (2012)
33. Hussain, N, Karapınar, E, Salimi, P, Akbar, F: α-Admissible mappings and related fixed point theorems. Fixed Point Theory Appl. **2013**, 114 (2013)
34. Karapınar, E, Kumam, P, Salimi, P: On α-ψ-Meir-Keeler contractive mappings. Fixed Point Theory Appl. **2013**, 94 (2013)
35. Salimi, P, Vetro, C, Vetro, P: Fixed point theorems for twisted (α, β)-ψ-contractive type mappings and applications. Filomat **27**(4), 605-615 (2013)
36. Salimi, P, Vetro, C, Vetro, P: Some new fixed point results in non-Archimedean fuzzy metric spaces. Nonlinear Anal.: Model. Control, **18**(3), 344-358 (2013)
37. Salimi, P, Karapınar, E: Suzuki-Edelstein type contractions via auxiliary functions. Math. Probl. Eng. **2013**, Article ID 648528 (2013)
38. Salimi, P, Latif, A, Hussain, N: Modified α-ψ-contractive mappings with applications. Fixed Point Theory Appl. **2013**, 151 (2013)

10.1186/1687-1812-2013-317
Cite this article as: Ćirić et al.: **Some fixed point results in ordered G_p-metric spaces.** *Fixed Point Theory and Applications* 2013, 2013:317