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Coupled coincidence point theorems for compatible mappings in partially ordered intuitionistic generalized fuzzy metric spaces

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Abstract

In this work, we introduce the notion of intuitionistic generalized fuzzy metric space by using the idea of intuitionistic fuzzy set due to Atanassov. We determine some coupled coincidence point results for compatibility of two mappings, that is, $F : X \times X \to X$ and $g : X \to X$, in the framework of intuitionistic generalized fuzzy metric spaces endowed with partial order. An interesting example is also displayed here in support of our result.

Keywords: *t*-norm; *t*-conorm; coupled coincidence point; compatible mappings; generalized metric space

1 Preliminaries, background and notation

Zadeh [1] suggests the creation of what are called fuzzy sets which base their development on the idea that the membership of an element to a set is indicated by a number between 0 and 1, having non-membership for 0, membership for 1 and different degrees of membership for the numbers between 0 and 1. Such sets have proved very useful in the description of phenomena governed by imprecise parameters as well as for the development of non-bivalent logic models. Using the idea of fuzzy set, many authors have introduced the concept of fuzzy metric in different point of views [2–4]. George and Veeramani [5] modified the concept of fuzzy metric space due to Kramosil and Michalek [4].

Atanassov [6] suggests a generalization of fuzzy sets making the degrees of membership and non-membership intervene to describe the vinculation of an element to a set, so that the sum of these degrees is always less or equal to 1, that is, an intuitionistic fuzzy set. Park [7] introduced and discussed the concept of intuitionistic fuzzy metric space which is based on the idea of intuitionistic fuzzy set and the notion of fuzzy metric space given by George and Veeramani [5]. Afterward, it was followed by the notion of intuitionistic fuzzy normed space and intuitionistic fuzzy bounded linear operators [8–18]. The authors of [19–21] established an interesting relationship between three various disciplines: the theory of fuzzy normed spaces, the theory of stability of functional equations and fixed point theory.

The concept of generalized metric space was introduced and studied by Mustafa and Sims [22] and was later used to determine coupled fixed point theorems and related results by a number of authors [23–33]. Sun and Yang [34] defined the notion of generalized fuzzy metric space with the help of generalized metric space and fuzzy sets, and further studied it in [35, 36] to deal with some fixed point theory.

In this work, we present an interesting generalization of generalized fuzzy metric space with the help of an intuitionistic fuzzy set and call it an intuitionistic generalized fuzzy metric space. We also define the notions of convergence, Cauchy sequences and compatibility of two mappings in this setup. Further, we establish coupled coincidence point and coupled fixed point results for compatible mappings in partially ordered intuitionistic generalized fuzzy metric spaces and construct an example in support of our result.

Now, we recall some definitions and notations which we will used throughout the article. We shall assume throughout this paper that the symbols $\mathbb{R}$ and $\mathbb{N}$ denote the set of real and natural numbers, respectively. In this section, we recall some definitions and preliminary results which we will use throughout the paper. Mustafa and Sims [22] defined the notion of generalized metric space as follows.

Let X be a nonempty set and a mapping $\mathcal{G} : X \times X \times X \to \mathbb{R}$. Then $\mathcal{G}$ is called a *generalized metric* (for short, $\mathcal{G}$*-metric*) on X and $(X, \mathcal{G})$ a *generalized metric space* or simply a $\mathcal{G}$*-metric space* if the following conditions are satisfied:

(i) $\mathcal{G}(x,y,z) = 0$ if $x = y = z$,

(ii) $\mathcal{G}(x,x,y) > 0$ for all $x, y \in X$ and $x \neq y$,

(iii) $\mathcal{G}(x,x,y) \le \mathcal{G}(x,y,z)$ for all $x,y,z \in X$ and $y \neq z$,

(iv) $\mathcal{G}(x,y,z) = \mathcal{G}(x,z,y) = \mathcal{G}(y,z,x) = \cdots$ (symmetry in all three variables),

(v) $\mathcal{G}(x,y,z) \le \mathcal{G}(x,a,a) + \mathcal{G}(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality).

We remark that every $\mathcal{G}$-metric on X defines a metric $d_{\mathcal{G}}$ on X by $d_{\mathcal{G}}(x,y) = \mathcal{G}(x,y,y) + \mathcal{G}(y,x,x)$ for all $x, y \in X$.

For example, let (X,d) be a metric space. The function $\mathcal{G} : X \times X \times X \to [0,\infty)$ is defined by

$$\mathcal{G}(x,y,z) = \max\{d(x,y), d(y,z), d(z,x)\}$$

or

$$\mathcal{G}(x,y,z) = d(x,y) + d(y,z) + d(z,x)$$

for all $x,y,z \in X$. Then $(X,\mathcal{G})$ is a $\mathcal{G}$-metric space [22].

Bhaskar and Lakshmikantham [37] presented the definitions of mixed monotone property and coupled fixed point for the contractive mapping $F : X \times X \to X$ and established some coupled fixed point theorems for a mixed monotone operator. As an application of the coupled fixed point theorems, they determined the existence and uniqueness of the solution of a periodic boundary value problem. Afterward, Lakshmikantham and Ćirić [38] presented the notions of mixed g-monotone property and coupled coincidence point and proved coupled coincidence and coupled common fixed point theorems for nonlinear contractive mappings in partially ordered complete metric spaces. Many authors obtained important fixed point theorems, for details and background of fixed point theory, we refer to [39–48] and references therein.

Definition 1.1 [37] Let $(X, \le)$ be a partially ordered set and $F : X \times X \to X$ be a mapping. Then a map F is said to have the *mixed monotone property* if $F(x,y)$ is monotone non-decreasing in x and is monotone non-increasing in y; that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad x_1 \leq x_2 \quad \text{implies} \quad F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad y_1 \leq y_2 \quad \text{implies} \quad F(x, y_1) \geq F(x, y_2).$$

Definition 1.2 [37] An element $(x, y) \in X \times X$ is said to be a *coupled fixed point* of the mapping $F : X \times X \to X$ if

$$F(x, y) = x \quad \text{and} \quad F(y, x) = y.$$

Definition 1.3 [38] Let $(X, \leq)$ be a partially ordered set, and let $F : X \times X \to X$ and $g : X \to X$ be two mappings. Then a map F is said to have the *mixed g-monotone property* if F is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad g(x_1) \leq g(x_2) \quad \text{implies} \quad F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad g(y_1) \leq g(y_2) \quad \text{implies} \quad F(x, y_1) \geq F(x, y_2).$$

Definition 1.4 [38] An element $(x, y) \in X \times X$ is said to be a *coupled coincidence point* of mappings $F : X \times X \to X$ and $g : X \to X$ if

$$F(x, y) = gx \quad \text{and} \quad F(y, x) = gy.$$

Definition 1.5 [38] Let X be a nonempty set, and let $F : X \times X \to X$ and $g : X \to X$ be two mappings. Then F and g are commutative if for all $x, y, z \in X$, we have

$$g\bigl(F(x, y)\bigr) = F(gx, gy).$$

Definition 1.6 [41] Let (X, d) be an IGFM-space. The mappings F and g, where $F : X \times X \to X$ and $g : X \to X$, are said to be *compatible* if

$$\lim_{n\to\infty} d\bigl(g\bigl(F(x_n, y_n)\bigr), F\bigl(g(x_n), g(y_n)\bigr), t\bigr) = 0$$

and

$$\lim_{n\to\infty} d\bigl(g\bigl(F(y_n, x_n)\bigr), F\bigl(g(y_n), g(x_n)\bigr), t\bigr) = 0,$$

whenever (x_n) and (y_n) are sequences in X such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$ for some $x, y \in X$.

Afterward, the concept of compatible mappings was introduced in fuzzy metric spaces by Hu [49]. In the recent past, Hu and Luo [36] defined and studied this notion in the framework of generalized fuzzy metric spaces.

2 Intuitionistic generalized fuzzy metric space

Let us recall [50] that a binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be a *continuous t-norm* if it satisfies the following conditions:

(a) $*$ is associative and commutative,
(b) $*$ is continuous,
(c) $a * 1 = a$ for all $a \in [0,1]$,
(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a,b,c,d \in [0,1]$.

Similarly, a binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is said to be a *continuous t-conorm* if it satisfies the following conditions:

(a′) $\diamond$ is associative and commutative,
(b′) $\diamond$ is continuous,
(c′) $a \diamond 0 = a$ for all $a \in [0,1]$,
(d′) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ for each $a,b,c,d \in [0,1]$.

Remark 2.1 The concepts of triangular norms (t-norms) and triangular conorms (t-conorms) are known as the axiomatic skeletons that we use for characterizing fuzzy intersections and unions, respectively.

In 2004, Park [7] presented the notion of intuitionistic fuzzy metric space as follows: The 5-tuple $X, M, N, *, \diamond$ is said to be an *intuitionistic fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times (0,\infty)$ satisfying the following conditions for all $x,y,z \in X$, $s,t > 0$: (i) $M(x,y,t) + N(x,y,t) \leq 1$, (ii) $M(x,y,t) > 0$, (iii) $M(x,y,t) = 1$ if and only if $x = y$, (iv) $M(x,y,t) = M(y,x,t)$, (v) $M(x,y,t) * M(y,z,t) \leq M(x,z,t+s)$, (vi) $M(x,y,\cdot) : (0,\infty) \to (0,1]$ is continuous, (vii) $N(x,y,t) < 1$, (viii) $N(x,y,t) = 0$ if and only if $x = y$, (ix) $N(x,y,t) = N(y,x,t)$, (x) $N(x,y,t) \diamond N(y,z,s) \geq N(x,z,t+s)$, and (xi) $N(x,y,\cdot) : (0,\infty) \to (0,1]$ is continuous.

Now, we introduce the notion of intuitionistic generalized fuzzy metric space by using the concepts of continuous t-norm and t-conorm.

Definition 2.2 The 5-tuple $(X,G,H,*,\diamond)$ is said to be an *intuitionistic generalized fuzzy metric space* (for short, IGFM-space) if X is an arbitrary nonempty set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, and G, H are fuzzy sets on $X^3 \times (0,\infty)$ satisfying the following conditions. For every $x,y,z,a \in X$ and $s,t > 0$,

(i) $G(x,y,z,t) + H(x,y,z,t) \leq 1$,
(ii) $G(x,x,y,t) > 0$ for $x \neq y$,
(iii) $G(x,x,y,t) \geq G(x,y,z,t)$ for $y \neq z$,
(iv) $G(x,y,z,t) = 1$ if and only if $x = y = z$,
(v) $G(x,y,z,t) = G(p(x,y,z),t)$, where p is a permutation function,
(vi) $G(x,a,a,t) * G(a,y,z,s) \leq G(x,y,z,t+s)$,
(vii) $G(x,y,z,\cdot) : (0,\infty) \to [0,1]$ is continuous,
(viii) G is a non-decreasing function on $\mathbb{R}^+$,

$$\lim_{t\to\infty} G(x,y,z,t) = 1, \qquad \lim_{t\to 0} G(x,y,z,t) = 0, \quad \forall x,y,z \in X, t > 0,$$

(xi) $H(x,x,y,t) < 1$ for $x \neq y$,
(x) $H(x,x,y,t) \leq H(x,y,z,t)$ for $y \neq z$,

(xi) $H(x,y,z,t) = 0$ if and only if $x = y = z$,
(xii) $H(x,y,z,t) = H(p(x,y,z),t)$, where p is a permutation function,
(xiii) $H(x,a,a,t) \diamond H(a,y,z,s) \geq H(x,y,z,t+s)$,
(xiv) $H(x,y,z,\cdot) : (0,\infty) \to [0,1]$ is continuous,
(xv) H is a non-increasing function on $\mathbb{R}^+$,

$$\lim_{t\to\infty} H(x,y,z,t) = 0, \qquad \lim_{t\to 0} H(x,y,z,t) = 1 \quad \forall x,y,z \in X, t > 0.$$

In this case, the pair (G,H) is called an *intuitionistic generalized fuzzy metric* on X.

Example 2.3 Let $(X,\mathcal{G})$ be a $\mathcal{G}$-metric space. For all $x,y,z \in X$ and every $t > 0$, consider G, H to be fuzzy sets on $X^3 \times (0,\infty)$ defined by

$$G(x,y,z,t) = \frac{t}{t+\mathcal{G}(x,y,z)} \quad \text{and} \quad H(x,y,z,t) = \frac{\mathcal{G}(x,y,z)}{t+\mathcal{G}(x,y,z)},$$

and denote

$$a * b = ab \quad \text{and} \quad a \diamond b = \min\{a+b,1\} \quad \bigl(a,b \in [0,1]\bigr).$$

Then $(X,G,H,*,\diamond)$ is an intuitionistic generalized fuzzy metric space. Notice that the above example holds even with the t-norm $a * b = \min(a,b)$ and the t-conorm $a \diamond b = \max(a,b)$. We remark that this intuitionistic generalized fuzzy metric is induced by a $\mathcal{G}$-metric $\mathcal{G}$, the standard intuitionistic generalized fuzzy metric.

Remark 2.4 In an intuitionistic generalized fuzzy metric space, $G(x,y,z,\cdot)$ is non-decreasing and $H(x,y,z,\cdot)$ is non-increasing for all $x,y,z \in X$.

Definition 2.5 Let $x \in X$, where $(X,G,H,*,\diamond)$ is an IGFM-space. Then, for $r \in (0,1)$ and $t > 0$, the set

$$B_{G,H}(x,r,t) = \bigl\{y \in X : G(x,y,y,t) > 1-r \text{ and } H(x,y,y,t) < r\bigr\}$$

is said to be an *open ball* with center x and radius r with respect to t. Note that every open ball $B_{G,H}(x,r,t)$ is an open set.

Remark 2.6 Let $(X,G,H,*,\diamond)$ be an IGFM-space. Define $\tau_{G,H} = \{A \subset X :$ for each $x \in A$, there exist $t > 0$ and $r \in (0,1)$ such that $B_{G,H}(x,r,t) \subset A\}$. Then $\tau_{G,H}$ is a topology on X (induced by the intuitionistic generalized fuzzy metric (G,H)).

Recall that a topological space is first countable if each point has a countable (decreasing) local base. Since $B_{G,H}(x,1/n,1/n)$ is a local base at x, then topology $\tau_{G,H}$ is first countable.

Definition 2.7 Let $(X,G,H,*,\diamond)$ be an IGFM-space. Then a sequence (x_j) is said to be *convergent* to $x \in X$ with respect to the intuitionistic generalized fuzzy metric (G,H) if for every $\epsilon > 0$ and $t > 0$, there exists $j_0 \in \mathbb{N}$ such that $G(x_j,x_k,x,t) > 1-\epsilon$ and $H(x_j,x_k,x,t) < \epsilon$ for all $j,k \geq j_0$. In this case, we write $x_j \stackrel{(G,H)}{\longrightarrow} x$ or (G,H)-$\lim x_j = x$.

Theorem 2.8 *Let $(X, G, H, *, \diamond)$ be an IGFM-space and $\tau_{G,H}$ be the topology induced by the fuzzy metric. Then, for a sequence (x_j) in X, $x_j \overset{(G,H)}{\longrightarrow} x$ if and only if $G(x_j, x_j, x, t) \to 1$ and $H(x_j, x_j, x, t) \to 0$ as $j \to \infty$.*

Proof Let (x_j) be convergent to x with respect to an intuitionistic generalized fuzzy metric (G, H), *i.e.*, $x_j \overset{(G,H)}{\longrightarrow} x$. Then, for every $r \in (0,1)$ and $t > 0$, there is a number $j_0 \in \mathbb{N}$ such that $x_j \in B_{G,H}(x, r, t)$ for all $j \geq j_0$. It follows that $G(x_j, x_j, x, t) > 1 - r$ and $H(x_j, x_j, x, t) < r$ and hence $1 - G(x_j, x_j, x, t) < r$ and $H(x_j, x_j, x, t) < r$. Thus, $G(x_j, x_j, x, t) \to 1$ and $H(x_j, x_j, x, t) \to 0$ as $j \to \infty$.

Conversely, suppose that $G(x_j, x_j, x, t) \to 1$ and $H(x_j, x_j, x, t) \to 0$ as $j \to \infty$ for each $t > 0$. Then, for any $t > 0$ and $r \in (0,1)$, there exists $j_0 \in \mathbb{N}$ such that $1 - G(x_j, x_j, x, t) < r$ and $H(x_j, x_j, x, t) < r$ for all $j \geq j_0$. It follows that $G(x_j, x_j, x, t) > 1 - r$ and $H(x_j, x_j, x, t) < r$ for all $j \geq j_0$. Therefore $x_j \in B_{G,H}(x, r, t)$ for all $j \geq j_0$. Hence (x_j) is convergent to x with respect to (G, H). □

Definition 2.9 Let $(X, G, H, *, \diamond)$ be an IGFM-space. Then (x_j) is a *Cauchy sequence* with respect to the intuitionistic generalized fuzzy metric (G, H) if, for every $\epsilon > 0$ and $t > 0$, there exists $j_0 \in \mathbb{N}$ such that $G(x_j, x_\ell, x_k, t) > 1 - \epsilon$ and $H(x_j, x_\ell, x_k, t) < \epsilon$ for all $j, \ell, k \geq j_0$. An IGFM-space $(X, G, H, *, \diamond)$ is said to be complete if every Cauchy sequence with respect to the intuitionistic generalized fuzzy metric (G, H) is convergent with respect to (G, H).

3 Coupled coincidence results for compatible mappings

In this section we establish coupled coincidence theorems for compatibility of two mappings in partially ordered intuitionistic generalized fuzzy metric spaces. Before proceeding further, first we define the notion of compatible mappings with respect to the intuitionistic generalized fuzzy metric (G, H) as follows.

Definition 3.1 Let $(X, G, H, *, \diamond)$ be an IGFM-space. The mappings F and g, where $F: X \times X \to X$ and $g: X \to X$, are said to be *compatible* with respect to (G, H) if for all $t > 0$,

$$\lim_{n\to\infty} G\big(g\big(F(x_n, y_n)\big), g\big(F(x_n, y_n)\big), F\big(g(x_n), g(y_n)\big), t\big) = 1,$$
$$\lim_{n\to\infty} H\big(g\big(F(x_n, y_n)\big), g\big(F(x_n, y_n)\big), F\big(g(x_n), g(y_n)\big), t\big) = 0$$

and

$$\lim_{n\to\infty} G\big(g\big(F(y_n, x_n)\big), g\big(F(y_n, x_n)\big), F\big(g(y_n), g(x_n)\big), t\big) = 1,$$
$$\lim_{n\to\infty} H\big(g\big(F(y_n, x_n)\big), g\big(F(y_n, x_n)\big), F\big(g(y_n), g(x_n)\big), t\big) = 0,$$

whenever (x_n) and (y_n) are sequences in X such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$ for some $x, y \in X$.

Secondly, we prove the following lemmas which we will used to prove our coupled coincidence theorem.

Lemma 3.2 *Let $(X, G, H, *, \diamondsuit)$ be an IGFM-space. Suppose that $\Im_\lambda : X^3 \to [0,\infty)$ is defined by*

$$\Im_\lambda(x,y,z) = \inf\bigl\{t > 0 : G(x,y,z,t) > 1-\lambda \text{ and } H(x,y,z,t) < \lambda\bigr\} \tag{3.1}$$

for all $x,y,z \in X$, $\lambda \in (0,1]$ and $t > 0$. Then, for each $\lambda \in (0,1]$, there exists $\mu \in (0,1]$ such that

$$\Im_\lambda(x_1,x_1,x_n) \le \sum_{j=1}^{n-1} \Im_\mu(x_j,x_j,x_{j+1})$$

for all $x_1, x_2, \dots, x_n \in X$.

Proof Given $\lambda \in (0,1]$ choose $\mu \in (0,1]$ such that $(1-\mu) * (1-\mu) * \cdots * (1-\mu) > 1-\lambda$ and $\mu \diamondsuit \mu \diamondsuit \cdots \diamondsuit \mu < \lambda$. Then, for any $\epsilon > 0$, write

$$\begin{aligned}
&G\Biggl(x_1,x_1,x_n, \sum_{j=1}^{n-1} \Im_\mu(x_j,x_j,x_{j+1}) + (\overbrace{\epsilon + \cdots + \epsilon}^{(n-1)\text{ times}})\Biggr) \\
&\quad\ge G\bigl(x_1,x_1,x_2,\Im_\mu(x_1,x_1,x_2)+\epsilon\bigr) * G\bigl(x_2,x_2,x_3,\Im_\mu(x_2,x_2,x_3)+\epsilon\bigr) * \cdots \\
&\qquad * G\bigl(x_{n-1},x_{n-1},x_n,\Im_\mu(x_{n-1},x_{n-1},x_n)+\epsilon\bigr) \\
&\quad > (1-\mu) * (1-\mu) * \cdots * (1-\mu) > 1-\lambda
\end{aligned}$$

and, similarly,

$$\begin{aligned}
&H\Biggl(x_1,x_1,x_n, \sum_{j=1}^{n-1} \Im_\mu(x_j,x_j,x_{j+1}) + (\overbrace{\epsilon + \cdots + \epsilon}^{(n-1)\text{ times}})\Biggr) \\
&\quad\le H\bigl(x_1,x_1,x_2,\Im_\mu(x_1,x_1,x_2)+\epsilon\bigr) \diamondsuit G\bigl(x_2,x_2,x_3,\Im_\mu(x_2,x_2,x_3)+\epsilon\bigr) \diamondsuit \cdots \\
&\qquad \diamondsuit H\bigl(x_{n-1},x_{n-1},x_n,\Im_\mu(x_{n-1},x_{n-1},x_n)+\epsilon\bigr) \\
&\quad < \mu \diamondsuit \mu \diamondsuit \cdots \diamondsuit \mu < \lambda.
\end{aligned}$$

This implies

$$\begin{aligned}
\Im_\lambda(x_1,x_1,x_n) &\le \Im_\mu(x_1,x_1,x_2) + \Im_\mu(x_2,x_2,x_3) + \cdots \\
&\quad + \Im_\mu(x_{n-1},x_{n-1},x_n) + (n-1)\epsilon.
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we have

$$\Im_\lambda(x_1,x_1,x_n) \le \Im_\mu(x_1,x_1,x_2) + \Im_\mu(x_2,x_2,x_3) + \cdots + \Im_\mu(x_{n-1},x_{n-1},x_n). \qquad \square$$

Denote by Φ the family of strictly increasing functions $\phi : [0,+\infty) \to [0,+\infty)$ such that $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for all $t > 0$, where ϕ^n is the nth iterate of ϕ and satisfies (i) ψ is upper semi-continuous, (ii) $\phi^{-1}(\{0\}) = \{0\}$, (iii) $\phi(t) < t$ for all $t > 0$.

Lemma 3.3 *Let $(X,G,H,*,\diamond)$ be an IGFM-space and (y_n) be a sequence in X. Suppose that there exists $\phi\in\Phi$ such that*

$$\left.\begin{aligned}&G(y_n,y_n,y_{n+1},\phi(t))\geq G(y_{n-1},y_{n-1},y_n,t)*G(y_n,y_n,y_{n+1},t)\quad \text{and}\\ &H(y_n,y_n,y_{n+1},\phi(t))\leq H(y_{n-1},y_{n-1},y_n,t)\diamond H(y_n,y_n,y_{n+1},t)\end{aligned}\right\}\tag{3.2}$$

for all $t>0$. Then (y_n) is a Cauchy sequence with respect to the intuitionistic generalized fuzzy metric (G,H).

Proof Let $\Im_\lambda(x,y,z)$ be given by (3.1). For simplicity in notation, write $a_n=\Im_\lambda(y_{n-1},y_{n-1},y_n)$ for each $\lambda\in(0,1]$. We have to show that

$$a_{n+1}\leq\phi(a_n)\quad\text{for all } n\in\mathbb{N}.\tag{3.3}$$

Since ϕ is upper semi-continuous from right, for given $\epsilon>0$ and each a_n, there exists $p_n>a_n$ such that $\phi(p_n)<\phi(a_n)+\epsilon$. It follows from (3.1) that

$$G(y_{n-1},y_{n-1},y_n,p_n)>1-\lambda\quad\text{and}\quad H(y_{n-1},y_{n-1},y_n,p_n)<\lambda.$$

From (3.2), we have

$$\begin{aligned}&G\bigl(y_n,y_n,y_{n+1},\phi\bigl(\max\{p_n,p_{n+1}\}\bigr)\bigr)\\ &\quad\geq G\bigl(y_{n-1},y_{n-1},y_n,\max\{p_n,p_{n+1}\}\bigr)*G\bigl(y_n,y_n,y_{n+1},\max\{p_n,p_{n+1}\}\bigr)\\ &\quad\geq G(y_{n-1},y_{n-1},y_n,p_n)*G(y_n,y_n,y_{n+1},p_{n+1})>1-\lambda\end{aligned}$$

and

$$\begin{aligned}&H\bigl(y_n,y_n,y_{n+1},\phi\bigl(\max\{p_n,p_{n+1}\}\bigr)\bigr)\\ &\quad\leq H\bigl(y_{n-1},y_{n-1},y_n,\max\{p_n,p_{n+1}\}\bigr)\diamond H\bigl(y_n,y_n,y_{n+1},\max\{p_n,p_{n+1}\}\bigr)\\ &\quad\leq H(y_{n-1},y_{n-1},y_n,p_n)\diamond H(y_n,y_n,y_{n+1},p_{n+1})<\lambda.\end{aligned}$$

Again, from (3.1), we have

$$\Im_\lambda(y_n,y_n,y_{n+1})\leq\phi\bigl(\max\{p_n,p_{n+1}\}\bigr)=\max\bigl\{\phi(p_n),\phi(p_{n+1})\bigr\}\leq\max\bigl\{\phi(a_n),\phi(a_{n+1})\bigr\}+\epsilon.$$

Since $\epsilon>0$ was arbitrary, we have

$$a_{n+1}=\Im_\lambda(y_n,y_n,y_{n+1})\leq\max\bigl\{\phi(a_n),\phi(a_{n+1})\bigr\}.\tag{3.4}$$

Then $a_{n+1}\leq\phi(a_n)$. Otherwise, $a_{n+1}\leq\phi(a_{n+1})<a_{n+1}$ which is not possible. Proceeding along the same lines as in the proof of Lemma 2.5 in [36], for a given $\epsilon>0$, there exists $n_0\in\mathbb{N}$ such that

$$G(y_n,y_n,y_m,\epsilon)>1-\lambda\quad\text{and}\quad H(y_n,y_n,y_m,\epsilon)<\lambda\tag{3.5}$$

for all $m,n\geq n_0$. Hence (y_n) is a Cauchy sequence with respect to the intuitionistic generalized fuzzy metric (G,H). □

Now, we are ready to determine a coupled coincidence theorem for compatible mappings in partially ordered intuitionistic generalized fuzzy metric spaces.

Theorem 3.4 *Let* $(X,\preceq)$ *be a partially ordered set and* $(X,G,H,*,\diamondsuit)$ *be a complete IGFM-space. Suppose that* $F:X\times X\to X$ *and* $g:X\to X$ *are mappings such that* F *has the mixed* g*-monotone property, and also assume that there exists* $\phi\in\Phi$ *such that*

$$\left.\begin{aligned} &G(F(x,y),F(x,y),F(u,v),\phi(t))\geq G(g(x),g(x),u,t)*G(g(x),g(x),F(x,y),t)\\ &\qquad\qquad *G(g(u),g(u),F(u,v),t)\\ &\text{and}\\ &H(F(x,y),F(x,y),F(u,v),\phi(t))\leq H(g(x),g(x),u,t)\diamondsuit H(g(x),g(x),F(x,y),t)\\ &\qquad\qquad \diamondsuit H(g(u),g(u),F(u,v),t)\end{aligned}\right\}\tag{3.6}$$

for all $x,y,u,v\in X$ *and* $t>0$ *with* $g(x)\preceq g(u)$ *and* $g(y)\succeq g(v)$, *or* $g(x)\succeq g(u)$ *and* $g(y)\preceq g(v)$. *Suppose that* $F(X\times X)\subseteq g(X)$, g *is continuous and* F *and* g *are compatible with respect to* (G,H), *and also suppose that either*

(a) F *is continuous, or*
(b) X *has the following property*:
  (i) *if a non-decreasing sequence* x_n *is convergent to* x *with respect to* (G,H), *then* $x_n\preceq x$ *for all* n,
  (ii) *if a non-increasing sequence* y_n *is convergent to* y *with respect to* (G,H), *then* $y_n\succeq y$ *for all* n.

If there exist $x_0,y_0\in X$ *with* $g(x_0)\preceq F(x_0,y_0)$ *and* $g(y_0)\succeq F(y_0,x_0)$, *then there exist* $x,y\in X$ *such that* $g(x)=F(x,y)$ *and* $g(y)=F(y,x)$; *that is,* F *and* g *have a coupled coincidence point.*

Proof Let $x_0,y_0\in X$ be such that $g(x_0)\preceq F(x_0,y_0)$ and $g(y_0)\succeq F(y_0,x_0)$. Since $F(X\times X)\subseteq g(X)$, we can choose $x_1,y_1\in X$ such that $g(x_1)=F(x_0,y_0)$ and $g(y_1)=F(y_0,x_0)$. Again, using the assumption $F(X\times X)\subseteq g(X)$, choose $x_2,y_2\in X$ such that $g(x_2)=F(x_1,y_1)$ and $g(y_2)=F(y_1,x_1)$. Continuing this process, we can construct two sequences (x_n) and (y_n) in X as follows:

$$g(x_{n+1})=F(x_n,y_n)\quad\text{and}\quad g(y_{n+1})=F(y_n,x_n)\tag{3.7}$$

for all $n\geq 0$. We have to show that

$$g(x_n)\preceq g(x_{n+1})\quad\text{and}\quad g(y_n)\succeq g(y_{n+1})\tag{3.8}$$

for all $n\geq 0$. Since $g(x_0)\preceq F(x_0,y_0)$ and $g(y_0)\succeq F(y_0,x_0)$ and as $g(x_1)=F(x_0,y_0)$ and $g(y_1)=F(y_0,x_0)$, we have $g(x_0)\preceq g(x_1)$ and $g(y_0)\succeq g(y_1)$. This proves that (3.7) and (3.8) hold for $n=0$. Let us suppose that (3.7) and (3.8) hold for some fixed $n>0$. Since $g(x_n)\preceq g(x_{n+1})$ and $g(y_n)\succeq gy_{n+1}$, so by the mixed g-monotone property of F, we have

$$g(x_{n+1})=F(x_n,y_n)\preceq F(x_{n+1},y_n)\quad\text{and}\quad F(y_{n+1},x_n)\preceq F(y_n,x_n)=g(y_{n+1}).$$

Also,

$$g(x_{n+2})=F(x_{n+1},y_{n+1})\succeq F(x_{n+1},y_n)\quad\text{and}\quad F(y_{n+1},x_n)\succeq F(y_{n+1},x_{n+1})=g(y_{n+2}).$$

It follows that

$$g(x_{n+1}) \le g(x_{n+2}) \quad \text{and} \quad g(y_{n+1}) \ge g(y_{n+2}).$$

Thus, by the mathematical induction, we conclude that (3.7) and (3.8) hold for all $n \ge 0$.

Substituting $x = x_{n-1}$, $y = y_{n-1}$, $u = x_n$ and $v = y_n$ in (3.6), we obtain

$$\begin{aligned} &G\big(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), F(x_n, y_n), \phi(t)\big) \\ &\quad \ge G\big(g(x_{n-1}), g(x_{n-1}), g(x_n), t\big) * G\big(g(x_{n-1}), g(x_{n-1}), F(x_{n-1}, y_{n-1}), t\big) \\ &\qquad * G\big(g(x_n), g(x_n), F(x_n, y_n), t\big) \end{aligned}$$

and

$$\begin{aligned} &H\big(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), F(x_n, y_n), \phi(t)\big) \\ &\quad \le H\big(g(x_{n-1}), g(x_{n-1}), g(x_n), t\big) \diamond H\big(g(x_{n-1}), g(x_{n-1}), F(x_{n-1}, y_{n-1}), t\big) \\ &\qquad \diamond H\big(g(x_n), g(x_n), F(x_n, y_n), t\big) \end{aligned}$$

for all $t > 0$. Using (3.7) in the above inequalities, we get

$$\begin{aligned} G\big(g(x_n), g(x_n), g(x_{n+1}), \phi(t)\big) &\ge G\big(g(x_{n-1}), g(x_{n-1}), g(x_n), t\big) * G\big(g(x_{n-1}), g(x_{n-1}), g(x_n), t\big) \\ &\quad * G\big(g(x_n), g(x_n), g(x_{n+1}), t\big) \\ &= G\big(g(x_{n-1}), g(x_{n-1}), g(x_n), t\big) * G\big(g(x_n), g(x_n), g(x_{n+1}), t\big) \end{aligned}$$

and

$$\begin{aligned} H\big(g(x_n), g(x_n), g(x_{n+1}), \phi(t)\big) &\le H\big(g(x_{n-1}), g(x_{n-1}), g(x_n), t\big) \diamond H\big(g(x_{n-1}), g(x_{n-1}), g(x_n), t\big) \\ &\quad \diamond H\big(g(x_n), g(x_n), g(x_{n+1}), t\big) \\ &= H\big(g(x_{n-1}), g(x_{n-1}), g(x_n), t\big) \diamond H\big(g(x_n), g(x_n), g(x_{n+1}), t\big) \end{aligned}$$

for all $t > 0$. Hence, by Lemma 3.3, $(g(x_n))$ is a Cauchy sequence with respect to the intuitionistic generalized fuzzy metric (G, H). Again, by substituting $x = y_n$, $y = x_n$, $u = y_{n-1}$, and $v = x_{n-1}$ in (3.6), we get

$$\begin{aligned} &G\big(F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1}), F(y_n, x_n), \phi(t)\big) \\ &\quad \ge G\big(g(y_{n-1}), g(y_{n-1}), g(y_n), t\big) * G\big(g(y_{n-1}), g(y_{n-1}), F(y_{n-1}, x_{n-1}), t\big) \\ &\qquad * G\big(g(y_n), g(y_n), F(y_n, x_n), t\big) \end{aligned}$$

and, similarly,

$$\begin{aligned} &H\big(F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1}), F(y_n, x_n), \phi(t)\big) \\ &\quad \le H\big(g(y_{n-1}), g(y_{n-1}), g(y_n), t\big) \diamond H\big(g(y_{n-1}), g(y_{n-1}), F(y_{n-1}, x_{n-1}), t\big) \\ &\qquad \diamond H\big(g(y_n), g(y_n), F(y_n, x_n), t\big). \end{aligned}$$

By using (3.7), the above inequalities become

$$\begin{aligned} G\bigl(g(y_n),g(y_n),g(y_{n+1}),\phi(t)\bigr) &\geq G\bigl(g(y_{n-1}),g(y_{n-1}),g(y_n),t\bigr) * G\bigl(g(y_{n-1}),g(y_{n-1}),g(y_n),t\bigr) \\ &\quad * G\bigl(g(y_n),g(y_n),g(y_{n+1}),t\bigr) \\ &= G\bigl(g(y_{n-1}),g(y_{n-1}),g(y_n),t\bigr) * G\bigl(g(y_n),g(y_n),g(y_{n+1}),t\bigr) \end{aligned}$$

and

$$\begin{aligned} H\bigl(g(y_n),g(y_n),g(y_{n+1}),\phi(t)\bigr) &\leq H(y_{n-1},y_{n-1},y_n,t) \diamond H\bigl(g(y_{n-1}),g(y_{n-1}),g(y_n),t\bigr) \\ &\quad \diamond H\bigl(g(y_n),g(y_n),g(y_{n+1}),t\bigr) \\ &= H\bigl(g(y_{n-1}),g(y_{n-1}),g(y_n),t\bigr) \diamond H\bigl(g(y_n),g(y_n),g(y_{n+1}),t\bigr) \end{aligned}$$

for all $t > 0$. This proves that $(g(y_n))$ is a Cauchy sequence with respect to (G,H). That is, $(g(x_n))$ and $(g(y_n))$ are Cauchy sequences with respect to (G,H). Since $(X,G,H,*,\diamond)$ is a complete IGFM-space, there exist $x,y \in X$ such that

$$(G,H)\text{-}\lim_{n\to\infty} g(x_n) = x \quad \text{and} \quad (G,H)\text{-}\lim_{n\to\infty} g(y_n) = y. \tag{3.9}$$

Therefore

$$\left.\begin{aligned} &(G,H)\text{-}\lim_{n\to\infty} g(x_n) = (G,H)\text{-}\lim_{n\to\infty} F(x_n,y_n) = x \quad \text{and} \\ &(G,H)\text{-}\lim_{n\to\infty} g(y_n) = (G,H)\text{-}\lim_{n\to\infty} F(y_n,x_n) = y. \end{aligned}\right\} \tag{3.10}$$

Since (g,F) is a compatible pair with respect to (G,H), we have

$$\left.\begin{aligned} &\lim_{n\to\infty} G(g(F(x_n,y_n)),g(F(x_n,y_n)),F(g(x_n),g(y_n))),t) = 1, \\ &\lim_{n\to\infty} H(g(F(x_n,y_n)),g(F(x_n,y_n)),F(g(x_n),g(y_n))),t) = 0, \\ &\lim_{n\to\infty} G(g(F(y_n,x_n)),g(F(y_n,x_n)),F(g(y_n),g(x_n)),t) = 1, \\ &\lim_{n\to\infty} H(g(F(y_n,x_n)),g(F(y_n,x_n)),F(g(y_n),g(x_n)),t) = 0 \end{aligned}\right\} \tag{3.11}$$

for all $t > 0$. Now, suppose that assumption (a) holds. From (3.10) and (3.11), by using the continuity of F and g, we have

$$G\bigl(g(x),g(x),F(x,y),t\bigr) = 1 \quad \text{and} \quad H\bigl(g(x),g(x),F(x,y),t\bigr) = 0 \tag{3.12}$$

for all $t > 0$. Similarly, for all $t > 0$, we get

$$G\bigl(g(y),g(y),F(y,x),t\bigr) = 1 \quad \text{and} \quad H\bigl(g(y),g(y),F(y,x),t\bigr) = 0. \tag{3.13}$$

From (3.12) and (3.13), we obtain $g(x) = F(x,y)$ and $g(y) = F(y,x)$. This proves that F and g have a coupled coincidence point with respect to the intuitionistic generalized fuzzy metric (G,H).

Lastly, let us assume that (b) holds. Since $(g(x_n))$ is a non-decreasing sequence and $g(x_n)$ is convergent to x with respect to the intuitionistic generalized fuzzy metric (G,H), and also $(g(y_n))$ is a non-increasing sequence and $g(y_n)$ is convergent to y with respect to (G,H), by our assumption we have $g(x_n) \leq x$ and $g(y_n) \geq y$ for all n. Since (g,F) is a compatible pair, using the continuity of g, we have

$$\begin{aligned} g(x) &= (G,H)\text{-}\lim_{n\to\infty} g\bigl(g(x_n)\bigr) = (G,H)\text{-}\lim_{n\to\infty} g\bigl(F(x_n,y_n)\bigr) \\ &= (G,H)\text{-}\lim_{n\to\infty} F\bigl(g(x_n),g(y_n)\bigr) \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} g(y) &= (G,H)\text{-}\lim_{n\to\infty} g\bigl(g(y_n)\bigr) = (G,H)\text{-}\lim_{n\to\infty} g\bigl(F(y_n,x_n)\bigr) \\ &= (G,H)\text{-}\lim_{n\to\infty} F\bigl(g(y_n),g(x_n)\bigr). \end{aligned} \tag{3.15}$$

Write

$$\begin{aligned} G\bigl(g(x),g(x),F(x,y),\phi(t)\bigr) \geq{} & G\bigl(g(x),g(x),g\bigl(g(x_{n+1})\bigr),\phi(t)-\phi(kt)\bigr) \\ & * G\bigl(g\bigl(g(x_{n+1})\bigr),g\bigl(g(x_{n+1})\bigr),F(x,y),\phi(kt)\bigr) \end{aligned}$$

and

$$\begin{aligned} H\bigl(g(x),g(x),F(x,y),\phi(t)\bigr) \leq{} & H\bigl(g(x),g(x),g\bigl(g(x_{n+1})\bigr),\phi(t)-\phi(kt)\bigr) \\ & \diamond H\bigl(g\bigl(g(x_{n+1})\bigr),g\bigl(g(x_{n+1})\bigr),F(x,y),\phi(kt)\bigr) \end{aligned}$$

for all k and $t > 0$. Taking limit as $n \to \infty$, using the fact that G is continuous and from the definition of IGFM-space, we have

$$\left.\begin{aligned} &G(x,x,F(x,y),\phi(t)) \geq \lim_{n\to\infty} G(g(g(x_{n+1})),g(g(x_{n+1})),F(x,y),\phi(kt)) \quad \text{and} \\ &H(x,x,F(x,y),\phi(t)) \leq \lim_{n\to\infty} H(g(g(x_{n+1})),g(g(x_{n+1})),F(x,y),\phi(kt)) \end{aligned}\right\} \tag{3.16}$$

for all $t > 0$. Since $g(x_{n+1}) = F(x_n,y_n)$ and $g(y_{n+1}) = F(y_n,x_n)$, using equalities (3.14) and (3.15) in (3.16), we get

$$\left.\begin{aligned} &G(g(x),g(x),F(x,y),\phi(t)) \\ &\quad \geq \lim_{n\to\infty} G(g(F(x_n,y_n)),g(F(x_n,y_n)),F(x,y),\phi(kt)) \\ &\quad = \lim_{n\to\infty} G(F(g(x_n),g(y_n)),F(g(x_n),g(y_n)),F(x,y),\phi(kt)) \\ &\text{and} \\ &H(g(x),g(x),F(x,y),\phi(t)) \\ &\quad \leq \lim_{n\to\infty} H(g(F(x_n,y_n)),g(F(x_n,y_n)),F(x,y),\phi(kt)) \\ &\quad = \lim_{n\to\infty} H(F(g(x_n),g(y_n)),F(g(x_n),g(y_n)),F(x,y),\phi(kt)). \end{aligned}\right\} \tag{3.17}$$

Using (3.6) in the right-hand side of (3.17) and then from (3.14) and (3.15), we obtain

$$\begin{aligned} &G\bigl(g(x),g(x),F(x,y),\phi(t)\bigr) \\ &\quad \geq \lim_{n\to\infty}\bigl\{G\bigl(g\bigl(g(x_n)\bigr),g\bigl(g(x_n)\bigr),g(x),kt\bigr) * G\bigl(g\bigl(g(x_n)\bigr),g\bigl(g(x_n)\bigr),F\bigl(g(x_n),g(y_n)\bigr),kt\bigr) \\ &\qquad * G\bigl(g(x),g(x),F(x,y),kt\bigr)\bigr\} \end{aligned}$$

$$\begin{aligned}
&= G\big(g(x), g(x), g(x), kt\big) * G\big(g(x), g(x), F(x,y), kt\big) * G\big(g(x), g(x), F(x,y), kt\big) \\
&= G\big(g(x), g(x), F(x,y), kt\big)
\end{aligned}$$

and, similarly, we have

$$\begin{aligned}
&H\big(g(x), g(x), F(x,y), \phi(t)\big) \\
&\quad \le \lim_{n\to\infty} \big\{H\big(g\big(g(x_n)\big), g\big(g(x_n)\big), g(x), kt\big) \diamond H\big(g\big(g(x_n)\big), g\big(g(x_n)\big), F\big(g(x_n), g(y_n)\big), kt\big) \\
&\qquad \diamond H\big(g(x), g(x), F(x,y), kt\big)\big\} \\
&\quad = H\big(g(x), g(x), F(x,y), kt\big).
\end{aligned}$$

Letting $k \to \infty$ in the last two inequalities, we get

$$G\big(g(x), g(x), F(x,y), \phi(t)\big) = 1 \quad \text{and} \quad H\big(g(x), g(x), F(x,y), \phi(t)\big) = 0$$

for all $t > 0$. From the definition of IGFM-space, we conclude that $g(x) = F(x,y)$ and $g(y) = F(y,x)$ and hence F and g have a coupled coincidence point in X. □

Corollary 3.5 *Let $(X, \le)$ be a partially ordered set and $(X, G, H, *, \diamond)$ be a complete IGFM-space. Suppose that $F : X \times X \to X$ is a mapping having the mixed monotone property such that there exist $x_0, y_0 \in X$ with $x_0 \le F(x_0, y_0)$ and $y_0 \ge F(y_0, x_0)$. Suppose that there exists $\phi \in \Phi$ such that*

$$\left.\begin{aligned}
&G(F(x,y), F(x,y), F(u,v), \phi(t)) \\
&\quad \ge G(x,x,u,t) * G(x,x,F(x,y),t) * G(u,u,F(u,v),t) \quad \textit{and} \\
&H(F(x,y), F(x,y), F(u,v), \phi(t)) \\
&\quad \le H(x,x,u,t) \diamond H(x,x,F(x,y),t) \diamond H(u,u,F(u,v),t)
\end{aligned}\right\} \tag{3.18}$$

for all $x, y, u, v \in X$ and $t > 0$, for which $x \le u$ and $y \ge v$, or $x \ge u$ and $y \le v$. Also, suppose that either

(a) *F is continuous, or*
(b) *X has the following property*:
 (i) *if a non-decreasing sequence x_n is convergent to x with respect to (G, H), then $x_n \le x$ for all n,*
 (ii) *if a non-increasing sequence y_n is convergent to y with respect to (G, H), then $y_n \ge y$ for all n.*

Then there exist $x, y \in X$ such that $x = F(x,y)$ and $y = F(y,x)$, that is, F has a coupled fixed point in X.

Proof The proof follows by putting $g = I$, the identity mapping, in Theorem 3.4. □

We consider the following example in support of our Theorem 3.4.

Example 3.6 Let $(X, \le)$ be a partially ordered set with $X = [0,1]$. Suppose that $a * b = \min\{a,b\}$ and $a \diamond b = \max(a,b)$. Consider G, H to be fuzzy sets on $X^3 \times (0,\infty)$ defined by

$$G(x,y,z,t) = \frac{t}{t + |x-y| + |y-z| + |z-x|} \quad \text{and}$$
$$H(x,y,z,t) = \frac{|x-y| + |y-z| + |z-x|}{t + |x-y| + |y-z| + |z-x|}$$

for all $x, y, z \in X$ and $t > 0$. Then $(X, G, H, *, \diamond)$ is a complete IGFM-space. Let the mapping $g : X \to X$ be defined by

$$g(x) = x^2 \quad \text{for all } x \in X,$$

and let the mapping $F : X \times X \to X$ be defined by

$$F(x,y) = \begin{cases} \frac{x^2 - y^2}{3} & \text{if } x \geq y, \\ 0 & \text{if } x < y, \end{cases}$$

for all $x, y \in X$. Then F satisfies the mixed g-monotone property F. Let $\phi(t) : \mathbb{R}^+ \to \mathbb{R}^+$ be such that $\phi(t) = \frac{2}{3}t$ for all $t \in \mathbb{R}^+$. Suppose that (x_n) and (y_n) are two sequences in X such that

$$\lim_{n\to\infty} F(x_n, y_n) = a, \qquad \lim_{n\to\infty} g(x_n) = a \quad \text{and} \quad \lim_{n\to\infty} F(y_n, x_n) = b, \qquad \lim_{n\to\infty} g(y_n) = b.$$

Then $a = 0$ and $b = 0$. For all $n \geq 0$, we define

$$g(x_n) = x_n^2, \qquad g(y_n) = y_n^2,$$
$$F(x_n, y_n) = \begin{cases} \frac{x_n^2 - y_n^2}{3} & \text{if } x_n \geq y_n, \\ 0 & \text{if } x_n < y_n, \end{cases}$$

and

$$F(y_n, x_n) = \begin{cases} \frac{y_n^2 - x_n^2}{3} & \text{if } y_n \geq x_n, \\ 0 & \text{if } y_n < x_n. \end{cases}$$

From the above, we see that

$$G\big(g\big(F(x_n,y_n)\big), g\big(F(x_n,y_n)\big), F\big(g(x_n), g(y_n)\big), t\big) \to 1 \quad \text{as } n \to \infty,$$
$$H\big(g\big(F(x_n,y_n)\big), g\big(F(x_n,y_n)\big), F\big(g(x_n), g(y_n)\big), t\big) \to 0 \quad \text{as } n \to \infty,$$

and

$$G\big(g\big(F(y_n,x_n)\big), g\big(F(y_n,x_n)\big), F\big(g(y_n), g(x_n)\big), t\big) \to 1 \quad \text{as } n \to \infty,$$
$$H\big(g\big(F(y_n,x_n)\big), g\big(F(y_n,x_n)\big), F\big(g(y_n), g(x_n)\big), t\big) \to 0 \quad \text{as } n \to \infty.$$

This proves that F and g are compatible with respect to (G,H). Also, suppose that $x_0 = 0$ and $y_0 = \alpha$ are two points in X such that

$$g(x_0) = g(0) = F(0,\alpha) = F(x_0,y_0)$$

and

$$g(y_0) = g(\alpha) = \alpha^2 \geq \frac{\alpha^2}{3} = F(\alpha,0) = F(y_0,x_0).$$

Now it is left to show that (3.6) of Theorem 3.4 is satisfied with $\phi(t) = \frac{2}{3}t$ as defined above. Let $x,y,u,v \in X$ be such that $g(x) \leq g(u)$ and $g(y) \geq g(v)$, that is, $x^2 \leq u^2$, $y^2 \geq v^2$. We have the following possible cases.

Case 1: When $x \geq y$ and $u \geq v$. Then we get

$$\begin{aligned}
&G\bigl(F(x,y),F(x,y),F(u,v),\phi(t)\bigr)\\
&\quad= G\biggl(\frac{x^2-y^2}{3},\frac{x^2-y^2}{3},\frac{u^2-v^2}{3},\frac{2}{3}t\biggr)\\
&\quad= \frac{\frac{2}{3}t}{\frac{2}{3}t+2|\frac{(x^2-u^2)-(y^2-v^2)}{3}|}\\
&\quad= \frac{t}{t+|(x^2-u^2)-(y^2-v^2)|}\\
&\quad\geq \frac{t}{t+2|u^2-\frac{u^2-v^2}{3}|}\\
&\quad= G\bigl(g(u),g(u),F(u,v),t\bigr)\\
&\quad\geq G\bigl(g(x),g(x),g(u),t\bigr) * G\bigl(g(x),g(x),F(x,y),t\bigr) * G\bigl(g(u),g(u),F(u,v),t\bigr)
\end{aligned}$$

and

$$\begin{aligned}
&H\bigl(F(x,y),F(x,y),F(u,v),\phi(t)\bigr)\\
&\quad= H\biggl(\frac{x^2-y^2}{3},\frac{x^2-y^2}{3},\frac{u^2-v^2}{3},\frac{2}{3}t\biggr)\\
&\quad= \frac{2|\frac{(x^2-u^2)-(y^2-v^2)}{3}|}{\frac{2}{3}t+2|\frac{(x^2-u^2)-(y^2-v^2)}{3}|}\\
&\quad= \frac{|(x^2-u^2)-(y^2-v^2)|}{t+|(x^2-u^2)-(y^2-v^2)|}\\
&\quad\leq \frac{2|u^2-\frac{u^2-v^2}{3}|}{t+2|u^2-\frac{u^2-v^2}{3}|}\\
&\quad= H\bigl(g(u),g(u),F(u,v),t\bigr)\\
&\quad\leq H\bigl(g(x),g(x),g(u),t\bigr) \diamond H\bigl(g(x),g(x),F(x,y),t\bigr) \diamond H\bigl(g(u),g(u),F(u,v),t\bigr).
\end{aligned}$$

Case 2: If $x \geq y$, $u < v$, then we see that this assumption cannot happen since $x \leq u$.

Case 3: When $x < y$ and $u \geq v$. Then

$$\begin{aligned}
&G\big(F(x,y),F(x,y),F(u,v),\phi(t)\big)\\
&\quad = G\left(0,0,\frac{u^2-v^2}{3},\frac{2}{3}t\right)\\
&\quad = \frac{\frac{2}{3}t}{\frac{2}{3}t+2|\frac{(u^2-v^2)}{3}|}\\
&\quad = \frac{t}{t+|u^2-v^2|}\\
&\quad \geq \frac{t}{t+2|u^2-x^2|}\\
&\quad = G\big(g(x),g(x),g(u),t\big)\\
&\quad \geq G\big(g(x),g(x),g(u),t\big)*G\big(g(x),g(x),F(x,y),t\big)*G\big(g(u),g(u),F(u,v),t\big)
\end{aligned}$$

and

$$\begin{aligned}
&H\big(F(x,y),F(x,y),F(u,v),\phi(t)\big)\\
&\quad = H\left(0,0,\frac{u^2-v^2}{3},\frac{2}{3}t\right)\\
&\quad = \frac{2|\frac{(u^2-v^2)}{3}|}{\frac{2}{3}t+2|\frac{(u^2-v^2)}{3}|}\\
&\quad = \frac{|u^2-v^2|}{t+|u^2-v^2|}\\
&\quad \leq \frac{2|u^2-x^2|}{t+2|u^2-x^2|}\\
&\quad = H\big(g(x),g(x),g(u),t\big)\\
&\quad \leq H\big(g(x),g(x),g(u),t\big)\diamond H\big(g(x),g(x),F(x,y),t\big)\diamond H\big(g(u),g(u),F(u,v),t\big).
\end{aligned}$$

Case 4. If $x < y$ and $u < v$, then both $F(x,y) = 0$ and $F(u,v) = 0$. Therefore

$$G\big(F(x,y),F(x,y),F(u,v),\phi(t)\big) = 1 \quad \text{and} \quad H\big(F(x,y),F(x,y),F(u,v),\phi(t)\big) = 0.$$

Obviously, assumption (3.6) is fulfilled.

Thus all the hypotheses of Theorem 3.4 are fulfilled. So, we conclude that F and g have a coupled coincidence point. In this case, $(0,0)$ is a coupled coincidence point of F and g in X.

Remark 3.7 Proceeding along the same technique as given by Sintunavarat *et al.* [16], we can also obtain our coupled coincidence point theorem without using the commutative condition.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

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