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Some fixed point theorems in locally p-convex spaces

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Abstract
In this paper we investigate the existence of a fixed point of multivalued maps on almost p-convex and p-convex subsets of topological vector spaces. Our results extend and generalize some fixed point theorems on the topic in the literature, such as the results of Himmelberg, Fan and Glicksberg.

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1 Introduction and preliminaries

In nonlinear analysis, one of the dynamic research areas is investigation of existence of a fixed point of maps on convex sets and p-convex sets. Recently, a number of fixed point theorems have appeared on the setting of p-convex sets. For instance, Alimohammady *et al.* [1] extended the Markov-Kakutani fixed point theorem for compact p-star shaped subsets in topological vector spaces by using p-convex sets instead of convex sets, see also [2, 3]. Further, in [4] authors achieved a fixed point theorem due to Park for a compact mapping on a p-star shaped subset of a topological vector space via Fan-KKM principle in a generalized convex space. In [5, 6], generalized versions of Brouwer and Kakutani fixed point theorems were characterized in the context of locally p-convex space.

On the other hand, in 1993 Park and Kim introduced the concept of generalized convex space, which extends many generalized convex structures on topological vector spaces [7]. This new concept, developed in connection with fixed point theory and KKM theory, generalizes topological vector spaces.

Maki [8] introduced the notion of minimal spaces which is a generalization of the concept of topological spaces (see also [9]). After these initial papers, many authors have paid attention to the subject and have published several results in this direction; see, *e.g.*, [10–13]. Very recently, Darzi *et al.* [14] introduced the notion of minimal generalized convex space as to extend the construction of the generalized convex space.

For the sake of completeness, we recall some basic definitions and fundamental results in the literature. All we need regarding topological vector spaces can be found in [15–18].

Let U be a subset of a vector space V and $x, y \in U$ and $0 < p \leq 1$. Bayoumi [5] introduced the notion of *arc segment joining* x and y, as follows:

$$A_x^y = \left\{s^{\frac{1}{p}}x + t^{\frac{1}{p}}y : s + t = 1\right\} = \left\{ux + vy : u^p + v^p = 1\right\}.$$

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A set X in a vector space V is said to be *p-convex* if $A_x^y \subseteq X$ for every $x, y \in X$. The *p-convex hull* of X denoted by $C_p(X)$ is the smallest p-convex set containing X [5]. Further, the *closed p-convex hull* of X denoted by $\overline{C}_p(X)$ is the smallest closed p-convex set containing $X \subseteq E$, where E is a topological vector space. Notice that if $p = 1$ and $s + t = 1$, then A_x^y turns out to be the line segment joining x and y. In this case, $C_p(X)$ and $\overline{C}_p(X)$ become the convex hull and the closed convex hull of X, respectively. For more details, we refer to, *e.g.*, [5, 6, 19–23] and references therein.

Let X be a nonempty set. Then a family $\mathcal{M} \subseteq \mathcal{P}(X)$ is said to be a *minimal structure* on X if $\emptyset, X \in \mathcal{M}$. Moreover, the pair $(X, \mathcal{M})$ is called a *minimal space*. The natural examples of minimal spaces can be listed as follows [8]: τ, the collection of all semi-open sets $SO(X)$, the collection of all pre-open sets $PO(X)$, the collection of all α-open sets $\alpha O(X)$ and the collection of all β-open sets $\beta O(X)$, where (X, τ) is a topological space. In a minimal space $(X, \mathcal{M})$, a set $A \in \mathcal{P}(X)$ is said to be an *m-open set* if $A \in \mathcal{M}$. Similarly, a set $B \in \mathcal{P}(X)$ is an *m-closed set* if $B^c \in \mathcal{M}$. Furthermore, m-interior and m-closure of a set A are defined as follows:

$$m\text{-Int}(A) = \bigcup\{U : U \subseteq A, U \in \mathcal{M}\} \quad \text{and} \quad m\text{-Cl}(A) = \bigcap\{F : A \subseteq F, F^c \in \mathcal{M}\}.$$

For more details on minimal structure and minimal space, we refer the reader to, *e.g.*, [8, 9, 12–14, 24, 25].

The continuity of maps in a minimal space is defined as follows.

Definition 1.1 [25] Suppose that (X, τ) is a topological space, and also suppose that $(Y, \mathcal{N})$ is a minimal space. A function $f : (X, \tau) \longrightarrow (Y, \mathcal{N})$ is called (τ, m)-*continuous* if $f^{-1}(U) \in \tau$ for any $U \in \mathcal{N}$.

Let X and Y be two nonempty sets and $\mathcal{P}(Y)$ be the set of all subsets of Y. A *set-valued map* or a *set-valued function* from X into Y is a function from X to $\mathcal{P}(Y)$ that assigns an element x of X to a nonempty subset $T(x)$ of Y and is denoted by $x \multimapdotinv T(x)$. The *lower inverse of a point* $y \in Y$ of a set-valued map T is the set-valued map T^l of Y into X defined by

$$T^l(y) = \{x \in X : y \in T(x)\}.$$

Analogously, *lower inverse of a subset of* $B \subset Y$ is defined as

$$T^l(B) = \{x \in X : T(x) \cap B \neq \emptyset\}.$$

We note that $T^l(\emptyset) = \emptyset$. The set $\{x \in X : T(x) \subseteq B\}$ is the *upper inverse* of B and is denoted by $T^u(B)$. A map T is *lower semicontinuous* if $T^l(U)$ is open in X for every open set $U \subseteq Y$. Similarly, a map T is *upper semicontinuous* if for every open set $U \subseteq Y$, the set $T^u(U)$ is open in X.

A set-valued map $T : X \multimap Y$ is said to be *closed* if its graph, $\text{Graph}(T) = \{(x, y) : y \in T(x)\}$, is a closed subset of $X \times Y$. Also, T is called *compact* if its range, $T(X)$, is contained in a compact subset of Y.

The notion of almost convex was introduced by Himmelberg [26]. A nonempty subset B of a topological vector space X is said to be *almost convex* if for any neighborhood V

of 0 and for any finite subset $\{b_1,\dots,b_n\}$ of B, there exists a finite subset $\{x_1,\dots,x_n\} \subseteq B$ such that $x_i - b_i \in V$ for each $i = 1,\dots,n$ and $co(\{x_1,\dots,x_n\}) \subseteq B$. It is clear that any convex subset is almost convex. Moreover, if we delete a certain subset of the boundary of a closed convex set, then we have an almost convex set. Another example of an almost convex set is the following: Let $C([0,1])$ be the Banach space of all continuous real functions defined on the unit interval $[0,1]$, and let $P([0,1])$ be a dense subset of all polynomials. Then any subset of $C([0,1])$ containing $P([0,1])$ is almost convex.

Let A be a subset of a topological vector space X. A set-valued map $T : A \multimap A$ is said to have the (*convexly*) *almost fixed point property* if for every (convex) neighborhood U of 0 in X, there exists a point $a_U \in A$ for which $a_U \in T(a_U) + U$ or $T(a_U) \cap (a_U + U) \neq \emptyset$.

Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D, and let Δ_n be the n-simplex with vertices $e_0, e_1, \dots, e_n$, Δ_J be the face of Δ_n corresponding to $J \in \langle A\rangle$, where $A \in \langle D\rangle$. For instance, if $A = \{a_0, a_1, \dots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \dots, a_{i_k}\} \subseteq A$, then $\Delta_J = co\{e_{i_0}, e_{i_1}, \dots, e_{i_k}\}$. A *minimal generalized convex space* (briefly *MG*-convex space) (X, D, Γ) consists of a minimal space $(X, \mathcal{M})$, a nonempty set D and a set-valued map $\Gamma : \langle D\rangle \multimap X$ in which for $A \in \langle D\rangle$ with $n+1$ elements, there exists a (τ, m)-continuous function $\phi_A : \Delta_n \longrightarrow \Gamma_A := \Gamma(A)$ for which $J \in \langle A\rangle$ implies that $\phi_A(\Delta_J) \subseteq \Gamma_J$. If $\mathcal{M} = \tau$, then the notion of *MG*-convex space turns into *G*-convex space (see, *e.g.*, [27]). On the other hand, suppose that $(X, \mathcal{M})$ is a minimal vector space which is not a topological vector space. Consider the set-valued map $\Gamma : \langle X\rangle \multimap X$ defined by $\Gamma(\{a_0, a_1, \dots, a_n\}) = \{\sum_{i=0}^n \lambda_i a_i : 0 \le \lambda_i \le 1, \sum_{i=0}^n \lambda_i = 1\}$. Then (X, Γ) is a minimal generalized convex space; of course, we know that (X, Γ) is not a generalized convex space [14].

Definition 1.2 Suppose that (X, D, Γ) is an *MG*-convex space. A set-valued map $F : D \multimap X$ is called a *KKM set-valued map* if $\Gamma_A \subseteq F(A)$ for any $A \in \langle D\rangle$.

We state two useful theorems of Alimohammady *et al.* [25] as follows.

Theorem 1.3 [25] *Suppose that* (X, D, Γ) *is an MG-convex space and* $F : D \multimap X$ *is a set-valued map satisfying*

(a) *for all* $x \in D$, $F(x) = m\text{-}\mathrm{Cl}(A_x)$ *for some* $A_x \subseteq X$,
(b) F *is a KKM map.*

Then $\{F(z) : z \in D\}$ *has the finite intersection property.*

Further, if

(c) $\bigcap_{z\in N} F(z)$ *is m-compact for some* $N \in \langle D\rangle$,

then $\bigcap_{z\in D} F(z) \neq \emptyset$.

Theorem 1.4 [25] *Suppose that* (X, D, Γ) *is an MG-convex space and* $F : D \multimap X$ *is a set-valued map satisfying*

(a) *for all* $x \in D$, $F(x) = m\text{-}\mathrm{Int}(A_x)$ *for some* $A_x \subseteq X$,
(b) F *is a KKM map.*

Then $\{F(z) : z \in D\}$ *has the finite intersection property.*

In this paper we investigate the existence of a fixed point on the setting of locally p-convex spaces. In particular, we establish a generalized version of Alexandroff-Pasynkoff theorem. Furthermore, we present a generalization of the Himmelberg fixed point theorem. We also prove Fan-Glicksberg result for p-convex sets.

2 Main results

We start this section with the following result which is inspired by Theorem 1.3 and Theorem 1.4.

Theorem 2.1 *Suppose that A is a subset of a topological vector space X and B is a nonempty subset of A with $C_p(B) \subseteq A$. Also suppose that $F : B \multimap A$ is a set-valued map satisfying*

(a) *$F(b)$ is closed (resp. open) in A for all $b \in B$,*

(b) *$C_p(N) \subseteq F(N)$ for each $N \in \langle B\rangle$.*

Then $\{F(b) : b \in B\}$ has the finite intersection property.

Proof Consider the set-valued map $\Gamma : \langle B\rangle \multimap A$ defined by

$$\Gamma\bigl(\{b_0, b_1, \ldots, b_n\}\bigr) = \left\{\sum_{i=0}^{n} \lambda_i b_i : 0 \le \lambda_i \le 1, \sum_{i=0}^{n} \lambda_i^p = 1\right\}.$$

Since $C_p(B) \subseteq A$, the set-valued map Γ is well defined. Condition (b) implies that F is a KKM map. For each $N = \{b_0, b_1, \ldots, b_n\} \subseteq B$, let us define

$$\phi_N : \Delta_n \longrightarrow \Gamma_N,$$
$$\sum_{i=0}^{n} t_i e_i \mapsto \sum_{i=0}^{n} (t_i)^{\frac{1}{p}} b_i.$$

Now, one can verify that (A, B, Γ) is a G-convex space. The fact that $\{F(b) : b \in B\}$ has the finite intersection property follows from Theorem 1.3 (resp. Theorem 1.4). □

Theorem 2.2 *Suppose that A is a subset of an MG-convex space (X, D, Γ), $\{A_0, A_1, \ldots, A_n\}$ is a family of m-closure valued (resp. m-interior valued) subsets of X such that $A \subseteq \bigcup_{i=0}^{n} A_i$, and also suppose that $N = \{z_0, z_1, \ldots, z_n\}$ is a family of points in D in which $\Gamma(N) \subseteq A$. If $\Gamma(N \setminus \{z_i\}) \subseteq A_i$ for each $i = 0, 1, \ldots, n$, then $\bigcap_{i=0}^{n} A_i \neq \emptyset$.*

Proof Set $C_0 = \Gamma(N \setminus z_n)$ and for $i = 1, 2, \ldots, n$, let $C_i = \Gamma(N \setminus \{z_{i-1}\})$. Consider the set-valued map $F : D \multimap X$ defined by $F(z_0) = A_n$, $F(z_i) = A_{i-1}$ for $i = 1, 2, \ldots, n$ and $F(z) = X$ for all $z \in D \setminus N$. We claim that F is a KKM map. To see this, we note that $\Gamma(N) \subseteq A \subseteq \bigcup_{i=0}^{n} A_i = F(N)$ and for any choice of a proper subset $\{z_{i_0}, z_{i_1}, \ldots, z_{i_k}\}$ of N with $0 \le k < n$ and $0 \le i_0 < \cdots < i_k \le n$, one can see that

$$\Gamma\bigl(\{z_{i_0}, z_{i_1}, \ldots, z_{i_k}\}\bigr) \subseteq C_{i_j} \subseteq A_{i_j - 1} = F(z_{i_j})$$

for some $j \in \{0, 1, \ldots, k\}$. Notice that $i_j = 0$ if and only if $i_j - 1 = n$, and so $\Gamma(\{z_{i_0}, z_{i_1}, \ldots, z_{i_k}\}) \subseteq \bigcup_{j=0}^{k} F(z_{i_j})$. The fact that $\bigcap_{i=0}^{n} A_i \neq \emptyset$ follows from Theorem 1.3 (resp. Theorem 1.4). □

Remark 2.3 It should be noted that

(a) Theorem 1.3 and Theorem 1.4 are extended versions of the corresponding results in [14, 24], and hence they are generalizations of Theorem 1 in [27, 28] and Ky Fan's lemma [29],

(b) Theorem 2.2 for closed (open) subsets of a topological vector space goes back to Park [30] and it is an extended version of Alexandroff-Pasynkoff theorem [31].

Definition 2.4 A nonempty subset B of a topological vector space X is said to be *almost p-convex* if for any neighborhood V of 0 and for any finite subset $\{b_1,\dots,b_n\}$ of B, there exists a finite subset $\{x_1,\dots,x_n\} \subseteq B$ such that $x_i - b_i \in V$ for each $i = 1,\dots,n$ and $C_p(\{x_1,\dots,x_n\}) \subseteq B$.

Example 2.5 It is easy to see that any p-convex subset of a topological vector space X is almost p-convex. If we delete a certain subset of the boundary of a closed p-convex set, then we have an almost p-convex set.

Definition 2.6 Let A be a subset of a topological vector space X. A set-valued map $T: A \multimap A$ is said to have the *p-convexly almost fixed point property* if for every p-convex neighborhood U of 0 in X, there exists a point $a_U \in A$ for which $a_U \in T(a_U) + U$ or $T(a_U) \cap (a_U + U) \neq \emptyset$.

Theorem 2.7 *Let A be a subset of a topological vector space X and B be an almost p-convex dense subset of A. Suppose that $T: A \multimap X$ is a lower* (*resp. upper*) *semicontinuous set-valued map such that $T(b)$ is p-convex for all $b \in B$, and also suppose that there is a precompact subset K of A such that $T(b) \cap K \neq \emptyset$ for all $b \in B$. Then T has the p-convexly almost fixed point property.*

Proof Suppose that U is a p-convex neighborhood of 0 and suppose that T is lower semicontinuous. There is a symmetric open neighborhood V of 0 for which $\overline{V} + \overline{V} \subseteq U$. Since K is precompact, so there are $x_0, x_1, \dots, x_n$ in K for which $K \subseteq \bigcup_{i=0}^{n}(x_i + V)$. By using the fact that B is almost p-convex and dense in A, we find $D = \{b_0, b_1, \dots, b_n\} \subseteq B$ for which $b_i - x_i \in V$ for all $i \in \{0,1,\dots,n\}$ and also $C = C_p(D) \subseteq B$. Since T is lower semicontinuous, the set $F(b_i) := \{c \in C : T(c) \cap (x_i + V) = \emptyset\}$ is closed in C for each $i \in \{0,\dots,n\}$. Regarding $\emptyset \neq T(c) \cap K \subseteq T(c) \cap \bigcup_{i=0}^{n}(x_i + V)$, we have $\bigcap_{i=0}^{n} F(b_i) = \emptyset$. Now, Theorem 2.1 implies that there is $N = \{b_{i_0}, b_{i_1}, \dots, b_{i_k}\} \in \langle D \rangle$ and $x_U \in C_p(N) \subseteq B$ for which $x_U \notin F(N)$, and so $T(x_u) \cap (x_{i_j} + \overline{V}) \neq \emptyset$ for all $j \in \{0,1,\dots,k\}$. Both $b_i - x_i \in V$ and $\overline{V} + \overline{V} \subseteq U$ imply that $x_{i_j} + \overline{V} \subseteq b_{i_j} + U$, which implies that $T(x_U) \cap (b_{i_j} + U) \neq \emptyset$. Therefore

$$N \subseteq M := \{c \in C : T(x_U) \cap (c + U) \neq \emptyset\}.$$

C, $T(x_U)$ and U are p-convex and hence M is p-convex. Consequently, $x_U \in M$, which implies that $T(x_U) \cap (x_U + U) \neq \emptyset$; *i.e.*, T has the p-convexly almost fixed point property. Finally, for the case that T is upper semicontinuous, we note that $F(b_i) := \{c \in C : T(c) \cap (x_i + \overline{V}) = \emptyset\}$ is open in C for each $i \in \{0,\dots,n\}$. The rest of the proof is similar to the proof of the case that T is l.s.c. Regarding the analogy, we skip the proof. □

Corollary 2.8 *Let A be a p-convex subset of a topological vector space X, and let $T: A \multimap X$ be a lower* (*resp. upper*) *semicontinuous set-valued map such that $T(a)$ is p-convex for all $a \in A$. Suppose that there is a precompact subset K of A such that $T(a) \cap K \neq \emptyset$ for all $a \in A$. Then T has the p-convexly almost fixed point property.*

Proof It is sufficient to take $A = B$ in Theorem 2.7. □

Corollary 2.9 *Let A be a subset of a topological vector space X, and let B be an almost p-convex dense subset of A. Suppose that $T: A \multimap X$ is a set-valued map satisfying*

(a) $T^l(x)$ (*resp.* $T^u(x)$) *is open for all* $x \in X$,
(b) $T(b)$ *is p-convex for all* $b \in B$,
(c) *there is a precompact subset* K *of* A *such that* $T(b) \cap K \neq \emptyset$ *for all* $b \in B$.

Then T *has the p-convexly almost fixed point property.*

Proof It is clear that (a) implies that T is a lower (resp. upper) semicontinuous set-valued map and hence T has the p-convexly almost fixed point property by Theorem 2.7. □

Corollary 2.10 *Let* A *be a p-convex subset of a topological vector space* X, *and let* $T : A \multimap X$ *be a compact set-valued map satisfying the following conditions*:
(a) $T^l(x)$ (*resp.* $T^u(x)$) *is open for all* $x \in X$,
(b) $T(a)$ *is nonempty and p-convex for all* $a \in A$.

Then T *has the p-convexly almost fixed point property.*

Proof Consider $A = B$, it is easy to see that all the conditions of Corollary 2.9 are satisfied. □

Remark 2.11 It should be noted that
(a) Corollary 2.8 for a lower semicontinuous set-valued map on a locally convex Hausdorff topological vector space goes back to Ky Fan [32]. Corollary 2.8 for a single-valued map might be regarded as a generalization of the Thychonoff fixed point theorem to a noncompact (or precompact) convex set [32]. Also, Lassonde obtained Corollary 2.8 for a compact upper semicontinuous set-valued map with nonempty convex values [33].
(b) Convex versions of Theorem 2.7, Corollary 2.9 and Corollary 2.10 are due to Park [30].

Theorem 2.12 *Suppose that* A *is a subset of a locally p-convex space* X *and* B *is an almost p-convex dense subset of* A. *Suppose that* $T : A \multimap A$ *satisfies the following*:
(a) T *is compact upper semicontinuous*,
(b) $T(a)$ *is closed for all* $a \in A$,
(c) $T(b)$ *is nonempty p-convex for all* $b \in B$.

Then T *has a fixed point.*

Proof Since all the conditions of Theorem 2.7 are satisfied and since X is a locally p-convex space, T has the almost fixed point property. Then, for an arbitrary neighborhood U of 0, there exist a_U and b_U in A for which $b_U \in T(a_U) \cap (a_U + U)$. Since T is compact, we conclude that there is $a_0 \in \overline{T(A)} \subseteq A$ in which the net $b_U \longrightarrow a_0$. Because X is Hausdorff, $a_U \longrightarrow a_0$. Since T is an upper semicontinuous set-valued map with closed values, Graph(T) is closed. Consequently, a_0 is a fixed point of T. □

Corollary 2.13 *Suppose that* A *is a p-convex subset of a locally p-convex space* X. *Suppose that* $T : A \multimap A$ *satisfies the following*:
(a) T *is compact upper semicontinuous*,
(b) $T(a)$ *is closed for all* $a \in A$,
(c) $T(a)$ *is nonempty p-convex for all* $a \in A$.

Then T *has a fixed point.*

Theorem 2.14 *Suppose that A is a p-convex subset of a locally p-convex space X. Suppose that $T : A \multimap A$ satisfies the following*:
(a) *T is compact and closed,*
(b) *T has the almost fixed point property.*
Then T has a fixed point.

Proof Suppose that $\mathcal{U}$ is the family of neighborhoods of 0 in X. For any element U of $\mathcal{U}$, since T has the almost fixed point property, so there exist $a_U, b_U \in A$ for which $b_U \in T(a_U)$ and $b_U \in a_U + U$. Now, consider the nets $\{a_U\}$ and $\{b_U\}$. By (a) we have $\overline{T(A)}$ is compact and hence $\{b_U\}$ has a subnet converging to b_0. We may assume that $b_U \longrightarrow b_0$. Since X is Hausdorff, there is a subnet of a_U converging to b_0. The fact that $b_0 \in T(b_0)$ follows from $(a_U, b_U) \in \mathrm{Graph}(T)$ and the fact that $\mathrm{Graph}(T)$ is closed. □

Corollary 2.15 *Suppose that A is a p-convex subset of a locally p-convex space X and that $T : A \multimap A$ satisfies the following*:
(a) *T is compact and closed,*
(b) *$T^l(x)$ (resp. $T^u(x)$) is open for all $x \in X$,*
(c) *$T(a)$ is nonempty and p-convex for all $a \in A$.*
Then T has a fixed point.

Proof It is an immediate consequence of Corollary 2.10 and Theorem 2.14. □

Remark 2.16 Corollary 2.13 is a generalization of the main results of Himmelberg [26]. Theorem 2.12 for $p = 1$ goes back to Park [30]. Further, Theorem 2.14 for $p = 1$ is an extension of Himmelberg's theorem (see, *e.g.*, [34]).

For a set-valued map $T : X \multimap Y$, set $T_B = \{x \in X : x \in T(x) + B\}$ for $B \subseteq Y$.

Lemma 2.17 *Suppose that A is a p-convex subset of a topological vector space X, and also suppose that $\mathcal{U}$ is a fundamental system of open neighborhoods of* 0. *Then, for a set-valued map $T : A \multimap X$, the following are equivalent*:
(a) *If $a \in A$ satisfies $a \notin T(a) + U$ for some $U \in \mathcal{U}$, then*

$$a \notin \mathrm{Cl}\big(\{a \in A : a \in T(a) + C_p(V)\}\big) \quad \textit{for some } V \in \mathcal{U},$$

(b) $\bigcap_{U\in\mathcal{U}} T_U = \bigcap_{U\in\mathcal{U}} \overline{T_{C_p(U)}}$.

Proof It is straightforward. □

Remark 2.18 The conditions (a) and (b) considered in Lemma 2.17 for $p = 1$ are due to Kim [35].

Theorem 2.19 *Let A be a p-convex compact subset of a topological vector space X, and let $T : A \multimap X$ be a mapping satisfying the following conditions*:
(a) *T has the p-convexly almost fixed point property,*
(b) $\bigcap_{U\in\mathcal{U}} T_U = \bigcap_{U\in\mathcal{U}} \overline{T_{C_p(U)}}$.
Then $\overline{T}$ has a fixed point.

Proof Suppose that $\mathcal{U}$ is a fundamental system of open neighborhoods of 0. Since T has the p-convexly almost fixed point property, for any $U \in \mathcal{U}$, there is an $a_U \in A$ such that $a_U \in T(a_U) + C_p(U)$. Hence, $T_{C_p(U)} \neq \emptyset$ for each $U \in \mathcal{U}$. Now, since $\mathcal{U}$ is a fundamental system of open neighborhoods of 0, we deduce that for any $U, V \in \mathcal{U}$, there is $W \in \mathcal{U}$ such that

$$T_{C_p(U)} \cap T_{C_p(V)} \supseteq T_{C_p(U \cap V)} \supseteq T_{C_p(W)} \neq \emptyset.$$

Therefore $\{T_{C_p(U)} : U \in \mathcal{U}\}$ has the finite intersection property. It follows from the compactness of A that $\bigcap_{U \in \mathcal{U}} \overline{T_{C_p(U)}} \neq \emptyset$. Therefore, by the condition (b) there is an $a_0 \in A$ for which $a_0 \in \bigcap_{U \in \mathcal{U}} T_U$, that is, $a_0 \in T(a_0) + U$ for all $U \in \mathcal{U}$. Regarding $\bigcap_{U \in \mathcal{U}} (T(a_0) + U) = \overline{T(a_0)}$, we derive that $\overline{T}$ has a fixed point. □

Corollary 2.20 *Let A be a p-convex compact subset of a topological vector space X, and let $T : A \multimap X$ be a mapping such that*
(a) *T has the p-convexly almost fixed point property,*
(b) $\bigcap_{U \in \mathcal{U}} T_U = \bigcap_{U \in \mathcal{U}} \overline{T_{C_p(U)}}$,
(c) *T has closed values.*
Then T has a fixed point.

Corollary 2.21 *Let A be a p-convex compact subset of a topological vector space X, and let $T : A \multimap A$ be a mapping such that*
(a) *T is lower (resp. upper) semicontinuous,*
(b) *T has p-convex values,*
(c) $\bigcap_{U \in \mathcal{U}} T_U = \bigcap_{U \in \mathcal{U}} \overline{T_{C_p(U)}}$.
Then $\overline{T}$ has a fixed point.

Proof Since A is a p-convex and compact, by (a) and (b) one can see that all the conditions of Corollary 2.8 hold. Then T has the p-convexly almost fixed point property. The fact that $\overline{T}$ has a fixed point follows from Theorem 2.19. □

Corollary 2.22 *Let A be a p-convex compact subset of a topological vector space X, and let $T : A \multimap A$ be a mapping satisfying the following conditions:*
(a) *T is lower (resp. upper) semicontinuous,*
(b) *T has closed p-convex values,*
(c) $\bigcap_{U \in \mathcal{U}} T_U = \bigcap_{U \in \mathcal{U}} \overline{T_{C_p(U)}}$.
Then T has a fixed point.

Remark 2.23 Corollary 2.22 for $p = 1$ and lower semicontinuous set-valued maps goes back to Kim [35] and Park [36], and also this result for $p = 1$ and upper semicontinuous set-valued maps is due to Huang and Jeng [37].

Theorem 2.24 *Let A be a compact p-convex subset of a locally p-convex space X, and let the set-valued map $T : A \multimap A$ be a mapping such that*
(a) *T has the p-convexly almost fixed point property,*
(b) *T is a closed set-valued map.*
Then T has a fixed point.

Proof Suppose that $\mathcal{U}$ is a fundamental system of p-convex open neighborhoods of 0. Then, for any $U \in \mathcal{U}$, there is $V \in \mathcal{U}$ for which $V \subseteq \overline{V} \subseteq U$. Now, we claim that $T_{C_p(\overline{V})} = T_{\overline{V}}$ is closed. To see this, let $a \in \overline{T_{\overline{V}}}$. There is a net $\{a_i : i \in I\} \subseteq T_{\overline{V}}$ for which $a_i \longrightarrow a$. Then, for each $i \in I$, there exists $b_i \in T(a_i)$ in which $a_i - b_i \in \overline{V}$. Since T is compact and since $b_i \in T(A)$, so one can assume that $b_i \longrightarrow b$ for some $b \in \overline{T(A)}$, and so $a - b \in \overline{V}$. $b \in T(a)$, because T is closed. Therefore,

$$a \in (b + \overline{V}) \cap A \subseteq \big(T(a) + \overline{V}\big) \cap A;$$

i.e., $a \in T_{\overline{V}}$. Finally, since $T_{\overline{V}}$ is closed, and $V \subseteq \overline{V} \subseteq U$, so

$$\bigcap_{U\in\mathcal{U}} T_{C_p(U)} = \bigcap_{U\in\mathcal{U}} T_U = \bigcap_{V\in\mathcal{U}} T_{\overline{V}} = \bigcap_{V\in\mathcal{U}} \overline{T_{\overline{V}}} = \bigcap_{U\in\mathcal{U}} \overline{T_U} = \bigcap_{U\in\mathcal{U}} \overline{T_{C_p(U)}}.$$

Consequently, all the conditions of Corollary 2.20 hold and hence T has a fixed point. □

Remark 2.25 Theorem 2.24 is a generalization of the Fan-Glicksberg theorem [38, 39] and its convex version can be found in [34]. Notice also that Theorem 2.24 can be derived from Theorem 2.14.

Competing interests
The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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