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Alternating mann iterative algorithms for the split common fixed-point problem of quasi-nonexpansive mappings

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Abstract

Very recently, Moudafi (Alternating CQ-algorithms for convex feasibility and split fixed-point problems, J. Nonlinear Convex Anal.) introduced an alternating CQ-algorithm with weak convergence for the following split common fixed-point problem. Let H_1, H_2, H_3 be real Hilbert spaces, let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators.

$$\text{Find } x \in F(U), y \in F(T) \text{ such that } Ax = By, \tag{1}$$

where $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are two firmly quasi-nonexpansive operators with nonempty fixed-point sets $F(U) = \{x \in H_1 : Ux = x\}$ and $F(T) = \{x \in H_2 : Tx = x\}$. Note that by taking $H_2 = H_3$ and $B = I$, we recover the split common fixed-point problem originally introduced in Censor and Segal (J. Convex Anal. 16:587-600, 2009) and used to model many significant real-world inverse problems in sensor net-works and radiation therapy treatment planning. In this paper, we will continue to consider the split common fixed-point problem (1) governed by the general class of quasi-nonexpansive operators. We introduce two alternating Mann iterative algorithms and prove the weak convergence of algorithms. At last, we provide some applications. Our results improve and extend the corresponding results announced by many others.

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Keywords: split common fixed-point problem; quasi-nonexpansive mapping; weak convergence; Mann iterative algorithm; Hilbert space

1 Introduction

Throughout this paper, we always assume that H is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let I denote the identity operator on H. Let C and Q be nonempty closed convex subset of real Hilbert spaces H_1 and H_2, respectively. The split feasibility problem (SFP) is to find a point

$$x \in C \text{ such that } Ax \in Q, \tag{1.1}$$

where $A : H_1 \to H_2$ is a bounded linear operator. The SFP in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [1] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [2]. The SFP attracts many authors' attention due to its application in signal processing. Various algorithms have been invented to solve it (see [3–12] and references therein).

Note that if the split feasibility problem (1.1) is consistent (*i.e.*, (1.1) has a solution), then (1.1) can be formulated as a fixed point equation by using the fact

$$P_C\big(I - \gamma A^*(I - P_Q)A\big)x^* = x^*. \tag{1.2}$$

That is, x^* solves the SFP (1.1) if and only if x^* solves the fixed point equation (1.2) (see [13] for the details). This implies that we can use fixed point algorithms (see [6, 13–15]) to solve SFP. A popular algorithm that solves the SFP (1.1) is due to Byrne's CQ algorithm [2], which is found to be a gradient-projection method (GPM) in convex minimization. Subsequently, Byrne [3] applied KM iteration to the CQ algorithm and Zhao [16] applied KM iteration to the perturbed CQ algorithm to solve the SFP.

Recently, Moudafi [17] introduced a new convex feasibility problem (CFP). Let H_1, H_2, H_3 be real Hilbert spaces, let $C \subset H_1$, $Q \subset H_2$ be two nonempty closed convex sets, let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators. The convex feasibility problem in [17] is to find

$$x \in C, y \in Q \text{ such that } Ax = By, \tag{1.3}$$

which allows asymmetric and partial relations between the variables x and y. The interest is to cover many situations, for instance, in decomposition methods for PDEs, applications in game theory and in intensity-modulated radiation therapy (IMRT). In decision sciences, this allows to consider agents who interplay only via some components of their decision variables (see [18]). In (IMRT), this amounts to envisaging a weak coupling between the vector of doses absorbed in all voxels and that of the radiation intensity (see [19]). If $H_2 = H_3$ and $B = I$, then the convex feasibility problem (1.3) reduces to the split feasibility problem (1.1).

For solving the CFP (1.3), Moudafi [17] studied the fixed point formulation of the solutions of the CFP (1.3). Assuming that the CFP (1.3) is consistent (*i.e.*, (1.3) has a solution), if (x, y) solves (1.3), then it solves the following fixed point equation system

$$\begin{cases} x = P_C(x - \gamma A^*(Ax - By)), \\ y = P_Q(y + \beta B^*(Ax - By)), \end{cases} \tag{1.4}$$

where $\gamma, \beta > 0$ are any positive constants. Moudafi [17] introduced the following alternating CQ algorithm

$$\begin{cases} x_{k+1} = P_C(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = P_Q(y_k + \gamma_k B^*(Ax_{k+1} - By_k)), \end{cases} \tag{1.5}$$

where $\gamma_k \in (\varepsilon, \min(\frac{1}{\lambda_A}, \frac{1}{\lambda_B}) - \varepsilon)$, λ_A and λ_B are the spectral radiuses of A^*A and B^*B, respectively.

The split common fixed-point problem (SCFP) is a generalization of the split feasibility problem (SFP) and the convex feasibility problem (CFP); see [20]. SCFP is in itself at the core of the modeling of many inverse problems in various areas of mathematics and physical sciences and has been used to model significant real-world inverse problems in sensor net-works, in radiation therapy treatment planning, in resolution enhancement, in wavelet-based denoising, in antenna design, in computerized tomography, in materials science, in watermarking, in data compression, in magnetic resonance imaging, in holography, in color imaging, in optics and neural networks, in graph matching… (see [21]). Censor and Segal consider the following problem:

$$\text{find } x^* \in C \text{ such that } Ax^* \in Q, \tag{1.6}$$

where $A : H_1 \to H_2$ is a bounded linear operator, $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are two nonexpansive operators with nonempty fixed-point sets $F(U) = C$ and $F(T) = Q$ and denote the solution set of the two-operator SCFP by

$$\Gamma = \{y \in C; Ay \in Q\}.$$

To solve (1.6), Censor and Segal [20] proposed and proved, in finite-dimensional spaces, the convergence of the following algorithm:

$$x_{k+1} = U\bigl(x_k + \gamma A^t(T - I)Ax_k\bigr), \quad k \in N,$$

where $\gamma \in (0, \frac{2}{\lambda})$, with λ being the largest eigenvalue of the matrix A^tA (A^t stands for matrix transposition). For solving SCFP of quasi-nonexpansive operators, Moudafi [22] introduced the following relaxed algorithm:

$$x_{k+1} = \alpha_k u_k + (1 - \alpha_k)U(u_k), \quad k \in N, \tag{1.7}$$

where $u_k = x_k + \gamma\beta A^*(T - I)Ax_k$, $\beta \in (0,1)$, $\alpha_k \in (0,1)$ and $\gamma \in (0, \frac{1}{\lambda\beta})$, with λ being the spectral radius of the operator A^*A. Moudafi proved weak convergence result of the algorithm in Hilbert spaces.

In [17], Moudafi introduced the following SCFP

$$\text{find } x \in F(U), y \in F(T) \text{ such that } Ax = By, \tag{1.8}$$

and considered the following alternating SCFP-algorithm

$$\begin{cases} x_{k+1} = U(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = T(y_k + \gamma_k B^*(Ax_{k+1} - By_k)) \end{cases} \tag{1.9}$$

for firmly quasi-nonexpansive operators U and T. Moudafi [17] obtained the following result.

Theorem 1.1 *Let H_1, H_2, H_3 be real Hilbert spaces, let $U : H_1 \to H_1$, $T : H_2 \to H_2$ be two firmly quasi-nonexpansive operators such that $I - U$, $I - T$ are demiclosed at* 0. *Let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators. Assume that the solution set Γ is nonempty, (γ_k) is a positive non-decreasing sequence such that $\gamma_k \in (\varepsilon, \min(\frac{1}{\lambda_A}, \frac{1}{\lambda_B}) - \varepsilon)$, where λ_A, λ_B stand for the spectral radiuses of A^*A and B^*B, respectively. Then the sequence (x_k, y_k) generated by* (1.9) *weakly converges to a solution $(\bar{x}, \bar{y})$ of* (1.8). *Moreover, $\|Ax_k - By_k\| \to 0$, $\|x_k - x_{k+1}\| \to 0$, and $\|y_k - y_{k+1}\| \to 0$ as $k \to \infty$.*

In this paper, inspired and motivated by the works mentioned above, firstly, we introduce the following alternating Mann iterative algorithm for solving the SCFP (1.8) for the general class of quasi-nonexpansive operators.

Algorithm 1.1 Let $x_0 \in H_1$, $y_0 \in H_2$ be arbitrary.

$$\begin{cases} u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = \alpha_k u_k + (1-\alpha_k)U(u_k), \\ v_{k+1} = y_k + \gamma_k B^*(Ax_{k+1} - By_k), \\ y_{k+1} = \beta_k v_{k+1} + (1-\beta_k)T(v_{k+1}). \end{cases}$$

By taking $B = I$, we recover (1.8) clearly the classical SCFP (1.6). In addition, if $\gamma_k = 1$ and $\beta_k = \beta \in (0,1)$ in Algorithm 1.1, we have $v_{k+1} = Ax_{k+1}$ and $y_{k+1} = \beta_k Ax_{k+1} + (1-\beta_k)T(Ax_{k+1})$. Thus, Algorithm 1.1 reduces to $u_k = x_k + (1-\beta)A^*(T-I)Ax_k$ and $x_{k+1} = \alpha_k u_k + (1-\alpha_k)U(u_k)$, which is algorithm (1.7) proposed by Moudafi [22].

The CQ algorithm is a special case of the K-M algorithm. Due to the fixed point formulation (1.4) of the CFP (1.3), we can apply the K-M algorithm to obtain the following other alterative Mann iterative sequence for solving the SCFP (1.8) for quasi-nonexpansive operators.

Algorithm 1.2 Let $x_0 \in H_1$, $y_0 \in H_2$ be arbitrary.

$$\begin{cases} u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = \alpha_k x_k + (1-\alpha_k)U(u_k), \\ v_{k+1} = y_k + \gamma_k B^*(Ax_{k+1} - By_k), \\ y_{k+1} = \alpha_k y_k + (1-\alpha_k)T(v_{k+1}). \end{cases}$$

The organization of this paper is as follows. Some useful definitions and results are listed for the convergence analysis of the iterative algorithm in Section 2. In Section 3, we prove the weak convergence of the alternating Mann iterative Algorithms 1.1 and 1.2. At last, we provide some applications of Algorithms 1.1 and 1.2.

2 Preliminaries

Let $T : H \to H$ be a mapping. A point $x \in H$ is said to be a fixed point of T provided $Tx = x$. In this paper, we use $F(T)$ to denote the fixed point set and use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively. We use $\omega_w(x_k) = \{x : \exists x_{k_j} \rightharpoonup x\}$ stand for the weak ω-limit set of $\{x_k\}$ and use Γ stand for the solution set of the SCFP (1.8).

- A mapping $T : H \to H$ belongs to the general class Φ_Q of (possibly discontinuous) quasi-nonexpansive mappings if

$$\|Tx - q\| \le \|x - q\|, \quad \forall (x,q) \in H \times F(T).$$

- A mapping $T : H \to H$ belongs to the set Φ_N of nonexpansive mappings if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall (x,y) \in H \times H.$$

- A mapping $T : H \to H$ belongs to the set Φ_{FN} of firmly nonexpansive mappings if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \left\|(x - y) - (Tx - Ty)\right\|^2, \quad \forall (x, y) \in H \times H.$$

- A mapping $T : H \to H$ belongs to the set Φ_{FQ} of firmly quasi-nonexpansive mappings if

$$\|Tx - q\|^2 \leq \|x - q\|^2 - \|x - Tx\|^2, \quad \forall (x, q) \in H \times F(T).$$

It is easily observed that $\Phi_{FN} \subset \Phi_N \subset \Phi_Q$ and that $\Phi_{FN} \subset \Phi_{FQ} \subset \Phi_Q$. Furthermore, Φ_{FN} is well known to include resolvents and projection operators, while Φ_{FQ} contains subgradient projection operators (see, for instance, [23] and the reference therein).

A mapping $T : H \to H$ is called demiclosed at the origin if, for any sequence $\{x_n\}$ which weakly converges to x, and if the sequence $\{Tx_n\}$ strongly converges to 0, then $Tx = 0$.

In real Hilbert space, we easily get the following equality:

$$2\langle x, y\rangle = \|x\|^2 + \|y\|^2 - \|x - y\|^2 = \|x + y\|^2 - \|x\|^2 - \|y\|^2, \quad \forall x, y \in H. \tag{2.1}$$

In what follows, we give some key properties of the relaxed operator $T_\alpha = \alpha I + (1 - \alpha)T$ which will be needed in the convergence analysis of our algorithms.

Lemma 2.1 ([22]) *Let H be a real Hilbert space, and let $T : H \to H$ be a quasi-nonexpansive mapping. Set $T_\alpha = \alpha I + (1 - \alpha)T$ for $\alpha \in [0, 1)$. Then the following properties are reached for all $(x, q) \in H \times F(T)$:*

(i) $\langle x - Tx, x - q\rangle \geq \frac{1}{2}\|x - Tx\|^2$ *and* $\langle x - Tx, q - Tx\rangle \leq \frac{1}{2}\|x - Tx\|^2$;

(ii) $\|T_\alpha x - q\|^2 \leq \|x - q\|^2 - \alpha(1 - \alpha)\|Tx - x\|^2$;

(iii) $\langle x - T_\alpha x, x - q\rangle \geq \frac{1-\alpha}{2}\|x - Tx\|^2$.

Remark 2.2 Let $T_\alpha = \alpha I + (1 - \alpha)T$, where $T : H \to H$ is a quasi-nonexpansive mapping and $\alpha \in [0, 1)$. We have $F(T_\alpha) = F(T)$ and $\|T_\alpha x - x\|^2 = (1 - \alpha)^2\|Tx - x\|^2$. It follows from (ii) of Lemma 2.1 that $\|T_\alpha x - q\|^2 \leq \|x - q\|^2 - \frac{\alpha}{1-\alpha}\|T_\alpha x - x\|^2$, which implies that T_α is firmly quasi-nonexpansive when $\alpha = \frac{1}{2}$. On the other hand, if $\hat{T}$ is a firmly quasi-nonexpansive mapping, we can obtain $\hat{T} = \frac{1}{2}I + \frac{1}{2}T$, where T is quasi-nonexpansive. This is proved by the following inequalities.

For all $x \in H$ and $q \in F(\hat{T}) = F(T)$,

$$\begin{aligned}
\|Tx - q\|^2 &= \left\|(2\hat{T} - I)x - q\right\|^2 = \left\|(\hat{T}x - q) + (\hat{T}x - x)\right\|^2 \\
&= \|\hat{T}x - q\|^2 + \|\hat{T}x - x\|^2 + 2\langle \hat{T}x - q, \hat{T}x - x\rangle \\
&= \|\hat{T}x - q\|^2 + \|\hat{T}x - x\|^2 + \|\hat{T}x - q\|^2 + \|\hat{T}x - x\|^2 - \|x - q\|^2 \\
&= 2\|\hat{T}x - q\|^2 + 2\|\hat{T}x - x\|^2 - \|x - q\|^2 \\
&\leq 2\|x - q\|^2 - 2\|\hat{T}x - x\|^2 + 2\|\hat{T}x - x\|^2 - \|x - q\|^2 \\
&= \|x - q\|^2,
\end{aligned}$$

where $\hat{T}$ is firmly quasi-nonexpansive.

Lemma 2.3 ([24]) *Let H be a real Hilbert space. Then for all $t \in [0,1]$ and $x, y \in H$,*

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2.$$

3 Convergence of the alternating Mann iterative Algorithms 1.1 and 1.2

Theorem 3.1 *Let H_1, H_2, H_3 be real Hilbert spaces. Given two bounded linear operators $A : H_1 \to H_3$, $B : H_2 \to H_3$, let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be quasi-nonexpansive mappings with nonempty fixed point set $F(U)$ and $F(T)$. Assume that $U - I$, $T - I$ are demiclosed at origin, and the solution set Γ of* (1.8) *is nonempty. Let $\{\gamma_k\}$ be a positive non-decreasing sequence such that $\gamma_k \in (\varepsilon, \min(\frac{1}{\lambda_A}, \frac{1}{\lambda_B}) - \varepsilon)$, where λ_A, λ_B stand for the spectral radiuses of A^*A and B^*B, respectively, and ε is small enough. Then, the sequence $\{(x_k, y_k)\}$ generated by Algorithm* 1.1 *weakly converges to a solution (x^*, y^*) of* (1.8), *provided that $\{\alpha_k\} \subset (\delta, 1-\delta)$ and $\{\beta_k\} \subset (\sigma, 1-\sigma)$ for small enough $\delta, \sigma > 0$. Moreover, $\|Ax_k - By_k\| \to 0$, $\|x_k - x_{k+1}\| \to 0$ and $\|y_k - y_{k+1}\| \to 0$ as $k \to \infty$.*

Proof Taking $(x, y) \in \Gamma$, *i.e.*, $x \in F(U)$; $y \in F(T)$ and $Ax = By$. We have

$$\begin{aligned} \|u_k - x\|^2 &= \|x_k - \gamma_k A^*(Ax_k - By_k) - x\|^2 \\ &= \|x_k - x\|^2 - 2\gamma_k\langle x_k - x, A^*(Ax_k - By_k)\rangle + \gamma_k^2\|A^*(Ax_k - By_k)\|^2. \end{aligned} \tag{3.1}$$

From the definition of λ_A, it follows that

$$\begin{aligned} \gamma_k^2\|A^*(Ax_k - By_k)\|^2 &= \gamma_k^2\langle A^*(Ax_k - By_k), A^*(Ax_k - By_k)\rangle \\ &= \gamma_k^2\langle Ax_k - By_k, AA^*(Ax_k - By_k)\rangle \\ &\le \lambda_A\gamma_k^2\langle Ax_k - By_k, Ax_k - By_k\rangle \\ &= \lambda_A\gamma_k^2\|Ax_k - By_k\|^2. \end{aligned} \tag{3.2}$$

Using equality (2.1), we have

$$\begin{aligned} -2\langle x_k - x, A^*(Ax_k - By_k)\rangle &= -2\langle Ax_k - Ax, Ax_k - By_k\rangle \\ &= -\|Ax_k - Ax\|^2 - \|Ax_k - By_k\|^2 + \|By_k - Ax\|^2. \end{aligned} \tag{3.3}$$

By (3.1)-(3.3), we obtain

$$\begin{aligned} \|u_k - x\|^2 \le{}& \|x_k - x\|^2 - \gamma_k(1 - \lambda_A\gamma_k)\|Ax_k - By_k\|^2 \\ &- \gamma_k\|Ax_k - Ax\|^2 + \gamma_k\|By_k - Ax\|^2. \end{aligned} \tag{3.4}$$

Similarly, by Algorithm 1.1, we have

$$\begin{aligned} \|v_{k+1} - y\|^2 \le{}& \|y_k - y\|^2 - \gamma_k(1 - \lambda_B\gamma_k)\|Ax_{k+1} - By_k\|^2 \\ &- \gamma_k\|By_k - By\|^2 + \gamma_k\|Ax_{k+1} - By\|^2. \end{aligned} \tag{3.5}$$

By adding the two last inequalities, and by taking into account assumptions on $\{\gamma_k\}$ and the fact that $Ax = By$, we obtain

$$\begin{aligned}
&\|u_k - x\|^2 + \|v_{k+1} - y\|^2 \\
&\quad \le \|x_k - x\|^2 + \|y_k - y\|^2 - \gamma_k\|Ax_k - Ax\|^2 + \gamma_{k+1}\|Ax_{k+1} - Ax\|^2 \\
&\qquad - \gamma_k(1 - \lambda_A\gamma_k)\|Ax_k - By_k\|^2 - \gamma_k(1 - \lambda_B\gamma_k)\|Ax_{k+1} - By_k\|^2.
\end{aligned} \tag{3.6}$$

Using the fact that U and T are quasi-nonexpansive mappings, it follows from the property (ii) of Lemma 2.1 that

$$\|x_{k+1} - x\|^2 \le \|u_k - x\|^2 - \alpha_k(1 - \alpha_k)\|U(u_k) - u_k\|^2$$

and

$$\|y_{k+1} - y\|^2 \le \|v_{k+1} - y\|^2 - \beta_k(1 - \beta_k)\|T(v_{k+1}) - v_{k+1}\|^2.$$

So, by (3.6), we have

$$\begin{aligned}
&\|x_{k+1} - x\|^2 + \|y_{k+1} - y\|^2 \\
&\quad \le \|x_k - x\|^2 + \|y_k - y\|^2 - \gamma_k\|Ax_k - Ax\|^2 + \gamma_{k+1}\|Ax_{k+1} - Ax\|^2 \\
&\qquad - \gamma_k(1 - \lambda_A\gamma_k)\|Ax_k - By_k\|^2 - \gamma_k(1 - \lambda_B\gamma_k)\|Ax_{k+1} - By_k\|^2 \\
&\qquad - \alpha_k(1 - \alpha_k)\|U(u_k) - u_k\|^2 - \beta_k(1 - \beta_k)\|T(v_{k+1}) - v_{k+1}\|^2.
\end{aligned} \tag{3.7}$$

Now, by setting $\rho_k(x,y) := \|x_k - x\|^2 + \|y_k - y\|^2 - \gamma_k\|Ax_k - Ax\|^2$, we obtain the following inequality:

$$\begin{aligned}
\rho_{k+1}(x,y) \le{}& \rho_k(x,y) - \gamma_k(1 - \lambda_A\gamma_k)\|Ax_k - By_k\|^2 - \gamma_k(1 - \lambda_B\gamma_k)\|Ax_{k+1} - By_k\|^2 \\
&- \alpha_k(1 - \alpha_k)\|U(u_k) - u_k\|^2 - \beta_k(1 - \beta_k)\|T(v_{k+1}) - v_{k+1}\|^2.
\end{aligned} \tag{3.8}$$

On the other hand, noting that

$$\gamma_k\|Ax_k - Ax\|^2 = \gamma_k\langle x_k - x, A^*(Ax_k - Ax)\rangle \le \gamma_k\lambda_A\|x_k - x\|^2,$$

we have

$$\rho_k(x,y) \ge (1 - \lambda_A\gamma_k)\|x_k - x\|^2 + \|y_k - y\|^2 \ge 0. \tag{3.9}$$

The sequence $\{\rho_k(x,y)\}$ being decreasing and lower bounded by 0, consequently it converges to some finite limit, says $\rho(x,y)$. Again from (3.8), we have $\rho_{k+1}(x,y) \le \rho_k(x,y) - \gamma_k(1 - \lambda_A\gamma_k)\|Ax_k - By_k\|^2$, and hence

$$\lim_{k\to\infty}\|Ax_k - By_k\| = 0$$

by the assumption on $\{\gamma_k\}$. Similarly, by the conditions on $\{\gamma_k\}$, $\{\alpha_k\}$ and $\{\beta_k\}$, we obtain

$$\lim_{k\to\infty}\|Ax_{k+1}-By_k\| = \lim_{k\to\infty}\big\|U(u_k)-u_k\big\| = \lim_{k\to\infty}\|Tv_{k+1}-v_{k+1}\| = 0.$$

Since

$$\|u_k-x_k\| = \gamma_k\big\|A^*(Ax_k-By_k)\big\|,$$

and $\{\gamma_k\}$ is bounded, we have $\lim_{k\to\infty}\|u_k-x_k\| = 0$. It follows from $\lim_{k\to\infty}\|U(u_k)-u_k\| = 0$ that $\lim_{k\to\infty}\|U(u_k)-x_k\| = 0$. So,

$$\|x_{k+1}-x_k\| \le \alpha_k\|u_k-x_k\| + (1-\alpha_k)\big\|U(u_k)-x_k\big\| \to 0$$

as $n\to\infty$, which infers that $\{x_k\}$ is asymptotically regular, namely $\lim_{k\to\infty}\|x_{k+1}-x_k\| = 0$. Similarly, $\lim_{k\to\infty}\|v_{k+1}-y_k\| = 0$, and $\{y_k\}$ is asymptotically regular, too. Now, relation (3.9) and the assumption on $\{\gamma_k\}$ imply that

$$\rho_k(x,y) \ge \varepsilon\lambda_A\|x_k-x\|^2 + \|y_k-y\|^2,$$

which ensures that both sequences $\{x_k\}$ and $\{y_k\}$ are bounded thanks to the fact that $\{\rho_k(x,y)\}$ converges to a finite limit.

Taking $x^*\in\omega_w(x_k)$ and $y^*\in\omega_w(y_k)$, from $\lim_{k\to\infty}\|u_k-x_k\| = 0$ and $\lim_{k\to\infty}\|v_{k+1}-y_k\| = 0$, we have $x^*\in\omega_w(u_k)$ and $y^*\in\omega_w(v_{k+1})$. Combined with the demiclosednesses of $U-I$ and $T-I$ at 0,

$$\lim_{k\to\infty}\big\|U(u_k)-u_k\big\| = \lim_{k\to\infty}\|Tv_{k+1}-v_{k+1}\| = 0$$

yields $Ux^* = x^*$ and $Ty^* = y^*$. So, $x\in F(U)$ and $y\in F(T)$. On the other hand, $Ax^*-By^*\in\omega_w(Ax_k-By_k)$ and lower semicontinuity of the norm imply that

$$\big\|Ax^*-By^*\big\| \le \liminf_{k\to\infty}\|Ax_k-By_k\| = 0,$$

hence $(x^*,y^*)\in\Gamma$.

Next, we will show the uniqueness of the weak cluster points of $\{x_k\}$ and $\{y_k\}$. Indeed, let $\bar{x}$, $\bar{y}$ be other weak cluster points of $\{x_k\}$ and $\{y_k\}$, respectively, then $(\bar{x},\bar{y})\in\Gamma$. From the definition of $\rho_k(x,y)$, we have

$$\begin{aligned}
\rho_k\big(x^*,y^*\big) &= \big\|x_k-x^*\big\|^2 + \big\|y_k-y^*\big\|^2 - \gamma_k\big\|Ax_k-Ax^*\big\|^2 \\
&= \|x_k-\bar{x}\|^2 + \big\|\bar{x}-x^*\big\|^2 + 2\big\langle x_k-\bar{x},\bar{x}-x^*\big\rangle \\
&\quad + \|y_k-\bar{y}\|^2 + \big\|\bar{y}-y^*\big\|^2 + 2\big\langle y_k-\bar{y},\bar{y}-y^*\big\rangle \\
&\quad - \gamma_k\big(\|Ax_k-A\bar{x}\|^2 + \big\|A\bar{x}-Ax^*\big\|^2 + 2\big\langle Ax_k-A\bar{x},A\bar{x}-Ax^*\big\rangle\big) \\
&= \rho_k(\bar{x},\bar{y}) + \big\|\bar{x}-x^*\big\|^2 + \big\|\bar{y}-y^*\big\|^2 - \gamma_k\big\|A\bar{x}-Ax^*\big\|^2 \\
&\quad + 2\big\langle x_k-\bar{x},\bar{x}-x^*\big\rangle + 2\big\langle y_k-\bar{y},\bar{y}-y^*\big\rangle - 2\gamma_k\big\langle Ax_k-A\bar{x},A\bar{x}-Ax^*\big\rangle.
\end{aligned} \tag{3.10}$$

Without loss of generality, we may assume that $x_k \rightharpoonup \bar{x}$, $y_k \rightharpoonup \bar{y}$ and $\gamma_k \to \gamma*$ because of the boundedness of the sequence $\{\gamma_k\}$. By passing to the limit in relation (3.10), we obtain

$$\rho(x^*, y^*) = \rho(\bar{x}, \bar{y}) + \|\bar{x} - x^*\|^2 + \|\bar{y} - y^*\|^2 - \gamma^* \|A\bar{x} - Ax^*\|^2.$$

Reversing the role of (x^*, y^*) and $(\bar{x}, \bar{y})$, we also have

$$\rho(\bar{x}, \bar{y}) = \rho(x^*, y^*) + \|x^* - \bar{x}\|^2 + \|y^* - \bar{y}\|^2 - \gamma^* \|Ax^* - A\bar{x}\|^2.$$

By adding the two last equalities, and having in mind that $\{\gamma_k\}$ is a non-decreasing sequence satisfying $1 - \gamma_k \lambda_A > \varepsilon \lambda_A$, we obtain

$$\varepsilon \lambda_A \|x^* - \bar{x}\|^2 + \|y^* - \bar{y}\|^2 \le 0.$$

Hence $x^* = \bar{x}$ and $y^* = \bar{y}$, this implies that the whole sequence $\{(x_k, y_k)\}$ weakly converges to a solution of problem (1.8), which completes the proof. □

Remark 3.2 Taking $\alpha_n = \beta_n = \frac{1}{2}$ in Algorithm 1.1, it follows from Remark 2.2 that Theorem 3.1 becomes Theorem 1.1, which is proved by Moudafi [17].

Theorem 3.3 *Let H_1, H_2, H_3 be real Hilbert spaces. Given two bounded linear operators $A : H_1 \to H_3$, $B : H_2 \to H_3$, let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be quasi-nonexpansive mappings with nonempty fixed point set $F(U)$ and $F(T)$. Assume that $U - I$, $T - I$ are demiclosed at origin, and the solution set Γ of* (1.8) *is nonempty. Let $\{\gamma_k\}$ be a positive non-decreasing sequence such that $\gamma_k \in (\varepsilon, \min(\frac{1}{\lambda_A}, \frac{1}{\lambda_B}) - \varepsilon)$, where λ_A, λ_B stand for the spectral radiuses of A^*A and B^*B, respectively, and ε is small enough. Then the sequence $\{(x_k, y_k)\}$ generated by Algorithm* 1.2 *weakly converges to a solution (x^*, y^*) of* (1.8), *provided that $\{\alpha_k\}$ is an non-increasing sequence such that $\{\alpha_k\} \subset (\delta, 1 - \delta)$ for small enough $\delta > 0$. Moreover, $\|Ax_k - By_k\| \to 0$, $\|x_k - x_{k+1}\| \to 0$ and $\|y_k - y_{k+1}\| \to 0$ as $k \to \infty$.*

Proof Taking $(x, y) \in \Gamma$, *i.e.*, $x \in F(U)$; $y \in F(T)$ and $Ax = By$. By repeating the proof of Theorem 3.1, we have that (3.6) is true.

Using the fact that U and T are quasi-nonexpansive mappings, it follows from Lemma 2.3 that

$$\begin{aligned}\|x_{k+1} - x\|^2 &= \alpha_k \|x_k - x\|^2 + (1 - \alpha_k)\|U(u_k) - x\|^2 - \alpha_k(1 - \alpha_k)\|U(u_k) - x_k\|^2 \\ &\le \alpha_k \|x_k - x\|^2 + (1 - \alpha_k)\|u_k - x\|^2 - \alpha_k(1 - \alpha_k)\|U(u_k) - x_k\|^2\end{aligned}$$

and

$$\|y_{k+1} - y\|^2 \le \alpha_k \|y_k - y\|^2 + (1 - \alpha_k)\|v_{k+1} - y\|^2 - \alpha_k(1 - \alpha_k)\|T(v_{k+1}) - y_k\|^2.$$

So, by (3.6) and the assumption on $\{\alpha_k\}$, we have

$$\begin{aligned}&\|x_{k+1} - x\|^2 + \|y_{k+1} - y\|^2 \\ &\quad \le \|x_k - x\|^2 + \|y_k - y\|^2 - \gamma_k(1 - \alpha_k)\|Ax_k - Ax\|^2 + \gamma_{k+1}(1 - \alpha_{k+1})\|Ax_{k+1} - Ax\|^2\end{aligned}$$

$$
\begin{aligned}
&- \gamma_k(1-\alpha_k)(1-\lambda_A\gamma_k)\|Ax_k - By_k\|^2 - \gamma_k(1-\alpha_k)(1-\lambda_B\gamma_k)\|Ax_{k+1} - By_k\|^2 \\
&- \alpha_k(1-\alpha_k)\big\|U(u_k) - x_k\big\|^2 - \alpha_k(1-\alpha_k)\big\|T(v_{k+1}) - y_k\big\|^2. \qquad (3.11)
\end{aligned}
$$

Now, by setting $\rho_k(x,y) := \|x_k - x\|^2 + \|y_k - y\|^2 - \gamma_k(1-\alpha_k)\|Ax_k - Ax\|^2$, we obtain the following inequality:

$$
\begin{aligned}
\rho_{k+1}(x,y) \le \rho_k(x,y) &- \gamma_k(1-\alpha_k)(1-\lambda_A\gamma_k)\|Ax_k - By_k\|^2 \\
&- \gamma_k(1-\alpha_k)(1-\lambda_B\gamma_k)\|Ax_{k+1} - By_k\|^2 \\
&- \alpha_k(1-\alpha_k)\big\|U(u_k) - x_k\big\|^2 - \alpha_k(1-\alpha_k)\big\|T(v_{k+1}) - y_k\big\|^2. \qquad (3.12)
\end{aligned}
$$

Following the lines of the proof of Theorem 3.1, by the conditions on $\{\gamma_k\}$ and $\{\alpha_k\}$, we have that the sequence $\{\rho_k(x,y)\}$ converges to some finite limit, say $\rho(x,y)$. Furthermore, we obtain

$$
\lim_{k\to\infty} \|Ax_k - By_k\| = \lim_{k\to\infty} \|Ax_{k+1} - By_k\| = \lim_{k\to\infty} \big\|U(u_k) - x_k\big\| = \lim_{k\to\infty} \|Tv_{k+1} - y_k\| = 0.
$$

Since

$$
\|u_k - x_k\| = \gamma_k \big\|A^*(Ax_k - By_k)\big\|,
$$

and $\{\gamma_k\}$ is bounded, we have $\lim_{k\to\infty} \|u_k - x_k\| = 0$. It follows from

$$
\lim_{k\to\infty} \|x_{k+1} - x_k\| = \lim_{k\to\infty} (1-\alpha_k)\big\|U(u_k) - x_k\big\| = 0
$$

that $\{x_k\}$ is asymptotically regular. Similarly, $\lim_{k\to\infty} \|v_{k+1} - y_k\| = 0$ and $\{y_k\}$ is asymptotically regular, too.

The rest of the proof is analogous to that of Theorem 3.1. □

4 Applications

We now turn our attention to providing some applications relying on some convex and nonlinear analysis notions, see, for example, [25].

4.1 Convex feasibility problem (1.3)

Taking $U = P_C$ and $T = P_Q$, we have the following alterative Mann iterative algorithms for CFP (1.3).

Algorithm 4.1 Let $x_0 \in H_1$, $y_0 \in H_2$ be arbitrary.

$$
\begin{cases}
u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\
x_{k+1} = \alpha_k u_k + (1-\alpha_k)P_C(u_k), \\
v_{k+1} = y_k + \gamma_k B^*(Ax_{k+1} - By_k), \\
y_{k+1} = \beta_k v_{k+1} + (1-\beta_k)P_Q(v_{k+1}).
\end{cases}
$$

Algorithm 4.2 Let $x_0 \in H_1$, $y_0 \in H_2$ be arbitrary.

$$\begin{cases} u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = \alpha_k x_k + (1-\alpha_k)P_C(u_k), \\ v_{k+1} = y_k + \gamma_k B^*(Ax_{k+1} - By_k), \\ y_{k+1} = \alpha_k y_k + (1-\alpha_k)P_Q(v_{k+1}). \end{cases}$$

4.2 Variational problems via resolvent mappings

Given a maximal monotone operator $M : H_1 \to 2^{H_1}$, it is well known that its associated resolvent mapping, $J_\mu^M(x) := (I + \mu M)^{-1}$, is quasi-nonexpansive and $0 \in M(x) \Leftrightarrow x = J_\mu^M(x)$. In other words, zeroes of M are exactly fixed-points of its resolvent mapping. By taking $U = J_\mu^M$, $T = J_\nu^S$, where $N : H_2 \to 2^{H_2}$ is another maximal monotone operator, the problem under consideration is nothing but

$$\text{find } x^* \in M^{-1}(0), y^* \in N^{-1}(0) \text{ such that } Ax^* = By^*, \tag{4.1}$$

and the algorithms take the following equivalent form.

Algorithm 4.3 Let $x_0 \in H_1$, $y_0 \in H_2$ be arbitrary.

$$\begin{cases} u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = \alpha_k u_k + (1-\alpha_k)J_\mu^M(u_k), \\ v_{k+1} = y_k + \gamma_k B^*(Ax_{k+1} - By_k), \\ y_{k+1} = \beta_k v_{k+1} + (1-\beta_k)J_\nu^S(v_{k+1}). \end{cases}$$

Algorithm 4.4 Let $x_0 \in H_1$, $y_0 \in H_2$ be arbitrary.

$$\begin{cases} u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = \alpha_k x_k + (1-\alpha_k)J_\mu^M(u_k), \\ v_{k+1} = y_k + \gamma_k B^*(Ax_{k+1} - By_k), \\ y_{k+1} = \alpha_k y_k + (1-\alpha_k)J_\nu^S(v_{k+1}). \end{cases}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors carried out the algorithm design and drafted the manuscript. The authors completed the proof. All authors read and approved the final manuscript.

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