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Fixed point theorems of contractive mappings of integral type

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Abstract
Three fixed point theorems for three general classes of contractive mappings of integral type in complete metric spaces are proved. Three examples are included.
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1 Introduction

Branciari [1] was the first to study the existence of fixed points for the contractive mapping of integral type. He established a nice integral version of the Banach contraction principle and proved the following fixed point theorem.

Theorem 1.1 *Let f be a mapping from a complete metric space (X,d) into itself satisfying*

$$\int_0^{d(fx,fy)} \varphi(t)\,dt \le c\int_0^{d(x,y)} \varphi(t)\,dt, \quad \forall x,y\in X,$$

where $c\in(0,1)$ is a constant and $\varphi\in\Phi_1$. Then f has a unique fixed point $a\in X$ such that $\lim_{n\to\infty} f^n x = a$ for each $x\in X$.

Afterwards, many authors continued the study of Branciari and obtained many fixed point theorems for several classes of contractive mappings of integral type; see, *e.g.*, [1–8] and the references therein. In particular, in 2011, Liu *et al.* [5] extended the result of Branciari [1] and deduced the following fixed point theorems.

Theorem 1.2 *Let f be a mapping from a complete metric space (X,d) into itself satisfying*

$$\int_0^{d(fx,fy)} \varphi(t)\,dt \le \alpha\bigl(d(x,y)\bigr)\int_0^{d(x,y)} \varphi(t)\,dt, \quad \forall x,y\in X,$$

where $\varphi\in\Phi_1$ and $\alpha:\mathbb{R}^+\to[0,1)$ is a function with

$$\limsup_{s\to t}\alpha(s)<1, \quad \forall t>0.$$

Then f has a unique fixed point $a\in X$ such that $\lim_{n\to\infty} f^n x = a$ for each $x\in X$.

Theorem 1.3 *Let f be a mapping from a complete metric space (X, d) into itself satisfying*

$$\int_0^{d(fx,fy)} \varphi(t)\,dt \le \alpha\big(d(x,y)\big)\int_0^{d(x,fx)} \varphi(t)\,dt + \beta\big(d(x,y)\big)\int_0^{d(y,fy)} \varphi(t)\,dt, \quad \forall x,y \in X,$$

where $\varphi \in \Phi_1$ and $\alpha, \beta : \mathbb{R}^+ \to [0,1)$ are two functions with

$$\alpha(t) + \beta(t) < 1, \quad \forall t \in \mathbb{R}^+, \qquad \limsup_{s\to 0^+} \beta(s) < 1, \qquad \limsup_{s\to t^+} \frac{\alpha(s)}{1-\beta(s)} < 1, \quad \forall t > 0.$$

Then f has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} f^n x = a$ for each $x \in X$.

In 2008, Dutta and Choudhuty [9] proved the following result.

Theorem 1.4 *Let f be a mapping from a complete metric space (X, d) into itself satisfying*

$$\psi\big(d(fx,fy)\big) \le \psi\big(d(x,y)\big) - \varphi\big(d(x,y)\big), \quad \forall x,y \in X,$$

where $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ are both continuous and monotone nondecreasing functions with $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$. Then f has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} f^n x = a$ for each $x \in X$.

However, to the best of our knowledge, no one studied the following contractive mappings of integral type:

$$\psi\left(\int_0^{d(fx,fy)} \varphi(t)\,dt\right) \le \psi\left(\int_0^{d(x,y)} \varphi(t)\,dt\right) - \phi\left(\int_0^{d(x,y)} \varphi(t)\,dt\right), \quad \forall x,y \in X, \tag{1.1}$$

where $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$;

$$\psi\left(\int_0^{d(fx,fy)} \varphi(t)\,dt\right) \le \alpha\big(d(x,y)\big)\psi\left(\int_0^{d(x,y)} \varphi(t)\,dt\right), \quad \forall x,y \in X, \tag{1.2}$$

where $(\varphi, \psi, \alpha) \in \Phi_1 \times \Phi_3 \times \Phi_5$;

$$\begin{aligned}\psi\left(\int_0^{d(fx,fy)} \varphi(t)\,dt\right) &\le \alpha\big(d(x,y)\big)\phi\left(\int_0^{d(x,fx)} \varphi(t)\,dt\right) \\ &\quad + \beta\big(d(x,y)\big)\psi\left(\int_0^{d(y,fy)} \varphi(t)\,dt\right), \quad \forall x,y \in X,\end{aligned} \tag{1.3}$$

where $(\varphi, \psi, \phi) \in \Phi_1 \times \Phi_3 \times \Phi_4$ and $(\alpha, \beta) \in \Phi_6$.

It is clear that the above contractive mappings of integral type include these mappings in Theorems 1.1-1.4 as special cases. The purpose of this paper is to investigate the existence of fixed points for contractive mappings (1.1)-(1.3) of integral type. Under certain conditions, we prove the existence, uniqueness and iterative approximations of fixed points for contractive mappings (1.1)-(1.3) of integral type in complete metric spaces. Three examples with uncountably many points are constructed.

2 Preliminaries

Throughout this paper, we assume that $\mathbb{R}^+ = [0,+\infty)$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N}$ denotes the set of all positive integers, (X,d) is a metric space, $f : X \to X$ is a self-mapping and

$$d_n = d\big(f^n x, f^{n+1}x\big), \quad \forall (n,x) \in \mathbb{N}_0 \times X,$$

$\Phi_1 = \{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is Lebesgue integrable, summable on each compact subset of $\mathbb{R}^+$ and $\int_0^\varepsilon \varphi(t)\,dt > 0$ for each $\varepsilon > 0\}$;
$\Phi_2 = \{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies that $\liminf_{n\to\infty} \varphi(a_n) > 0 \Leftrightarrow \liminf_{n\to\infty} a_n > 0$ for each $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+\}$;
$\Phi_3 = \{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing continuous and $\varphi(t) = 0 \Leftrightarrow t = 0\}$;
$\Phi_4 = \{\varphi : \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies that $\varphi(0) = 0\}$;
$\Phi_5 = \{\varphi : \varphi : \mathbb{R}^+ \to [0,1)$ satisfies that $\limsup_{s\to t} \varphi(s) < 1$ for each $t > 0\}$;
$\Phi_6 = \{(\alpha,\beta) : \alpha,\beta : \mathbb{R}^+ \to [0,1)$ satisfy that $\limsup_{s\to 0^+} \beta(s) < 1$, $\limsup_{s\to t^+} \frac{\alpha(s)}{1-\beta(s)} < 1$ and $\alpha(t) + \beta(t) < 1$ for each $t > 0\}$.

The following lemmas play important roles in this paper.

Lemma 2.1 ([5]) *Let $\varphi \in \Phi_1$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence with $\lim_{n\to\infty} r_n = a$. Then*

$$\lim_{n\to\infty} \int_0^{r_n} \varphi(t)\,dt = \int_0^a \varphi(t)\,dt.$$

Lemma 2.2 ([5]) *Let $\varphi \in \Phi_1$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence. Then*

$$\lim_{n\to\infty} \int_0^{r_n} \varphi(t)\,dt = 0$$

if and only if $\lim_{n\to\infty} r_n = 0$.

Lemma 2.3 *Let $\varphi \in \Phi_2$. Then $\varphi(t) > 0$ if and only if $t > 0$.*

Proof Let $t > 0$. Put $a_n = t$ for each $n \in \mathbb{N}$. It is easy to see that $t = \liminf_{n\to\infty} a_n > 0$, which together with $\varphi \in \Phi_2$ ensures that

$$\varphi(t) = \liminf_{n\to\infty} \varphi(a_n) > 0.$$

Conversely, suppose that $\varphi(t) > 0$ for some $t \in \mathbb{R}^+$. Set $a_n = t$ for each $n \in \mathbb{N}$. It is clear that $\varphi(t) = \liminf_{n\to\infty} \varphi(a_n) > 0$, which together with $\varphi \in \Phi_2$ guarantees that

$$t = \liminf_{n\to\infty} a_n > 0.$$

This completes the proof. □

3 Main results

In this section we show the existence, uniqueness and iterative approximations of fixed points for contractive mappings (1.1)-(1.3) of integral type, respectively.

Theorem 3.1 *Let f be a mapping from a complete metric space (X, d) into itself satisfying* (1.1). *Then f has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} f^n x = a$ for each $x \in X$.*

Proof Let x be an arbitrary point in X. Firstly, we show that

$$d_n \leq d_{n-1}, \quad \forall n \in \mathbb{N}. \tag{3.1}$$

Suppose that (3.1) does not hold. It follows that there exists some $n_0 \in \mathbb{N}$ satisfying

$$d_{n_0} > d_{n_0-1}. \tag{3.2}$$

Note that (3.2) and $\varphi \in \Phi_1$ imply that

$$\int_0^{d_{n_0}} \varphi(t)\,dt > 0. \tag{3.3}$$

Using (1.1), (3.2) and $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$, we conclude immediately that

$$\begin{aligned}
\psi\left(\int_0^{d_{n_0-1}} \varphi(t)\,dt\right) &\leq \psi\left(\int_0^{d_{n_0}} \varphi(t)\,dt\right) \\
&= \psi\left(\int_0^{d(f^{n_0}x, f^{n_0+1}x)} \varphi(t)\,dt\right) \\
&\leq \psi\left(\int_0^{d(f^{n_0-1}x, f^{n_0}x)} \varphi(t)\,dt\right) - \phi\left(\int_0^{d(f^{n_0-1}x, f^{n_0}x)} \varphi(t)\,dt\right) \\
&= \psi\left(\int_0^{d_{n_0-1}} \varphi(t)\,dt\right) - \phi\left(\int_0^{d_{n_0-1}} \varphi(t)\,dt\right) \\
&\leq \psi\left(\int_0^{d_{n_0-1}} \varphi(t)\,dt\right),
\end{aligned}$$

which yields that

$$\psi\left(\int_0^{d_{n_0}} \varphi(t)\,dt\right) = \psi\left(\int_0^{d_{n_0-1}} \varphi(t)\,dt\right) \tag{3.4}$$

and

$$\phi\left(\int_0^{d_{n_0-1}} \varphi(t)\,dt\right) = 0. \tag{3.5}$$

Combining (3.5) and Lemma 2.3, we get that

$$\int_0^{d_{n_0-1}} \varphi(t)\,dt = 0,$$

which together with $\psi \in \Phi_3$ and (3.4) means that

$$\psi\left(\int_0^{d_{n_0}} \varphi(t)\,dt\right) = \psi\left(\int_0^{d_{n_0-1}} \varphi(t)\,dt\right) = \psi(0) = 0,$$

that is,

$$\int_0^{d_{n_0}} \varphi(t)\,dt = 0,$$

which contradicts (3.3). Hence (3.1) holds.

Secondly, we show that

$$\lim_{n\to\infty} d_n = 0. \tag{3.6}$$

In view of (3.1), we deduce that the nonnegative sequence $\{d_n\}_{n\in\mathbb{N}_0}$ is nonincreasing, which means that there exists a constant c with $\lim_{n\to\infty} d_n = c \geq 0$. Suppose that $c > 0$. It follows from (1.1) that

$$\begin{aligned}
\psi\left(\int_0^{d_n} \varphi(t)\,dt\right) &= \psi\left(\int_0^{d(f^n x, f^{n+1}x)} \varphi(t)\,dt\right) \\
&\leq \psi\left(\int_0^{d(f^n x, f^{n-1}x)} \varphi(t)\,dt\right) - \phi\left(\int_0^{d(f^n x, f^{n-1}x)} \varphi(t)\,dt\right) \\
&= \psi\left(\int_0^{d_{n-1}} \varphi(t)\,dt\right) - \phi\left(\int_0^{d_{n-1}} \varphi(t)\,dt\right), \quad \forall n \in \mathbb{N}.
\end{aligned} \tag{3.7}$$

Taking upper limit in (3.7) and using Lemma 2.1 and $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$, we conclude that

$$\begin{aligned}
\psi\left(\int_0^{c} \varphi(t)\,dt\right) &= \limsup_{n\to\infty} \psi\left(\int_0^{d_n} \varphi(t)\,dt\right) \\
&\leq \limsup_{n\to\infty}\left[\psi\left(\int_0^{d_{n-1}} \varphi(t)\,dt\right) - \phi\left(\int_0^{d_{n-1}} \varphi(t)\,dt\right)\right] \\
&\leq \limsup_{n\to\infty} \psi\left(\int_0^{d_{n-1}} \varphi(t)\,dt\right) - \liminf_{n\to\infty} \phi\left(\int_0^{d_{n-1}} \varphi(t)\,dt\right) \\
&= \psi\left(\int_0^{c} \varphi(t)\,dt\right) - \liminf_{n\to\infty} \phi\left(\int_0^{d_{n-1}} \varphi(t)\,dt\right) \\
&< \psi\left(\int_0^{c} \varphi(t)\,dt\right),
\end{aligned}$$

which is a contradiction. Hence $c = 0$.

Thirdly, we show that $\{f^n x\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Suppose that $\{f^n x\}_{n\in\mathbb{N}}$ is not a Cauchy sequence, which means that there is a constant $\varepsilon > 0$ such that for each positive integer k, there are positive integers $m(k)$ and $n(k)$ with $m(k) > n(k) > k$ satisfying

$$d\big(f^{m(k)}x, f^{n(k)}x\big) > \varepsilon. \tag{3.8}$$

For each positive integer k, let $m(k)$ denote the least integer exceeding $n(k)$ and satisfying (3.8). It follows that

$$d\big(f^{m(k)}x, f^{n(k)}x\big) > \varepsilon \quad \text{and} \quad d\big(f^{m(k)-1}x, f^{n(k)}x\big) \leq \varepsilon, \quad \forall k \in \mathbb{N}. \tag{3.9}$$

Note that

$$\begin{aligned}
&d\bigl(f^{m(k)}x,f^{n(k)}x\bigr) \le d\bigl(f^{n(k)}x,f^{m(k)-1}x\bigr) + d_{m(k)-1}, \quad \forall k \in \mathbb{N};\\
&\bigl|d\bigl(f^{m(k)}x,f^{n(k)+1}x\bigr) - d\bigl(f^{m(k)}x,f^{n(k)}x\bigr)\bigr| \le d_{n(k)}, \quad \forall k \in \mathbb{N};\\
&\bigl|d\bigl(f^{m(k)+1}x,f^{n(k)+1}x\bigr) - d\bigl(f^{m(k)}x,f^{n(k)+1}x\bigr)\bigr| \le d_{m(k)}, \quad \forall k \in \mathbb{N};\\
&\bigl|d\bigl(f^{m(k)+1}x,f^{n(k)+1}x\bigr) - d\bigl(f^{m(k)+1}x,f^{n(k)+2}x\bigr)\bigr| \le d_{n(k)+1}, \quad \forall k \in \mathbb{N}.
\end{aligned} \tag{3.10}$$

In light of (3.9) and (3.10), we get that

$$\begin{aligned}
\varepsilon &= \lim_{k\to\infty} d\bigl(f^{n(k)}x,f^{m(k)}x\bigr) = \lim_{k\to\infty} d\bigl(f^{m(k)}x,f^{n(k)+1}x\bigr)\\
&= \lim_{k\to\infty} d\bigl(f^{m(k)+1}x,f^{n(k)+1}x\bigr) = \lim_{k\to\infty} d\bigl(f^{m(k)+1}x,f^{n(k)+2}x\bigr).
\end{aligned} \tag{3.11}$$

In view of (1.1), we deduce that

$$\begin{aligned}
&\psi\left(\int_0^{d(f^{m(k)+1}x,f^{n(k)+2}x)} \varphi(t)\,dt\right)\\
&\quad\le \psi\left(\int_0^{d(f^{m(k)}x,f^{n(k)+1}x)} \varphi(t)\,dt\right) - \phi\left(\int_0^{d(f^{m(k)}x,f^{n(k)+1}x)} \varphi(t)\,dt\right), \quad \forall k \in \mathbb{N}.
\end{aligned} \tag{3.12}$$

Taking upper limit in (3.12) and using (3.11), $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemma 2.1, we deduce that

$$\begin{aligned}
&\psi\left(\int_0^{\varepsilon} \varphi(t)\,dt\right)\\
&\quad= \limsup_{k\to\infty} \psi\left(\int_0^{d(f^{m(k)+1}x,f^{n(k)+2}x)} \varphi(t)\,dt\right)\\
&\quad\le \limsup_{k\to\infty} \left[\psi\left(\int_0^{d(f^{m(k)}x,f^{n(k)+1}x)} \varphi(t)\,dt\right) - \phi\left(\int_0^{d(f^{m(k)}x,f^{n(k)+1}x)} \varphi(t)\,dt\right)\right]\\
&\quad\le \limsup_{k\to\infty} \psi\left(\int_0^{d(f^{m(k)}x,f^{n(k)+1}x)} \varphi(t)\,dt\right) - \liminf_{k\to\infty} \phi\left(\int_0^{d(f^{m(k)}x,f^{n(k)+1}x)} \varphi(t)\,dt\right)\\
&\quad= \psi\left(\int_0^{\varepsilon} \varphi(t)\,dt\right) - \liminf_{k\to\infty} \phi\left(\int_0^{d(f^{m(k)}x,f^{n(k)+1}x)} \varphi(t)\,dt\right)\\
&\quad< \psi\left(\int_0^{\varepsilon} \varphi(t)\,dt\right),
\end{aligned}$$

which is impossible. Thus $\{f^n x\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Since (X,d) is complete, it follows that there exists a point $a \in X$ satisfying $\lim_{n\to\infty} f^n x = a$. By virtue of (1.1), we infer that

$$\psi\left(\int_0^{d(f^{n+1}x,fa)} \varphi(t)\,dt\right) \le \psi\left(\int_0^{d(f^n x,a)} \varphi(t)\,dt\right) - \phi\left(\int_0^{d(f^n x,a)} \varphi(t)\,dt\right), \quad \forall n \in \mathbb{N},$$

which together with $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ and Lemmas 2.1 and 2.2 gives that

$$\begin{aligned}
\psi\left(\int_0^{d(a,fa)} \varphi(t)\,dt\right) &= \limsup_{n\to\infty} \psi\left(\int_0^{d(f^{n+1}x,fa)} \varphi(t)\,dt\right) \\
&\le \limsup_{n\to\infty}\left[\psi\left(\int_0^{d(f^n x,a)} \varphi(t)\,dt\right) - \phi\left(\int_0^{d(f^n x,a)} \varphi(t)\,dt\right)\right] \\
&\le \limsup_{n\to\infty} \psi\left(\int_0^{d(f^n x,a)} \varphi(t)\,dt\right) - \liminf_{n\to\infty} \phi\left(\int_0^{d(f^n x,a)} \varphi(t)\,dt\right) \\
&= \psi(0) - 0 \\
&= 0,
\end{aligned}$$

which together with $\psi \in \Phi_3$ yields that

$$\int_0^{d(a,fa)} \varphi(t)\,dt = 0,$$

that is, $a = fa$.

Finally, we show that a is a unique fixed point of f in X. Suppose that f has another fixed point $b \in X \setminus \{a\}$. It follows from (1.1) and $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$ that

$$\begin{aligned}
\psi\left(\int_0^{d(a,b)} \varphi(t)\,dt\right) &= \psi\left(\int_0^{d(fa,fb)} \varphi(t)\,dt\right) \\
&\le \psi\left(\int_0^{d(a,b)} \varphi(t)\,dt\right) - \phi\left(\int_0^{d(a,b)} \varphi(t)\,dt\right) \\
&< \psi\left(\int_0^{d(a,b)} \varphi(t)\,dt\right),
\end{aligned}$$

which is a contradiction. This completes the proof. □

Theorem 3.2 *Let f be a mapping from a complete metric space (X, d) into itself satisfying* (1.2). *Then f has a unique fixed point $a \in X$ such that* $\lim_{n\to\infty} f^n x = a$ *for each $x \in X$.*

Proof Let x be an arbitrary point in X. Suppose that (3.2) holds for some $n_0 \in \mathbb{N}$. Using (1.2), (3.2) and $(\varphi, \psi, \alpha) \in \Phi_1 \times \Phi_3 \times \Phi_5$, we get that

$$\psi\left(\int_0^{d_{n_0}} \varphi(t)\,dt\right) > 0$$

and

$$\begin{aligned}
\psi\left(\int_0^{d_{n_0-1}} \varphi(t)\,dt\right) \le \psi\left(\int_0^{d_{n_0}} \varphi(t)\,dt\right) &= \psi\left(\int_0^{d(f^{n_0}x,f^{n_0+1}x)} \varphi(t)\,dt\right) \\
&\le \alpha\big(d\big(f^{n_0-1}x, f^{n_0}x\big)\big)\psi\left(\int_0^{d(f^{n_0-1}x,f^{n_0}x)} \varphi(t)\,dt\right) \\
&= \alpha(d_{n_0-1})\psi\left(\int_0^{d_{n_0-1}} \varphi(t)\,dt\right) < \psi\left(\int_0^{d_{n_0-1}} \varphi(t)\,dt\right),
\end{aligned}$$

which is a contradiction, and hence (3.2) does not hold. Consequently, (3.1) is true. Notice that the nonnegative sequence $\{d_n\}_{n\in\mathbb{N}_0}$ is nonincreasing, which implies that there exists a constant $c \geq 0$ with $\lim_{n\to\infty} d_n = c$. Suppose that $c > 0$. In light of (1.2), we infer that

$$\begin{aligned}
\psi\left(\int_0^{d_n}\varphi(t)\,dt\right) &= \psi\left(\int_0^{d(f^n x, f^{n+1}x)}\varphi(t)\,dt\right) \\
&\leq \alpha\left(d\left(f^{n-1}x, f^n x\right)\right)\psi\left(\int_0^{d(f^{n-1}x, f^n x)}\varphi(t)\,dt\right) \\
&= \alpha(d_{n-1})\psi\left(\int_0^{d_{n-1}}\varphi(t)\,dt\right), \quad \forall n\in\mathbb{N}.
\end{aligned} \tag{3.13}$$

Taking upper limit in (3.13) and using Lemma 2.1 and $(\varphi,\psi,\alpha)\in\Phi_1\times\Phi_3\times\Phi_5$, we know that

$$\begin{aligned}
\psi\left(\int_0^{c}\varphi(t)\,dt\right) &= \limsup_{n\to\infty}\psi\left(\int_0^{d_n}\varphi(t)\,dt\right) \\
&\leq \limsup_{n\to\infty}\left[\alpha(d_{n-1})\psi\left(\int_0^{d_{n-1}}\varphi(t)\,dt\right)\right] \\
&\leq \limsup_{n\to\infty}\alpha(d_{n-1})\cdot\limsup_{n\to\infty}\psi\left(\int_0^{d_{n-1}}\varphi(t)\,dt\right) \\
&< \psi\left(\int_0^{c}\varphi(t)\,dt\right),
\end{aligned}$$

which is a contradiction, and hence $c = 0$, that is, (3.6) holds.

Now we show that $\{f^n x\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Suppose that $\{f^n x\}_{n\in\mathbb{N}}$ is not a Cauchy sequence. As in the proof of Theorem 3.1, we conclude that there exist $\varepsilon > 0$ and $\{m(k), n(k) : k\in\mathbb{N}\}\subseteq\mathbb{N}$ with $m(k) > n(k) > k$ for each $k\in\mathbb{N}$ satisfying (3.8)-(3.11). By means of (1.2), (3.11), Lemma 2.1 and $(\varphi,\psi,\alpha)\in\Phi_1\times\Phi_3\times\Phi_5$, we get that

$$\begin{aligned}
\psi\left(\int_0^{\varepsilon}\varphi(t)\,dt\right) &= \limsup_{k\to\infty}\psi\left(\int_0^{d(f^{m(k)+1}x, f^{n(k)+2}x)}\varphi(t)\,dt\right) \\
&= \limsup_{k\to\infty}\left[\alpha\left(d\left(f^{m(k)}x, f^{n(k)+1}x\right)\right)\psi\left(\int_0^{d(f^{m(k)}x, f^{n(k)+1}x)}\varphi(t)\,dt\right)\right] \\
&\leq \limsup_{k\to\infty}\alpha\left(d\left(f^{m(k)}x, f^{n(k)+1}x\right)\right)\cdot\limsup_{k\to\infty}\psi\left(\int_0^{d(f^{m(k)}x, f^{n(k)+1}x)}\varphi(t)\,dt\right) \\
&< \psi\left(\int_0^{\varepsilon}\varphi(t)\,dt\right),
\end{aligned}$$

which is a contradiction. Hence $\{f^n x\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

It follows from completeness of (X,d) that there exists $a\in X$ with $\lim_{n\to\infty} f^n x = a$. In view of (1.2), we have

$$\psi\left(\int_0^{d(f^{n+1}x, fa)}\varphi(t)\,dt\right) \leq \alpha\left(d\left(f^n x, a\right)\right)\psi\left(\int_0^{d(f^n x, a)}\varphi(t)\,dt\right), \quad \forall n\in\mathbb{N}_0. \tag{3.14}$$

Taking upper limit in (3.14) and making use of $(\varphi,\psi,\alpha)\in\Phi_1\times\Phi_3\times\Phi_5$ and Lemmas 2.1 and 2.2, we get that

$$\begin{aligned}\psi\left(\int_0^{d(a,fa)}\varphi(t)\,dt\right)&=\limsup_{n\to\infty}\psi\left(\int_0^{d(f^{n+1}x,fa)}\varphi(t)\,dt\right)\\&\le\limsup_{n\to\infty}\left[\alpha\big(d(f^nx,a)\big)\psi\left(\int_0^{d(f^nx,a)}\varphi(t)\,dt\right)\right]\\&\le\limsup_{n\to\infty}\alpha\big(d(f^nx,a)\big)\cdot\limsup_{n\to\infty}\psi\left(\int_0^{d(f^nx,a)}\varphi(t)\,dt\right)\\&=0,\end{aligned}$$

which means that

$$\psi\left(\int_0^{d(a,fa)}\varphi(t)\,dt\right)=0,$$

that is, $fa=a$.

Next we prove that a is a unique fixed point of f in X. Suppose that f has another fixed point $b\in X\setminus\{a\}$. It follows from (1.2) and $(\varphi,\psi,\alpha)\in\Phi_1\times\Phi_3\times\Phi_5$ that

$$\begin{aligned}\psi\left(\int_0^{d(a,b)}\varphi(t)\,dt\right)&=\psi\left(\int_0^{d(fa,fb)}\varphi(t)\,dt\right)\le\alpha\big(d(a,b)\big)\psi\left(\int_0^{d(a,b)}\varphi(t)\,dt\right)\\&<\psi\left(\int_0^{d(a,b)}\varphi(t)\,dt\right),\end{aligned}$$

which is a contradiction. This completes the proof. □

Theorem 3.3 *Let f be a mapping from a complete metric space (X,d) into itself satisfying* (1.3) *and*

$$\phi(t)\le\psi(t),\quad\forall t\in\mathbb{R}^+.\tag{3.15}$$

Then f has a unique fixed point $a\in X$ such that $\lim_{n\to\infty}f^nx=a$ for each $x\in X$.

Proof Let x be an arbitrary point in X. If there exists $n_0\in\mathbb{N}_0$ satisfying $d_{n_0}=0$, it is clear that $f^{n_0}x$ is a fixed point of f and $\lim_{n\to\infty}f^nx=f^{n_0}x$. Now we assume that $d_n\ne0$ for all $n\in\mathbb{N}_0$. Suppose that (3.2) holds for some $n_0\in\mathbb{N}$. It follows from (1.3) that

$$\begin{aligned}\psi\left(\int_0^{d_{n_0}}\varphi(t)\,dt\right)&=\psi\left(\int_0^{d(f^{n_0}x,f^{n_0+1}x)}\varphi(t)\,dt\right)\\&\le\alpha\big(d(f^{n_0-1}x,f^{n_0}x)\big)\phi\left(\int_0^{d(f^{n_0-1}x,f^{n_0}x)}\varphi(t)\,dt\right)\\&\quad+\beta\big(d(f^{n_0-1}x,f^{n_0}x)\big)\psi\left(\int_0^{d(f^{n_0}x,f^{n_0+1}x)}\varphi(t)\,dt\right)\\&=\alpha(d_{n_0-1})\phi\left(\int_0^{d_{n_0-1}}\varphi(t)\,dt\right)+\beta(d_{n_0-1})\psi\left(\int_0^{d_{n_0}}\varphi(t)\,dt\right),\end{aligned}$$

which together with (3.2), (3.15), $(\varphi,\psi,\phi) \in \Phi_1 \times \Phi_3 \times \Phi_4$ and $(\alpha,\beta) \in \Phi_6$ implies that

$$\begin{aligned}
0 &< \psi\left(\int_0^{d_{n_0-1}} \varphi(t)\,dt\right) \le \psi\left(\int_0^{d_{n_0}} \varphi(t)\,dt\right) \\
&\le \frac{\alpha(d_{n_0-1})}{1-\beta(d_{n_0-1})}\phi\left(\int_0^{d_{n_0-1}} \varphi(t)\,dt\right) \\
&\le \frac{\alpha(d_{n_0-1})}{1-\beta(d_{n_0-1})}\psi\left(\int_0^{d_{n_0-1}} \varphi(t)\,dt\right) \\
&< \psi\left(\int_0^{d_{n_0-1}} \varphi(t)\,dt\right),
\end{aligned}$$

which is a contradiction, and hence (3.2) does not hold. Consequently, (3.1) holds.

Next we show that $\lim_{n\to\infty} d_n = 0$. Note that the nonnegative sequence $\{d_n\}_{n\in\mathbb{N}}$ is nonincreasing, which implies that there exists a constant $c \ge 0$ with $\lim_{n\to\infty} d_n = c$. Suppose that $c > 0$. It follows from (1.3) that

$$\begin{aligned}
\psi\left(\int_0^{d_n} \varphi(t)\,dt\right) &= \psi\left(\int_0^{d(f^n x, f^{n+1}x)} \varphi(t)\,dt\right) \\
&\le \alpha\bigl(d(f^{n-1}x, f^n x)\bigr)\phi\left(\int_0^{d(f^{n-1}x, f^n x)} \varphi(t)\,dt\right) \\
&\quad + \beta\bigl(d(f^{n-1}x, f^n x)\bigr)\psi\left(\int_0^{d(f^n x, f^{n+1}x)} \varphi(t)\,dt\right) \\
&= \alpha(d_{n-1})\phi\left(\int_0^{d_{n-1}} \varphi(t)\,dt\right) + \beta(d_{n-1})\psi\left(\int_0^{d_n} \varphi(t)\,dt\right), \quad \forall n \in \mathbb{N},
\end{aligned}$$

which means that

$$\psi\left(\int_0^{d_n} \varphi(t)\,dt\right) \le \frac{\alpha(d_{n-1})}{1-\beta(d_{n-1})}\phi\left(\int_0^{d_{n-1}} \varphi(t)\,dt\right), \quad \forall n \in \mathbb{N}. \tag{3.16}$$

Taking upper limit in (3.16) and using (3.15), $(\varphi,\psi,\phi) \in \Phi_1 \times \Phi_3 \times \Phi_4$, $(\alpha,\beta) \in \Phi_6$ and Lemma 2.1, we arrive at

$$\begin{aligned}
\psi\left(\int_0^{c} \varphi(t)\,dt\right) &= \limsup_{n\to\infty} \psi\left(\int_0^{d_n} \varphi(t)\,dt\right) \\
&\le \limsup_{n\to\infty}\left[\frac{\alpha(d_{n-1})}{1-\beta(d_{n-1})}\phi\left(\int_0^{d_{n-1}} \varphi(t)\,dt\right)\right] \\
&\le \limsup_{n\to\infty} \frac{\alpha(d_{n-1})}{1-\beta(d_{n-1})} \cdot \limsup_{n\to\infty} \psi\left(\int_0^{d_{n-1}} \varphi(t)\,dt\right) \\
&\le \limsup_{s\to c^+} \frac{\alpha(s)}{1-\beta(s)} \cdot \psi\left(\int_0^{c} \varphi(t)\,dt\right) \\
&< \psi\left(\int_0^{c} \varphi(t)\,dt\right),
\end{aligned}$$

which is impossible. Therefore $c = 0$, that is, $\lim_{n\to\infty} d_n = 0$.

Next we show that $\{f^n x\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Suppose that $\{f^n x\}_{n\in\mathbb{N}}$ is not a Cauchy sequence. As in the proof of Theorem 3.1, we conclude that there exist $\varepsilon > 0$ and $\{m(k), n(k) : k \in \mathbb{N}\} \subseteq \mathbb{N}$ with $m(k) > n(k) > k$ for each $k \in \mathbb{N}$ satisfying (3.8)-(3.11). By means of (3.12), we deduce that

$$\begin{aligned}
&\psi\left(\int_0^{d(f^{m(k)+1}x,f^{n(k)+2}x)} \varphi(t)\,dt\right) \\
&\quad\le \alpha\big(d\big(f^{m(k)}x,f^{n(k)+1}x\big)\big)\phi\left(\int_0^{d(f^{m(k)}x,f^{m(k)+1}x)} \varphi(t)\,dt\right) \\
&\qquad + \beta\big(d\big(f^{m(k)}x,f^{n(k)+1}x\big)\big)\psi\left(\int_0^{d(f^{n(k)+1}x,f^{n(k)+2}x)} \varphi(t)\,dt\right) \\
&\quad= \alpha\big(d\big(f^{m(k)}x,f^{n(k)+1}x\big)\big)\phi\left(\int_0^{d_{m(k)}} \varphi(t)\,dt\right) \\
&\qquad + \beta\big(d\big(f^{m(k)}x,f^{n(k)+1}x\big)\big)\psi\left(\int_0^{d_{n(k)}} \varphi(t)\,dt\right), \quad \forall k \in \mathbb{N}.
\end{aligned} \tag{3.17}$$

Taking upper limit in (3.17) and making use of (1.3), (3.11), Lemma 2.1, $(\varphi,\psi,\phi) \in \Phi_1 \times \Phi_3 \times \Phi_4$ and $(\alpha,\beta) \in \Phi_6$, we deduce that

$$\begin{aligned}
0 &< \psi\left(\int_0^{\varepsilon} \varphi(t)\,dt\right) = \limsup_{k\to\infty} \psi\left(\int_0^{d(f^{m(k)+1}x,f^{n(k)+2}x)} \varphi(t)\,dt\right) \\
&\le \limsup_{k\to\infty}\left[\alpha\big(d\big(f^{m(k)}x,f^{n(k)+1}x\big)\big)\phi\left(\int_0^{d_{m(k)}} \varphi(t)\,dt\right)\right. \\
&\qquad \left. + \beta\big(d\big(f^{m(k)}x,f^{n(k)+1}x\big)\big)\psi\left(\int_0^{d_{n(k)}} \varphi(t)\,dt\right)\right] \\
&\le \limsup_{k\to\infty} \alpha\big(d\big(f^{m(k)}x,f^{n(k)+1}x\big)\big) \cdot \limsup_{k\to\infty} \psi\left(\int_0^{d_{m(k)}} \varphi(t)\,dt\right) \\
&\qquad + \limsup_{k\to\infty} \beta\big(d\big(f^{m(k)}x,f^{n(k)+1}x\big)\big) \cdot \limsup_{k\to\infty} \psi\left(\int_0^{d_{n(k)}} \varphi(t)\,dt\right) \\
&\le \limsup_{s\to\varepsilon} \alpha(s) \cdot \psi\left(\int_0^{0} \varphi(t)\,dt\right) + \limsup_{s\to\varepsilon} \beta(s) \cdot \psi\left(\int_0^{0} \varphi(t)\,dt\right) \\
&= 0,
\end{aligned}$$

which is a contradiction. Hence $\{f^n x\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Completeness of (X,d) implies that there exists a point $a \in X$ such that $\lim_{n\to\infty} f^n x = a$. In view of (1.3), $(\varphi,\psi,\phi) \in \Phi_1 \times \Phi_3 \times \Phi_4$, $(\alpha,\beta) \in \Phi_6$ and Lemma 2.1, we infer that

$$\begin{aligned}
\psi\left(\int_0^{d(a,fa)} \varphi(t)\,dt\right) &= \limsup_{n\to\infty} \psi\left(\int_0^{d(f^{n+1}x,fa)} \varphi(t)\,dt\right) \\
&\le \limsup_{n\to\infty}\left[\alpha\big(d\big(f^n x,a\big)\big)\phi\left(\int_0^{d(f^n x,f^{n+1}x)} \varphi(t)\,dt\right)\right. \\
&\qquad \left. + \beta\big(d\big(f^n x,a\big)\big)\psi\left(\int_0^{d(a,fa)} \varphi(t)\,dt\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&\le \limsup_{n\to\infty} \alpha\bigl(d(f^n x, a)\bigr) \cdot \limsup_{n\to\infty} \psi\left(\int_0^{d_n} \varphi(t)\,dt\right) \\
&\quad + \limsup_{n\to\infty} \beta\bigl(d(f^n x, a)\bigr) \cdot \psi\left(\int_0^{d(a,fa)} \varphi(t)\,dt\right) \\
&\le \limsup_{s\to 0^+} \beta(s) \cdot \psi\left(\int_0^{d(a,fa)} \varphi(t)\,dt\right),
\end{aligned}
$$

which together with $(\alpha, \beta) \in \Phi_6$ yields that

$$
\psi\left(\int_0^{d(a,fa)} \varphi(t)\,dt\right) = 0,
$$

which gives that $d(fa, a) = 0$, that is, $fa = a$.

Finally, we prove that a is a unique fixed point of f in X. Suppose that f has another fixed point $b \in X \setminus \{a\}$. It follows from (1.3) and $(\varphi, \psi, \phi) \in \Phi_1 \times \Phi_3 \times \Phi_4$ and $(\alpha, \beta) \in \Phi_6$ that

$$
\begin{aligned}
0 &\le \psi\left(\int_0^{d(fa,fb)} \varphi(t)\,dt\right) \\
&\le \alpha\bigl(d(a,b)\bigr)\phi\left(\int_0^{d(a,fa)} \varphi(t)\,dt\right) + \beta\bigl(d(a,b)\bigr)\psi\left(\int_0^{d(b,fb)} \varphi(t)\,dt\right) \\
&= 0,
\end{aligned}
$$

which is a contradiction. This completes the proof. □

4 Three examples

Now we construct three examples to explain Theorems 3.1-3.3.

Example 4.1 Let $X = [0, \frac{1}{2}] \cup \{1\} \cup \{3\}$ be endowed with the Euclidean metric $d = |\cdot|$. Assume that $f : X \to X$ and $\varphi, \phi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ are defined by

$$
f(x) = \begin{cases} \frac{x}{2}, & \forall x \in [0, \frac{1}{2}], \\ 0, & x = 1, \\ 1, & x = 3, \end{cases} \qquad \varphi(t) = \begin{cases} \frac{t}{2}, & \forall t \in [0, 1], \\ 1, & \forall t \in (1, +\infty), \end{cases}
$$

$$
\phi(t) = \begin{cases} \frac{t^2}{4}, & \forall t \in [0, 1], \\ \frac{t^2}{8}, & \forall t \in (1, +\infty), \end{cases} \qquad \psi(t) = \begin{cases} t, & \forall t \in [0, 1], \\ \frac{t^2+1}{2}, & \forall t \in (1, +\infty). \end{cases}
$$

Clearly, (X, d) is a complete metric and $(\varphi, \phi, \psi) \in \Phi_1 \times \Phi_2 \times \Phi_3$. Let $x, y \in X$ with $x < y$. In order to verify (1.1), we have to consider the following four cases.

Case 1. Let $x, y \in [0, \frac{1}{2}]$. Note that

$$
\begin{aligned}
\psi\left(\int_0^{d(fx,fy)} \varphi(t)\,dt\right) &= \psi\left(\int_0^{\frac{1}{2}|x-y|} \varphi(t)\,dt\right) = \psi\left(\frac{|x-y|^2}{16}\right) = \frac{|x-y|^2}{16} \\
&\le \frac{|x-y|^2}{4} - \frac{|x-y|^4}{16} \\
&= \psi\left(\frac{|x-y|^2}{4}\right) - \phi\left(\frac{|x-y|^2}{4}\right)
\end{aligned}
$$

$$= \psi\left(\int_0^{|x-y|} \varphi(t)\,dt\right) - \phi\left(\int_0^{|x-y|} \varphi(t)\,dt\right)$$
$$= \psi\left(\int_0^{d(x,y)} \varphi(t)\,dt\right) - \phi\left(\int_0^{d(x,y)} \varphi(t)\,dt\right).$$

Case 2. Let $x \in [0, \frac{1}{2}]$ and $y = 1$. It follows that

$$\begin{aligned}
\psi\left(\int_0^{d(fx,fy)} \varphi(t)\,dt\right) &= \psi\left(\int_0^{\frac{x}{2}} \varphi(t)\,dt\right) = \psi\left(\frac{x^2}{16}\right) = \frac{x^2}{16} \le \frac{(1-x)^2}{4} - \frac{(1-x)^4}{16} \\
&= \psi\left(\frac{(1-x)^2}{4}\right) - \phi\left(\frac{(1-x)^2}{4}\right) \\
&= \psi\left(\int_0^{|x-1|} \varphi(t)\,dt\right) - \phi\left(\int_0^{|x-1|} \varphi(t)\,dt\right) \\
&= \psi\left(\int_0^{d(x,y)} \varphi(t)\,dt\right) - \phi\left(\int_0^{d(x,y)} \varphi(t)\,dt\right).
\end{aligned}$$

Case 3. Let $x \in [0, \frac{1}{2}]$ and $y = 3$. It follows that

$$\begin{aligned}
\psi\left(\int_0^{d(fx,fy)} \varphi(t)\,dt\right) &= \psi\left(\int_0^{\frac{2-x}{2}} \varphi(t)\,dt\right) = \psi\left(\frac{(2-x)^2}{16}\right) = \frac{(2-x)^2}{16} < \frac{1}{2} \\
&\le \frac{1}{2}\left[\left(\frac{9}{4} - x\right)^2 + 1\right] - \frac{1}{8}\left(\frac{9}{4} - x\right)^2 = \psi\left(\frac{9}{4} - x\right) - \phi\left(\frac{9}{4} - x\right) \\
&= \psi\left(\int_0^1 \varphi(t)\,dt + \int_1^{3-x} \varphi(t)\,dt\right) - \phi\left(\int_0^1 \varphi(t)\,dt + \int_1^{3-x} \varphi(t)\,dt\right) \\
&= \psi\left(\int_0^{3-x} \varphi(t)\,dt\right) - \phi\left(\int_0^{3-x} \varphi(t)\,dt\right) \\
&= \psi\left(\int_0^{d(x,y)} \varphi(t)\,dt\right) - \phi\left(\int_0^{d(x,y)} \varphi(t)\,dt\right).
\end{aligned}$$

Case 4. Let $x = 1$ and $y = 3$. Note that

$$\begin{aligned}
\psi\left(\int_0^{d(fx,fy)} \varphi(t)\,dt\right) &= \psi\left(\int_0^1 \varphi(t)\,dt\right) = \psi\left(\frac{1}{4}\right) = \frac{1}{4} < \frac{139}{128} \\
&= \frac{1}{2}\left(\frac{25}{16} + 1\right) - \frac{1}{8} \cdot \frac{25}{16} = \psi\left(\frac{5}{4}\right) - \phi\left(\frac{5}{4}\right) \\
&= \psi\left(\int_0^1 \varphi(t)\,dt + \int_1^2 \varphi(t)\,dt\right) - \phi\left(\int_0^1 \varphi(t)\,dt + \int_1^2 \varphi(t)\,dt\right) \\
&= \psi\left(\int_0^2 \varphi(t)\,dt\right) - \phi\left(\int_0^2 \varphi(t)\,dt\right) \\
&= \psi\left(\int_0^{d(x,y)} \varphi(t)\,dt\right) - \phi\left(\int_0^{d(x,y)} \varphi(t)\,dt\right).
\end{aligned}$$

That is, (1.1) holds. Thus Theorem 3.1 guarantees that f has a unique fixed point $0 \in X$ such that $\lim_{n\to\infty} f^n x = 0$ for each $x \in X$.

Example 4.2 Let $X = [0,1] \cup [4,5]$ be endowed with the Euclidean metric $d = |\cdot|$. Assume that $f : X \to X$ and $\varphi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ and $\alpha : \mathbb{R}^+ \to [0,1)$ are defined by

$$f(x) = \begin{cases} \frac{x^2}{4}, & \forall x \in [0,1], \\ \frac{x^2}{26}, & \forall x \in [4,5], \end{cases} \qquad \varphi(t) = \begin{cases} 4t^3, & \forall t \in [0,1], \\ 2t, & \forall t \in [4,5], \end{cases}$$

$$\psi(t) = t^{\frac{1}{2}}, \quad \forall t \in \mathbb{R}^+, \qquad \alpha(t) = \begin{cases} \frac{1}{3} + \frac{t^2}{2}, & \forall t \in [0,1], \\ \frac{1}{2t}, & \forall t \in (1,3), \\ \frac{1}{\sqrt{t}}, & \forall t \in (3,+\infty). \end{cases}$$

Obviously, $(\varphi, \psi, \alpha) \in \Phi_1 \times \Phi_3 \times \Phi_5$. Put $x, y \in X$ with $x < y$. In order to verify (1.2), we have to consider three possible cases as follows.

Case 1. Let $x, y \in [0,1]$. It is clear that

$$\begin{aligned} \psi\left(\int_0^{d(fx,fy)} \varphi(t)\,dt\right) &= \left(\int_0^{\frac{y^2-x^2}{4}} 4t^3\,dt\right)^{\frac{1}{2}} = \frac{(x+y)^2}{16}|x-y|^2 \le \frac{1}{4}|x-y|^2 \\ &\le \left(\frac{1}{3} + \frac{1}{2}|x-y|^2\right)|x-y|^2 \\ &= \alpha\big(d(x,y)\big)\psi\left(\int_0^{d(x,y)} \varphi(t)\,dt\right). \end{aligned}$$

Case 2. Let $x, y \in [4,5]$. It follows that

$$\begin{aligned} \psi\left(\int_0^{d(fx,fy)} \varphi(t)\,dt\right) &= \left(\int_0^{\frac{y^2-x^2}{26}} 4t^3\,dt\right)^{\frac{1}{2}} = \left(\frac{x+y}{26}\right)^2|x-y|^2 \le \frac{25}{169}|x-y|^2 \\ &\le \left(\frac{1}{3} + \frac{1}{2}|x-y|^2\right)|x-y|^2 \\ &= \alpha\big(d(x,y)\big)\psi\left(\int_0^{d(x,y)} \varphi(t)\,dt\right). \end{aligned}$$

Case 3. Let $x \in [0,1]$ and $y \in [4,5]$. It follows that

$$\begin{aligned} \psi\left(\int_0^{d(fx,fy)} \varphi(t)\,dt\right) &= \left(\int_0^{\frac{y^2}{26}-\frac{x^2}{4}} 4t^3\,dt\right)^{\frac{1}{2}} = \left(\frac{y^2}{26} - \frac{x^2}{4}\right)^2 \le \left(\frac{25}{26}\right)^2 < 1 < \sqrt{|x-y|} \\ &= \alpha\big(|x-y|\big)|x-y| = \alpha\big(|x-y|\big)\left(\int_0^1 4t^3\,dt + \int_1^{|x-y|} 2t\,dt\right)^{\frac{1}{2}} \\ &= \alpha\big(d(x,y)\big)\left(\int_0^{|x-y|} \varphi(t)\,dt\right)^{\frac{1}{2}} \\ &= \alpha\big(d(x,y)\big)\psi\left(\int_0^{d(x,y)} \varphi(t)\,dt\right). \end{aligned}$$

That is, (1.2) holds. Consequently, the conditions of Theorem 3.2 are satisfied. It follows from Theorem 3.2 that f has a unique fixed point $0 \in X$ such that $\lim_{n\to\infty} f^n x = 0$ for each $x \in X$.

Example 4.3 Let $X = [\frac{1}{2},1] \cup [\frac{3}{2},2]$ be endowed with the Euclidean metric $d = |\cdot|$. Assume that $f : X \to X$, $\varphi, \phi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ and $\alpha, \beta : \mathbb{R}^+ \to [0,1)$ are defined by

$$f(x) = \begin{cases} 1, & \forall x \in [\frac{1}{2},1], \\ \frac{x}{2}, & \forall x \in [\frac{3}{2},2], \end{cases} \qquad \phi(t) = \begin{cases} 0, & t \in [0,\frac{9}{16}) \\ \frac{32t^2}{9}, & t \in [\frac{9}{16},+\infty), \end{cases}$$

$$\varphi(t) = 2t, \qquad \psi(t) = 4t^2, \qquad \alpha(t) = \frac{t}{(\frac{1}{2}+t)^2}, \qquad \beta(t) = \frac{t^2}{(\frac{1}{2}+t)^2}, \quad \forall t \in \mathbb{R}^+.$$

It is easy to see that $(\varphi,\psi,\phi) \in \Phi_1 \times \Phi_3 \times \Phi_4$, $(\alpha,\beta) \in \Phi_6$ and (3.15) holds. In order to verify (1.3), we have to consider the five possible cases below.

Case 1. Let $x, y \in [\frac{3}{2},2]$ with $x \geq y$. Note that

$$\begin{aligned}
\psi\left(\int_0^{d(fx,fy)} \varphi(t)\,dt\right) &= \psi\left(\int_0^{\frac{|x-y|}{2}} \varphi(t)\,dt\right) = \psi\left(\frac{(x-y)^2}{4}\right) = \frac{(x-y)^4}{4} \leq \frac{x-y}{2} \\
&\leq \frac{x-y}{2} \cdot \frac{x^4}{(\frac{1}{2}+x-y)^2} \leq \frac{x-y}{(\frac{1}{2}+x-y)^2} \cdot \frac{32}{9}\left(\frac{x^2}{4}\right)^2 \\
&= \alpha\big(d(x,y)\big)\phi\left(\int_0^{d(x,fx)} \varphi(t)\,dt\right) \\
&\leq \alpha\big(d(x,y)\big)\phi\left(\int_0^{d(x,fx)} \varphi(t)\,dt\right)d + \beta\big(d(x,y)\big)\psi\left(\int_0^{d(y,fy)} \varphi(t)\,dt\right).
\end{aligned}$$

Case 2. Let $x, y \in [\frac{3}{2},2]$ with $y > x$. Note that

$$\begin{aligned}
\psi\left(\int_0^{d(fx,fy)} \varphi(t)\,dt\right) &= \psi\left(\int_0^{\frac{|y-x|}{2}} \varphi(t)\,dt\right) = \frac{(y-x)^4}{4} \leq \frac{(y-x)^2}{4} \cdot \frac{y^4}{(\frac{1}{2}+y-x)^2} \\
&= \frac{(y-x)^2}{(\frac{1}{2}+y-x)^2} \cdot \frac{y^4}{4} = \beta\big(d(x,y)\big)\psi\left(\frac{y^2}{4}\right) \\
&= \beta\big(d(x,y)\big)\psi\left(\int_0^{\frac{y}{2}} \varphi(t)\,dt\right) = \beta\big(d(x,y)\big)\psi\left(\int_0^{d(y,fy)} \varphi(t)\,dt\right) \\
&\leq \alpha\big(d(x,y)\big)\phi\left(\int_0^{d(x,fx)} \varphi(t)\,dt\right) + \beta\big(d(x,y)\big)\psi\left(\int_0^{d(y,fy)} \varphi(t)\,dt\right).
\end{aligned}$$

Case 3. Let $x \in [\frac{3}{2},2]$ and $y \in [\frac{1}{2},1]$. It follows that

$$\begin{aligned}
\psi\left(\int_0^{d(fx,fy)} \varphi(t)\,dt\right) &= \psi\left(\int_0^{\frac{x-2}{2}} \varphi(t)\,dt\right) = \psi\left(\frac{(x-2)^2}{4}\right) = \frac{(x-2)^4}{4} \leq \frac{1}{64} < \frac{27}{64} \\
&= \frac{3}{8} \cdot \frac{2}{9} \cdot \frac{81}{16} \leq \frac{x-y}{(\frac{1}{2}+x-y)^2} \cdot \frac{2}{9} \cdot x^4 = \alpha\big(d(x,y)\big)\phi\left(\frac{x^2}{4}\right) \\
&= \alpha\big(d(x,y)\big)\phi\left(\int_0^{d(x,fx)} \varphi(t)\,dt\right) \\
&\leq \alpha\big(d(x,y)\big)\phi\left(\int_0^{d(x,fx)} \varphi(t)\,dt\right) + \beta\big(d(x,y)\big)\psi\left(\int_0^{d(y,fy)} \varphi(t)\,dt\right).
\end{aligned}$$

Case 4. Let $x \in [\frac{1}{2}, 1]$ and $y \in [\frac{3}{2}, 2]$. Note that

$$\begin{aligned}
\psi\left(\int_0^{d(fx,fy)} \varphi(t)\,dt\right) &= \psi\left(\int_0^{\frac{|y-2|}{2}} 2t\,dt\right) = \frac{(y-2)^4}{4} \le \frac{1}{64} < \frac{1}{4}\cdot\frac{81}{64} \\
&\le \frac{(y-x)^2}{(\frac{1}{2}+y-x)^2}\cdot\frac{y^4}{4} = \beta\big(d(x,y)\big)\psi\left(\frac{y^2}{4}\right) \\
&= \beta\big(d(x,y)\big)\psi\left(\int_0^{d(y,fy)} \varphi(t)\,dt\right) \\
&\le \alpha\big(d(x,y)\big)\phi\left(\int_0^{d(x,fx)} \varphi(t)\,dt\right) + \beta\big(d(x,y)\big)\psi\left(\int_0^{d(y,fy)} \varphi(t)\,dt\right).
\end{aligned}$$

Case 5. Let $x, y \in [\frac{1}{2}, 1]$. Notice that $fx = fy = 1$. It follows that

$$\begin{aligned}
&\psi\left(\int_0^{d(fx,fy)} \varphi(t)\,dt\right) \\
&\quad = 0 \le \alpha\big(d(x,y)\big)\phi\left(\int_0^{d(x,fx)} \varphi(t)\,dt\right) + \beta\big(d(x,y)\big)\psi\left(\int_0^{d(y,fy)} \varphi(t)\,dt\right).
\end{aligned}$$

That is, (1.3) holds. Thus all the conditions of Theorem 3.3 are satisfied. It follows from Theorem 3.3 that f has a unique fixed point $1 \in X$ such that $\lim_{n\to\infty} f^n x = 1$ for each $x \in X$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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