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Fixed point theory for generalized Ćirić quasi-contraction maps in metric spaces

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Abstract
In this paper, we first give a new fixed point theorem for generalized Ćirić quasi-contraction maps in generalized metric spaces. Then we derive a common fixed point result for quasi-contractive type maps. Some examples are given to support our results. Our results extend and improve some fixed point and common fixed point theorems in the literature.

MSC: 47H10

Keywords: fixed points; common fixed point; generalized Ćirić quasi-contraction maps

1 Introduction and preliminaries

The well-known Banach fixed point theorem asserts that if (X,d) is a complete metric space and $T: X \to X$ is a map such that

$$d(Tx,Ty) \le cd(x,y) \quad \text{for each } x,y \in X,$$

where $0 \le c < 1$, then T has a unique fixed point $\bar{x} \in X$ and for any $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to $\bar{x}$.

In recent years, a number of generalizations of the above Banach contraction principle have appeared. Of all these, the following generalization of Ćirić [1] stands at the top.

Theorem 1.1 *Let (X,d) be a complete metric space. Let $T: X \to X$ be a Ćirić quasi-contraction map, that is, there exists $c < 1$ such that*

$$d(Tx,Ty) \le c \max\bigl\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\bigr\}$$

for any $x,y \in X$. Then T has a unique fixed point $\bar{x} \in X$ and for any $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to $\bar{x}$.

For other generalizations of the above theorem, see [2] and the references therein.

2 Main results

Let X be a nonempty set and let $d: X \times X \to [0,\infty]$ be a mapping. If d satisfies all of the usual conditions of a metric except that the value of d may be infinity, we say that (X,d) is a *generalized metric space*.

We now introduce the concept of a *generalized Ćirić quasi-contraction* map in generalized metric spaces.

Definition 2.1 Let (X,d) be a generalized metric space. The self-map $T: X \to X$ is said to be a *generalized Ćirić quasi-contraction* if

$$d(Tx,Ty) \le \alpha\big(d(x,y)\big)\max\big\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big\}$$

for any $x,y \in X$, where $\alpha: [0,\infty] \to [0,1)$ is a mapping.

As the following simple example due to Sastry and Naidu [3] shows, Theorem 1.1 is not true for generalized Ćirić quasi-contraction maps even if we suppose α is continuous and increasing.

Example 2.2 Let $X = [1,\infty)$ with the usual metric, $T: X \to X$ be given by $Tx = 2x$. Define $\alpha: [0,\infty) \to [0,1)$ by $\alpha(t) = \frac{2t}{1+2t}$. Then, clearly, α is continuous and increasing, and

$$|Tx - Ty| \le \alpha\big(|x-y|\big)\max\big\{|x-y|, |x-Tx|, |y-Ty|, |x-Ty|, |y-Tx|\big\}$$

for each $x,y \in X$, but T has no fixed point.

Now, a natural question is what further conditions are to be imposed on T or α to guarantee the existence of a fixed point for T? For some partial answers to this question and application of quasi-contraction maps to variational inequalities, see [4] and the references therein.

Now, we are ready to state our main result.

Theorem 2.3 *Let (X,d) be a complete generalized metric space. Let $T: X \to X$ be a generalized Ćirić quasi-contraction map such that α satisfies*

$$\limsup_{t\to r}\alpha(t) < 1 \quad \textit{for each } r \in [0,\infty).$$

Assume that there exists an $x_0 \in X$ with the bounded orbit, that is, the sequence $\{T^n x_0\}$ is bounded. Furthermore, suppose that $d(x,Tx) < \infty$ for each $x \in X$. Then T has a fixed point $\bar{x} \in X$ and $\lim_{n\to\infty} T^n x_0 = \bar{x}$. Moreover, if $\bar{y}$ is a fixed point of T, then either $d(\bar{x},\bar{y}) = \infty$ or $\bar{x} = \bar{y}$.

Proof If for some $n_0 \in \mathbb{N}$, $T^{n_0-1}x_0 = T^{n_0}x = T(T^{n_0-1}x_0)$, then $T^n x_0 = T^{n_0-1}x_0$ for $n \ge n_0$. Thus, $T^{n_0-1}x_0$ is a fixed point of T, the sequence $\{T^n x_0\}$ is convergent to $T^{n_0-1}x_0$, and we are finished (note that $T^n x_0 = T^{n_0-1}x_0$ for each $n \ge n_0$). So, we may assume that $T^{n-1}x_0 \ne T^n x_0$ for each $n \in \mathbb{N}$. Now, we show that there exists $0 < c < 1$ such that

$$\alpha\big(d\big(T^{n-1}x_0, T^n x_0\big)\big) < c \quad \text{for each } n = 0,1,2,3,\ldots. \tag{2.1}$$

On the contrary, assume that

$$\lim_{k\to\infty}\alpha\big(d\big(T^{n_k-1}x_0, T^{n_k}x_0\big)\big) = 1$$

for some subsequence $\{\alpha(d(T^{n_k-1}x_0, T^{n_k}x_0))\}$ of $\{\alpha(d(T^{n-1}x_0, T^n x_0))\}$. Since by our assumption the sequence $\{d(T^{n-1}x_0, T^n x_0)\}$ is bounded, then the subsequence $\{d(T^{n_k-1}x_0, T^{n_k}x_0)\}$ is bounded too, and so, by passing to subsequences if necessary, we may assume that it is convergent. Let $r_0 = \lim_{k\to\infty} d(T^{n_k-1}x_0, T^{n_k}x_0)$. Then from (2.1), we have $\limsup_{t\to r_0} \alpha(t) = 1$, a contradiction. Thus, (2.1) holds.

Now, we show that $\{T^n x_0\}$ is a Cauchy sequence. To prove the claim, we first show by induction that for each $n \geq 2$,

$$d\left(T^{n-1}x_0, T^n x_0\right) \leq Kc^{n-1}, \tag{2.2}$$

where K is a bound for the bounded sequence $\{d(x_0, T^n x_0)\}_n$. If $n = 2$ then, we get

$$\begin{aligned} d\left(Tx_0, T^2x_0\right) &\leq \alpha\left(d(x_0, Tx_0)\right) \max\left\{d(x_0, Tx_0), d\left(Tx_0, T^2x_0\right), d\left(x_0, T^2x_0\right)\right\} \\ &= \alpha\left(d(x_0, Tx_0)\right) \max\left\{d(x_0, Tx_0), d\left(x_0, T^2x_0\right)\right\} \leq Kc. \end{aligned}$$

Thus, (2.2) holds for $n = 2$. Suppose that (2.2) holds for each $k < n$, and we show that it holds for $k = n$. Since T is a generalized Ćirić quasi-contraction map, then we have

$$d\left(T^{n-1}x_0, T^n x_0\right) \leq \alpha\left(T^{n-2}x_0, T^{n-1}x_0\right)u \leq cu,$$

where

$$u \in \left\{d\left(T^{n-2}x_0, T^{n-1}x_0\right), d\left(T^{n-2}x_0, T^n x_0\right)\right\}.$$

It is trivial that (2.2) holds if $u = d(T^{n-2}x_0, T^{n-1}x_0)$. Now, suppose that $u = d(T^{n-2}x_0, T^n x_0)$. In this case, we have

$$d\left(T^{n-2}x_0, T^n x_0\right) \leq cu_1,$$

where

$$\begin{aligned} u_1 \in \Big\{ &d\left(T^{n-3}x_0, T^{n-1}x_0\right), d\left(T^{n-2}x_0, T^{n-1}x_0\right), \\ &d\left(T^{n-3}x_0, T^{n-2}x_0\right), d\left(T^{n-3}x_0, T^n x_0\right), d\left(T^{n-1}x_0, T^n x_0\right)\Big\}. \end{aligned}$$

Again, it is trivial that (2.2) holds if $u_1 = d(T^{n-1}x_0, T^n x_0)$ or $u_1 = d(T^{n-3}x_0, T^{n-2}x_0)$. If $u_1 = d(T^{n-2}x_0, T^{n-1}x_0)$, then

$$d\left(T^{n-1}x_0, T^n x_0\right) \leq c^2 d\left(T^{n-2}x_0, T^{n-1}x_0\right).$$

By the assumption of induction,

$$d\left(T^{n-2}x_0, T^{n-1}x_0\right) \leq Kc^{n-2}.$$

Hence,

$$d\left(T^{n-1}x_0, T^n x_0\right) \leq Kc^n \leq Kc^{n-1}.$$

If $u_1 = d(T^{n-3}x_0, T^{n-1}x_0)$, then

$$d\big(T^{n-1}x_0, T^n x_0\big) \leq c^2 d\big(T^{n-3}x_0, T^{n-1}x_0\big).$$

If $u_1 = d(T^{n-3}x_0, T^n x_0)$, then

$$d\big(T^{n-1}x_0, T^n x_0\big) \leq c^2 d\big(T^{n-3}x_0, T^n x_0\big).$$

Therefore, by continuing this process, we see that (2.2) holds for each $n \geq 2$. From (2.2), we deduce that $\{T^n x_0\}$ is a Cauchy sequence and since (X, d) is a generalized complete metric space, then there exists an $\bar{x} \in X$ such that $\lim_{n\to\infty} T^n x_0 = \bar{x}$. Now, we show that $\bar{x}$ is a fixed point of T. To show the claim, we first show that there exists $0 < k < 1$ such that $\alpha(d(\bar{x}, T^n x_0)) < k$ for each $n \in \mathbb{N}$. On the contrary, assume that $\lim_{j\to\infty} \alpha(d(\bar{x}, T^{n_j}x_0)) = 1$ for some subsequence n_j. Since $\lim_{j\to\infty} d(\bar{x}, T^{n_j}x_0) = 0$, then from the above, we get $\limsup_{t\to 0^+} \alpha(t) = 1$, a contradiction. Since T is a generalized Ćirić quasi-contraction, then we have

$$\begin{aligned}
&d\big(T\bar{x}, T^{n+1}x_0\big) \\
&\quad \leq \alpha\big(d\big(\bar{x}, T^n x_0\big)\big) \max\big\{d\big(\bar{x}, T^n x_0\big), d(\bar{x}, T\bar{x}), \\
&\qquad d\big(T^n x_0, T^{n+1}x_0\big), d\big(\bar{x}, T^{n+1}x_0\big), d\big(T^n x_0, T\bar{x}\big)\big\} \\
&\quad \leq k \max\big\{d\big(\bar{x}, T^n x_0\big), d(\bar{x}, T\bar{x}), d\big(T^n x_0, T^{n+1}x_0\big), d\big(\bar{x}, T^{n+1}x_0\big), d\big(T^n x_0, T\bar{x}\big)\big\}.
\end{aligned}$$

Then we have

$$d(T\bar{x}, \bar{x}) = \limsup_{n\to\infty} d\big(T\bar{x}, T^{n+1}x_0\big) \leq k \limsup_{n\to\infty} d\big(T\bar{x}, T^n x_0\big) = kd(T\bar{x}, \bar{x}),$$

which yields $d(T\bar{x}, \bar{x}) = 0$, and so $\bar{x} = T\bar{x}$ (note that $0 < k < 1$ and $d(T\bar{x}, \bar{x}) < \infty$ by our assumptions). Now, let us assume that $\bar{x}$ and $\bar{y}$ are fixed points of T such that $d(\bar{x}, \bar{y}) < \infty$. Then

$$\begin{aligned}
d(\bar{x}, \bar{y}) &= d(T\bar{x}, T\bar{y}) \\
&\leq \alpha\big(d(\bar{x}, \bar{y})\big) \max\big\{d(\bar{x}, \bar{y}), d(\bar{x}, T\bar{x}), d(\bar{y}, T\bar{y}), d(\bar{x}, T\bar{y}), d(\bar{y}, T\bar{x})\big\} \\
&= \alpha\big(d(\bar{x}, \bar{y})\big) d(\bar{x}, \bar{y}),
\end{aligned}$$

and so $\bar{x} = \bar{y}$ (note that $\alpha(d(\bar{x}, \bar{y})) < 1$). □

The following example shows that in the statement of Theorem 2.3, the condition $d(x, Tx) < \infty$ for each $x \in X$ is necessary.

Example 2.4 Let $X = \{0, \infty\}$, $d(0, 0) = d(\infty, \infty) = 0$ and let $d(0, \infty) = \infty$. Let $T : X \to X$ be given by $T0 = \infty$ and $T\infty = 0$. Then

$$d(Tx, Ty) \leq \frac{1}{2} d(x, y) \leq \frac{1}{2} \max\big\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\big\}$$

for each $x, y \in X$, but T is fixed point free.

Example 2.5 Let $X = [0,\infty]$, $d(x,y) = |x-y|$ for each $x,y \in [0,\infty)$, $d(x,\infty) = \infty$ for each $x \in [0,\infty)$ and let $d(\infty,\infty) = 0$. Then (X,d) is a complete generalized metric space. Let $T : X \to X$ be given by $Tx = 2x$ for each $x \in [0,\infty)$ and $T\infty = \infty$. Define $\alpha : [0,\infty] \to [0,1)$ by $\alpha(t) = \frac{2t}{1+2t}$ for each $t \in [0,\infty)$ and $\alpha(\infty) = \frac{1}{2}$. Then we have

$$|Tx - Ty| \le \alpha\big(|x-y|\big) \max\big\{|x-y|, |x-Tx|, |y-Ty|, |x-Ty|, |y-Tx|\big\},$$

and $d(x,Tx) < \infty$ for each $x,y \in X$. Thus, all of the assumptions of Theorem 2.3 are satisfied, and so T has a unique fixed point ($x = \infty$ is a unique fixed point of T). But we cannot invoke the above mentioned theorem of Ćirić to show the existence of a fixed point for T.

To prove the following common fixed point result, we use the technique in [5].

Corollary 2.6 *Let (X,d) be a complete metric space and let the self-maps T and S satisfy the contractive condition*

$$\begin{aligned} &d(Tx,Ty) \\ &\quad \le \alpha\big(d(Sx,Sy)\big) \max\big\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), d(Sx,Ty), d(Sy,Tx)\big\} \end{aligned}$$

for each $x,y \in X$, where α satisfies $\limsup_{t\to r^+} \alpha(t) < 1$ for each $r \in [0,\infty)$. If $TX \subseteq SX$ and SX is a complete subset of X, then T and S have a unique coincidence point in X. Moreover, if T and S are weakly compatible (i.e., they commute at their coincidence points), then T and S have a unique common fixed point.

Proof It is well known that there exists $E \subseteq X$ such that $SE = SX$ and $S : E \to X$ is one-to-one. Now, define a map $U : SE \to SE$ by $U(Sx) = Tx$. Since S is one-to-one on E, U is well defined. Note that

$$\begin{aligned} &d\big(U(Sx), U(Sy)\big) \\ &\quad = U(Tx,Ty) \\ &\quad \le \alpha\big(d(Sx,Sy)\big) \max\big\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), d(Sx,Ty), d(Sy,Tx)\big\} \end{aligned}$$

for all $Sx, Sy \in SE$. Since $SE = SX$ is complete, by using Theorem 2.3, there exists $\bar{x} \in X$ such that $U(S\bar{x}) = S\bar{x}$. Then $T\bar{x} = S\bar{x}$, and so T and S have a coincidence point, which is also unique. Since $T\bar{x} = S\bar{x}$ and T and S commute, then we have

$$T(T\bar{x}) = T^2\bar{x} = TS\bar{x} = ST\bar{x} = S^2\bar{x} = S(S\bar{x}).$$

Thus, $T\bar{x} = S\bar{x}$ is also a coincidence point of T and S. By the uniqueness of a coincidence point of T and S, we get $T\bar{x} = S\bar{x} = \bar{x}$. $\square$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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