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Strong convergence to a fixed point of a total asymptotically nonexpansive mapping

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Abstract

In this paper, we prove strong convergence for the modified Ishikawa iteration process of a total asymptotically nonexpansive mapping satisfying condition (**A**) in a real uniformly convex Banach space. Our result generalizes the results due to Rhoades (J. Math. Anal. Appl. 183:118-120, 1994).

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1 Introduction

Let X be a real Banach space, let C be a nonempty closed convex subset of X, and let T be a mapping of C into itself. Then T is said to be *asymptotically nonexpansive* [2] if there exists a sequence $\{k_n\}$, $k_n \geq 1$, with $\lim_{n\to\infty} k_n = 1$, such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \tag{1.1}$$

for all $x, y \in C$ and $n \geq 1$. T is said to be *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|$$

for all $x, y \in C$ and $n \geq 1$. If T is asymptotically nonexpansive, then it is uniformly L-Lipschitzian. We denote by $\mathbb{N}$ the set of all positive integers. T is said to be *total asymptotically nonexpansive* (in brief, TAN) [3] if there exist two nonnegative real sequences $\{c_n\}$ and $\{d_n\}$ with $c_n, d_n \to 0$ as $n \to \infty$, $\phi \in \Gamma(R^+)$ such that

$$\|T^n x - T^n y\| \leq \|x - y\| + c_n \phi\big(\|x - y\|\big) + d_n, \tag{1.2}$$

for all $x, y \in C$ and $n \geq 1$, where $R^+ := [0, \infty)$ and $\phi \in \Gamma(R^+)$ if and only if ϕ is strictly increasing, continuous on R^+ and $\phi(0) = 0$. It is clear that if we take $\phi(t) = t$ for all $t \geq 0$ and $d_n = 0$ for all $n \geq 1$ in (1.2), it is reduced to (1.1). Approximating fixed points of the modified Ishikawa iterative scheme under total asymptotically nonexpansive mappings has been investigated by several authors; see, for example, Chidume and Ofoedu [4, 5], Kim [6], Kim and Kim [7] and others. For a mapping T of C into itself in a Hilbert space, Schu [8] considered the following modified Ishikawa iteration process (*cf.* Ishikawa [9]) in C defined by

$$\begin{aligned} &x_1 \in C, \\ &x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, \\ &y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \end{aligned} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0,1]$. If $\beta_n = 0$ for all $n \geq 1$, then iteration process (1.3) becomes the following modified Mann iteration process (*cf.* Mann [10]):

$$\begin{aligned} &x_1 \in C, \\ &x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \end{aligned} \tag{1.4}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$.

Rhoades [1] proved the following results which extended Theorems 1.5 and 2.3 of Schu [8] to uniformly convex Banach spaces.

Theorem 1.1 *Let X be a uniformly convex Banach space, let C be a nonempty bounded closed convex subset of X, and let $T : C \to C$ be a completely continuous asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $k_n \geq 1$, $\sum_{n=1}^{\infty}(k_n^r - 1) < \infty$, $r = \max\{2, p\}$. Then, for any $x_1 \in C$, the sequence $\{x_n\}$ defined by* (1.4), *where $\{\alpha_n\}$ satisfies $a \leq \alpha_n \leq 1-a$ for all $n \geq 1$ and some $a > 0$, converges strongly to some fixed point of T.*

Theorem 1.2 *Let X be a uniformly convex Banach space, let C be a nonempty bounded closed convex subset of E, and let $T : C \to C$ be a completely continuous asymptotically nonexpansive mapping with $\{k_n\}$ satisfying $k_n \geq 1$, $\sum_{n=1}^{\infty}(k_n^r - 1) < \infty$, $r = \max\{2, p\}$. Then, for any $x_1 \in C$, the sequence $\{x_n\}$ defined by* (1.3), *where $\{\alpha_n\}$, $\{\beta_n\}$ satisfy $a \leq (1-\alpha_n), (1-\beta_n) \leq 1-a$ for all $n \geq 1$ and some $a > 0$, converges strongly to some fixed point of T.*

On the other hand, Kim [11] proved the following result which generalized Theorem 1 of Senter and Dotson [12].

Theorem 1.3 *Let X be a real uniformly convex Banach space, let C be a nonempty closed convex subset of X, and let T be a nonexpansive mapping of C into itself satisfying condition* (**A**) *with $F(T) \neq \emptyset$. Suppose that for any x_1 in C, the sequence $\{x_n\}$ is defined by $x_{n+1} = (1-\alpha_n)x_n + \alpha_n[\beta_n x_n + (1-\beta_n)Tx_n]$, for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n) = \infty$ and $\sum_{n=1}^{\infty}\beta_n < \infty$. Then $\{x_n\}$ converges strongly to some fixed point of T.*

In this paper, we prove that if T is a total asymptotically nonexpansive self-mapping satisfying condition (**A**), the iteration $\{x_n\}$ defined by (1.3) converges strongly to some fixed point of T, which generalizes the results due to Rhoades [1].

2 Preliminaries

Throughout this paper, we denote by X a real Banach space. Let C be a nonempty closed convex subset of X, and let T be a mapping from C into itself. Then we denote by $F(T)$ the set of all fixed points of T, *i.e.*, $F(T) = \{x \in C : Tx = x\}$. We also denote by $a \vee b := \max\{a, b\}$. A Banach space X is said to be *uniformly convex* if the modulus of convexity $\delta_X = \delta_X(\epsilon)$, $0 < \epsilon \leq 2$, of X defined by

$$\delta_X(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : x, y \in X, \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon\right\}$$

satisfies the inequality $\delta_X(\epsilon) > 0$ for every $\epsilon \in (0, 2]$. When $\{x_n\}$ is a sequence in X, then $x_n \to x$ will denote strong convergence of the sequence $\{x_n\}$ to x.

Definition 2.1 [12] A mapping $T : C \to C$ with $F(T) \neq \emptyset$ is said to satisfy condition (**A**) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$\|x - Tx\| \geq f\bigl(d\bigl(x, F(T)\bigr)\bigr)$$

for all $x \in C$, where $d(x, F(T)) = \inf_{z \in F(T)} \|x - z\|$.

3 Strong convergence theorem

We first begin with the following lemma.

Lemma 3.1 [13] *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of nonnegative real numbers such that $\sum_{n=1}^{\infty} b_n < \infty$, $\sum_{n=1}^{\infty} c_n < \infty$ and*

$$a_{n+1} \leq (1 + b_n)a_n + c_n$$

for all $n \geq 1$. Then $\lim_{n\to\infty} a_n$ exists.

Lemma 3.2 [14] *Let X be a uniformly convex Banach space. Let $x, y \in X$. If $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \epsilon > 0$, then $\|\lambda x + (1 - \lambda)y\| \leq 1 - 2\lambda(1 - \lambda)\delta(\epsilon)$ for $0 \leq \lambda \leq 1$.*

Lemma 3.3 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X, and let $T : C \to C$ be a TAN mapping with $F(T) \neq \emptyset$. Suppose that $\{c_n\}$, $\{d_n\}$ and ϕ satisfy the following two conditions*:

(I) $\exists \alpha, \beta > 0$ *such that $\phi(t) \leq \alpha t$ for all $t \geq \beta$.*

(II) $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$.

Suppose that the sequence $\{x_n\}$ is defined by (1.3). *Then $\lim_{n\to\infty} \|x_n - z\|$ exists for any $z \in F(T)$.*

Proof For any $z \in F(T)$, we set

$$M := 1 \vee \phi(\beta) < \infty.$$

From (I) and strict increasing of ϕ, we obtain

$$\phi(t) \leq \phi(\beta) + \alpha t, \quad t \geq 0. \tag{3.1}$$

By using (3.1), we have

$$\begin{aligned}\|T^n x_n - z\| &\le \|x_n - z\| + c_n\phi(\|x_n - z\|) + d_n \\ &\le \|x_n - z\| + c_n\{\phi(\beta) + \alpha\|x_n - z\|\} + d_n \\ &\le (1+\alpha c_n)\|x_n - z\| + \kappa_n M,\end{aligned}$$

where $\kappa_n = c_n + d_n$ and $\sum_{n=1}^{\infty}\kappa_n < \infty$. Since

$$\begin{aligned}\|y_n - z\| &= \|\beta_n T^n x_n + (1-\beta_n)x_n - z\| \\ &\le \beta_n\|T^n x_n - z\| + (1-\beta_n)\|x_n - z\| \\ &\le \beta_n\{(1+\alpha c_n)\|x_n - z\| + \kappa_n M\} + (1-\beta_n)\|x_n - z\| \\ &\le (1+\alpha c_n)\|x_n - z\| + \kappa_n M,\end{aligned}$$

and thus

$$\begin{aligned}&\|y_n - z\| + c_n\phi(\|y_n - z\|) \\ &\quad\le (1+\alpha c_n)\|x_n - z\| + \kappa_n M + c_n\{\phi(\beta) + \alpha\|y_n - z\|\} \\ &\quad\le (1+\alpha c_n)\|x_n - z\| + \kappa_n M + c_n\phi(\beta) + \alpha c_n(1+\alpha c_n)\|x_n - z\| + \alpha c_n\kappa_n M \\ &\quad\le (1+\sigma_n)\|x_n - z\| + \delta_n M,\end{aligned}$$

where $\sigma_n = 2\alpha c_n + \alpha^2 c_n^2$, $\delta_n = \kappa_n + c_n + \alpha c_n\kappa_n$, $\sum_{n=1}^{\infty}\sigma_n < \infty$ and $\sum_{n=1}^{\infty}\delta_n < \infty$. So, we have

$$\begin{aligned}\|T^n y_n - z\| &\le \|y_n - z\| + c_n\phi(\|y_n - z\|) + d_n \\ &\le (1+\sigma_n)\|x_n - z\| + \delta_n M + d_n \\ &\le (1+\sigma_n)\|x_n - z\| + \eta_n M,\end{aligned}$$

where $\eta_n = \delta_n + d_n$ and $\sum_{n=1}^{\infty}\eta_n < \infty$. Hence

$$\begin{aligned}\|x_{n+1} - z\| &= \|(1-\alpha_n)x_n + \alpha_n T^n y_n - z\| \\ &\le (1-\alpha_n)\|x_n - z\| + \alpha_n\|T^n y_n - z\| \\ &\le (1-\alpha_n)\|x_n - z\| + \alpha_n\{(1+\sigma_n)\|x_n - z\| + \eta_n M\} \\ &\le (1+\sigma_n)\|x_n - z\| + \eta_n M.\end{aligned}$$

By Lemma 3.1, we see that $\lim_{n\to\infty}\|x_n - z\|$ exists. □

Theorem 3.4 *Let X be a uniformly convex Banach space, and let C be a nonempty closed convex subset of X. Let $T : C \to C$ be a uniformly continuous and TAN mapping with $F(T) \neq \emptyset$. Suppose that $\{c_n\}$, $\{d_n\}$ and ϕ satisfy the following two conditions*:

(I) $\exists\alpha, \beta > 0$ *such that* $\phi(t) \le \alpha t$ *for all* $t \ge \beta$.

(II) $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$.

Suppose that for any x_1 in C, the sequence $\{x_n\}$ defined by (1.3) *satisfies* $\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n)=\infty$ *and* $\lim\beta_n=0$. *Then $\{x_n\}$ converges strongly to some fixed point of T.*

Proof For any $z\in F(T)$, by Lemma 3.3, $\{x_n\}$ is bounded. We set

$$M:=1\vee\phi(\beta)\vee\sup_{n\geq 1}\|x_n-z\|<\infty.$$

By Lemma 3.3, we see that $\lim_{n\to\infty}\|x_n-z\|$ ($\equiv r$) exists. Without loss of generality, we assume $r>0$. As in the proof of Lemma 3.3, we obtain

$$\begin{aligned}\left\|T^ny_n-z\right\| &\leq(1+\sigma_n)\|x_n-z\|+\eta_nM\\ &\leq\|x_n-z\|+\nu_nM,\end{aligned}$$

where $\nu_n=\sigma_n+\eta_n$ and $\sum_{n=1}^{\infty}\nu_n<\infty$. By using Lemma 3.2 and Takahashi [15], we obtain

$$\begin{aligned}\|x_{n+1}-z\| &= \left\|(1-\alpha_n)x_n+\alpha_nT^ny_n-z\right\|\\ &= \left\|(1-\alpha_n)(x_n-z)+\alpha_n\left(T^ny_n-z\right)\right\|\\ &\leq\left(\|x_n-z\|+\nu_nM\right)\left[1-2\alpha_n(1-\alpha_n)\delta_X\left(\frac{\|T^ny_n-x_n\|}{\|x_n-z\|+\nu_nM}\right)\right].\end{aligned}$$

Hence we obtain

$$\begin{aligned}&2\alpha_n(1-\alpha_n)\left(\|x_n-z\|+\nu_nM\right)\delta_X\left(\frac{\|T^ny_n-x_n\|}{\|x_n-z\|+\nu_nM}\right)\\ &\quad\leq\|x_n-z\|-\|x_{n+1}-z\|+\nu_nM.\end{aligned}$$

Thus

$$2\alpha_n(1-\alpha_n)\left(\|x_n-z\|+\nu_nM\right)\delta_X\left(\frac{\|T^ny_n-x_n\|}{\|x_n-z\|+\nu_nM}\right)<\infty.$$

Since δ_X is strictly increasing, continuous and $\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n)=\infty$, we obtain

$$\liminf_{n\to\infty}\left\|T^ny_n-x_n\right\|=0. \tag{3.2}$$

By using (3.1) in the proof of Lemma 3.3, we have

$$\begin{aligned}\left\|T^{n-1}x_{n-1}-z\right\| &\leq\|x_{n-1}-z\|+c_{n-1}\phi\left(\|x_{n-1}-z\|\right)+d_{n-1}\\ &\leq\|x_{n-1}-z\|+c_{n-1}\left\{\phi(\beta)+\alpha\|x_{n-1}-z\|\right\}+d_{n-1}\\ &\leq(1+\alpha c_{n-1})\|x_{n-1}-z\|+\rho_{n-1}M,\end{aligned}$$

where $\rho_{n-1}=c_{n-1}+d_{n-1}$ and $\sum_{n=2}^{\infty}\rho_{n-1}<\infty$. Thus

$$\begin{aligned}\|y_{n-1}-z\| &= \left\|\beta_{n-1}T^{n-1}x_{n-1}+(1-\beta_{n-1})x_{n-1}-z\right\|\\ &\leq\beta_{n-1}\left\|T^{n-1}x_{n-1}-z\right\|+(1-\beta_{n-1})\|x_{n-1}-z\|\end{aligned}$$

$$\begin{aligned}
&\leq \beta_{n-1}\big\{(1+\alpha c_{n-1})\|x_{n-1}-z\| + \rho_{n-1}M\big\} + (1-\beta_{n-1})\|x_{n-1}-z\| \\
&\leq (1+\alpha c_{n-1})\|x_{n-1}-z\| + \rho_{n-1}M,
\end{aligned}$$

and hence

$$\begin{aligned}
&\|y_{n-1}-z\| + c_{n-1}\phi\big(\|y_{n-1}-z\|\big) \\
&\quad\leq (1+\alpha c_{n-1})\|x_{n-1}-z\| + \rho_{n-1}M + c_{n-1}\big\{\phi(\beta) + \alpha\|y_{n-1}-z\|\big\} \\
&\quad\leq (1+\alpha c_{n-1})\|x_{n-1}-z\| + \rho_{n-1}M + c_{n-1}\phi(\beta) + \alpha c_{n-1}(1+\alpha c_{n-1})\|x_{n-1}-z\| \\
&\qquad + \alpha c_{n-1}\rho_{n-1}M \\
&\quad\leq (1+\mu_{n-1})\|x_{n-1}-z\| + \varphi_{n-1}M,
\end{aligned}$$

where $\mu_{n-1} = 2\alpha c_{n-1} + \alpha^2 c_{n-1}^2$, $\varphi_{n-1} = \rho_{n-1} + c_{n-1} + \alpha c_{n-1}\rho_{n-1}$, $\sum_{n=2}^{\infty}\mu_{n-1} < \infty$ and $\sum_{n=2}^{\infty}\varphi_{n-1} < \infty$. So, we have

$$\begin{aligned}
\big\|T^{n-1}y_{n-1}-z\big\| &\leq \|y_{n-1}-z\| + c_{n-1}\phi\big(\|y_{n-1}-z\|\big) + d_{n-1} \\
&\leq (1+\mu_{n-1})\|x_{n-1}-z\| + \varphi_{n-1}M + d_{n-1} \\
&\leq \|x_{n-1}-z\| + \omega_{n-1}M,
\end{aligned}$$

where $\omega_{n-1} = \mu_{n-1} + \varphi_{n-1} + d_{n-1}$ and $\sum_{n=2}^{\infty}\omega_{n-1} < \infty$. By using Lemma 3.2 and Takahashi [15], we obtain

$$\begin{aligned}
\|x_n - z\| &= \big\|(1-\alpha_{n-1})x_{n-1} + \alpha_{n-1}T^{n-1}y_{n-1} - z\big\| \\
&= \big\|(1-\alpha_{n-1})(x_{n-1}-z) + \alpha_{n-1}\big(T^{n-1}y_{n-1}-z\big)\big\| \\
&\leq \big(\|x_{n-1}-z\| + \omega_{n-1}M\big)\left[1 - 2\alpha_n(1-\alpha_n)\delta_X\left(\frac{\|T^{n-1}y_{n-1}-x_{n-1}\|}{\|x_{n-1}-z\| + \omega_{n-1}M}\right)\right].
\end{aligned}$$

By the same method as above, we obtain

$$\liminf_{n\to\infty}\big\|T^{n-1}y_{n-1} - x_{n-1}\big\| = 0. \tag{3.3}$$

Since $\{x_n\}$ is bounded and T is a TAN mapping, we obtain

$$\begin{aligned}
\|y_n - x_n\| &= \big\|\beta_n T^n x_n + (1-\beta_n)x_n - x_n\big\| \\
&\leq \beta_n\big\|T^n x_n - x_n\big\| \\
&\leq \beta_n M',
\end{aligned}$$

where $M' = \sup_{n\geq 1}\|T^n x_n - x_n\| < \infty$. By using $\lim\beta_n = 0$, we have

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{3.4}$$

Since

$$\big\|T^n y_n - y_n\big\| \leq \big\|T^n y_n - x_n\big\| + \|x_n - y_n\|,$$

by (3.2) and (3.4), we obtain

$$\liminf_{n\to\infty}\left\|T^n y_n - y_n\right\| = 0. \tag{3.5}$$

By using (3.3) and (3.4), we obtain

$$\liminf_{n\to\infty}\left\|T^{n-1} y_{n-1} - y_{n-1}\right\| = 0. \tag{3.6}$$

Since

$$\begin{aligned}\left\|T^{n-1}x_{n-1} - x_{n-1}\right\| &\le \left\|T^{n-1}x_{n-1} - T^{n-1}y_{n-1}\right\| + \left\|T^{n-1}y_{n-1} - x_{n-1}\right\| \\ &\le \|x_{n-1} - y_{n-1}\| + c_{n-1}\phi\big(\|x_{n-1} - y_{n-1}\|\big) + d_{n-1} \\ &\quad + \left\|T^{n-1}y_{n-1} - x_{n-1}\right\|,\end{aligned}$$

by using (3.3) and (3.4), we have

$$\liminf_{n\to\infty}\left\|T^{n-1} x_{n-1} - x_{n-1}\right\| = 0. \tag{3.7}$$

Since

$$\begin{aligned}\|x_n - x_{n-1}\| &= \left\|(1-\alpha_{n-1})x_{n-1} + \alpha_{n-1}T^{n-1}y_{n-1} - x_{n-1}\right\| \\ &= \alpha_{n-1}\left\|T^{n-1}y_{n-1} - x_{n-1}\right\| \\ &\le \left\|T^{n-1}y_{n-1} - y_{n-1}\right\| + \|y_{n-1} - x_{n-1}\|,\end{aligned}$$

by (3.4) and (3.6), we get

$$\liminf_{n\to\infty}\|x_n - x_{n-1}\| = 0. \tag{3.8}$$

From

$$\begin{aligned}\left\|T^{n-1}x_n - x_n\right\| &\le \left\|T^{n-1}x_n - T^{n-1}x_{n-1}\right\| + \left\|T^{n-1}x_{n-1} - x_{n-1}\right\| + \|x_{n-1} - x_n\| \\ &\le 2\|x_n - x_{n-1}\| + c_{n-1}\phi\big(\|x_n - x_{n-1}\|\big) + d_{n-1} + \left\|T^{n-1}x_{n-1} - x_{n-1}\right\|,\end{aligned}$$

by (3.7) and (3.8), we obtain

$$\liminf_{n\to\infty}\left\|T^{n-1} x_n - x_n\right\| = 0. \tag{3.9}$$

Since

$$\begin{aligned}&\|x_n - Tx_n\| \\ &\quad\le \|x_n - y_n\| + \left\|y_n - T^n y_n\right\| + \left\|T^n y_n - T^n x_n\right\| + \left\|T^n x_n - Tx_n\right\| \\ &\quad\le \left\|y_n - T^n y_n\right\| + 2\|x_n - y_n\| + c_n\phi\big(\|x_n - y_n\|\big) + d_n + \left\|T^n x_n - Tx_n\right\|\end{aligned}$$

and by the uniform continuity of T, (3.4), (3.5) and (3.9), we have

$$\liminf_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.10}$$

By using condition (**A**), we obtain

$$f\bigl(d\bigl(x_n, F(T)\bigr)\bigr) \le \|x_n - Tx_n\| \tag{3.11}$$

for all $n \ge 1$. As in the proof of Lemma 3.3, we obtain

$$\|x_{n+1} - z\| \le (1 + \sigma_n)\|x_n - z\| + \eta_n M. \tag{3.12}$$

Thus

$$\inf_{z\in F(T)} \|x_{n+1} - z\| \le (1 + \sigma_n) \inf_{z\in F(T)} \|x_n - z\| + \eta_n M.$$

By using Lemma 3.1, we see that $\lim_{n\to\infty} d(x_n, F(T))$ ($\equiv c$) exists. We first claim that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. In fact, assume that $c = \lim_{n\to\infty} d(x_n, F(T)) > 0$. Then we can choose $n_0 \in \mathbb{N}$ such that $0 < \frac{c}{2} < d(x_n, F(T))$ for all $n \ge n_0$. By using condition (**A**), (3.10) and (3.11), we obtain

$$0 < f\left(\frac{c}{2}\right) \le f\bigl(d\bigl(x_{n_i}, F(T)\bigr)\bigr) \le \|x_{n_i} - Tx_{n_i}\| \to 0$$

as $i \to \infty$. This is a contradiction. So, we obtain $c = 0$. Next, we claim that $\{x_n\}$ is a Cauchy sequence. Since $\sum_{n=1}^{\infty} \sigma_n < \infty$, we obtain $\prod_{n=1}^{\infty}(1 + \sigma_n) := U < \infty$. Let $\epsilon > 0$ be given. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$ and $\sum_{n=1}^{\infty} \eta_n < \infty$, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, we obtain

$$d\bigl(x_n, F(T)\bigr) < \frac{\epsilon}{4U + 4} \quad \text{and} \quad \sum_{i=n_0}^{\infty} \eta_i < \frac{\epsilon}{4M}. \tag{3.13}$$

Let $n, m \ge n_0$ and $p \in F(T)$. Then, by (3.12), we obtain

$$\begin{aligned} \|x_n - x_m\| &\le \|x_n - p\| + \|x_m - p\| \\ &\le \prod_{i=n_0}^{n-1}(1 + \sigma_i)\|x_{n_0} - p\| + M\sum_{i=n_0}^{n-1}\eta_i + \prod_{i=n_0}^{m-1}(1 + \sigma_i)\|x_{n_0} - p\| + M\sum_{i=n_0}^{m-1}\eta_i \\ &\le 2\left[\prod_{i=n_0}^{\infty}(1 + \sigma_i)\|x_{n_0} - p\| + M\sum_{i=n_0}^{\infty}\eta_i\right]. \end{aligned}$$

Taking the infimum over all $p \in F(T)$ on both sides and by (3.13), we obtain

$$\begin{aligned} \|x_n - x_m\| &\le 2\left[\prod_{i=n_0}^{\infty}(1 + \sigma_i)d\bigl(x_{n_0}, F(T)\bigr) + M\sum_{i=n_0}^{\infty}\eta_i\right] \\ &< 2\left[(U + 1)\frac{\epsilon}{4U + 4} + M\frac{\epsilon}{4M}\right] = \epsilon \end{aligned}$$

for all $n, m \geq n_0$. This implies that $\{x_n\}$ is a Cauchy sequence. Let $\lim_{n\to\infty} x_n = q$. Then $d(q, F(T)) = 0$. Since $F(T)$ is closed, we obtain $q \in F(T)$. Hence $\{x_n\}$ converges strongly to some fixed point of T. □

Remark 3.5 If $T : C \to C$ is completely continuous, then it satisfies demicompact and, if T is continuous and demicompact, it satisfies condition (**A**); see Senter and Dotson [12].

Remark 3.6 If $\{\alpha_n\}$ is bounded away from both 0 and 1, *i.e.*, $a \leq \alpha_n \leq b$ for all $n \geq 1$ and some $a, b \in (0,1)$, then $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$ and $\lim_{n\to\infty} \beta_n = 0$ hold. However, the converse is not true. For example, consider $\alpha_n = \frac{1}{n}$.

We give an example of a mapping $T : C \to C$ which satisfies all the assumptions of T in Theorem 3.4, *i.e.*, $T : C \to C$ is a uniformly continuous mapping with $F(T) \neq \emptyset$ which is TAN on C, not Lipschitzian and hence not asymptotically nonexpansive.

Example 3.7 Let $X := \mathbb{R}$ and $C := [0,2]$. Define $T : C \to C$ by

$$Tx = \begin{cases} 1, & x \in [0,1]; \\ \frac{1}{\sqrt{3}}\sqrt{4 - x^2}, & x \in [1,2]. \end{cases}$$

Note that $T^n x = 1$ for all $x \in C$ and $n \geq 2$ and $F(T) = \{1\}$. Clearly, T is both uniformly continuous and TAN on C. We show that T satisfies condition (**A**). In fact, if $x \in [0,1]$, then $|x - 1| = |x - Tx|$. Similarly, if $x \in [1,2]$, then

$$|x - 1| = x - 1 \leq x - \frac{1}{\sqrt{3}}\sqrt{4 - x^2} = |x - Tx|.$$

So, we get $d(x, F(T)) = |x - 1| \leq |x - Tx|$ for all $x \in C$. But T is not Lipschitzian. Indeed, suppose not, *i.e.*, there exists $L > 0$ such that

$$|Tx - Ty| \leq L|x - y|$$

for all $x, y \in C$. If we take $x = 2 - \frac{1}{3(L+1)^2} > 1$ and $y = 2$, then

$$\frac{1}{\sqrt{3}}\sqrt{4 - x^2} \leq L(2 - x) \quad \Leftrightarrow \quad \frac{1}{3L^2} \leq \frac{2 - x}{2 + x} = \frac{1}{12L^2 + 24L + 1}.$$

This is a contradiction.

Competing interests
The author declares that they have no competing interests.

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