Hussain et al. *Fixed Point Theory and Applications* 2013, 2013:261
http://www.fixedpointtheoryandapplications.com/content/2013/1/261

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Coupled fixed point results on quasi-Banach spaces with application to a system of integral equations

Nawab Hussain[1], Peyman Salimi[2] and Saleh Al-Mezel[1*]

Dedicated to Professor Wataru Takahashi on the occasion of his seventieth birthday

*Correspondence:
salmezel@kau.edu.sa
[1]Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia
Full list of author information is available at the end of the article

Abstract
The aim of this paper is to obtain coupled fixed point theorems for self-mappings defined on an ordered closed and convex subset of a quasi-Banach space. Our method of proof is different and constructive in nature. As an application of our coupled fixed point results, we establish corresponding coupled coincidence point results without any type of commutativity of underlying maps. Moreover, an application to integral equations is given to illustrate the usability of the obtained results.

MSC: Primary 47H10; 54H25; 55M20

Keywords: quasi-Banach space; metric-type space; coupled fixed point; mixed monotone property; ordered set; integral equation

1 Introduction

It is well known that the Banach contraction principle is one of the most important results in classical functional analysis. It is widely considered as the source of metric fixed point theory. Also, its significance lies in its vast applicability in a number of branches of mathematics (see [1–15]). The study of coupled fixed points in partially ordered metric spaces was first investigated in 1987 by Guo and Lakshmikantham [16], and then it attracted many researchers; see, for example, [3, 5] and references therein. Recently, Bhaskar and Lakshmikantham [12] presented coupled fixed point theorems for contractions in partially ordered metric spaces. Luong and Thuan [17] presented nice generalizations of these results. Alsulami *et al.* [3] further extended the work of Luong and Thuan to coupled coincidences in partial metric spaces. For more related work on coupled fixed points and coincidences, we refer the readers to recent results in [13–30].

In recent years, several authors have obtained coupled fixed point results for various classes of mappings on the setting of many generalized metric spaces. The concept of metric-type space appeared in some works, such as Czerwik [8], Khamsi [9] and Khamsi and Hussain [10]. Metric-type space is a symmetric space with some special properties. A metric-type space can also be regarded as a triplet (X,d,K), where (X,d) is a symmetric space and $K \geq 1$ is a real number such that

$$d(x,z) \leq K\big(d(x,y) + d(y,z)\big)$$

for any $x, y, z \in X$. In this paper, we adopt a different and constructive method to prove some coupled fixed and coincidence point results for contractive mappings defined on an ordered closed convex subset of a quasi-Banach space. Moreover, an application to integral equations is given to illustrate the usability of the obtained results.

2 Preliminaries

The aim of this section is to present some notions and results used in the paper.

Definition 2.1 Let X be a non-empty set and $d : X \times X \to [0, +\infty)$. (X, d) is a symmetric space (also called E-space) if and only if it satisfies the following conditions:

(W1) $d(x, y) = 0$ if and only if $x = y$;

(W2) $d(x, y) = d(y, x)$ for any $x, y \in X$.

Symmetric spaces differ from more convenient metric spaces in the absence of triangle inequality. Nevertheless, many notions can be defined similar to those in metric spaces. For instance, in a symmetric space (X, d), the limit point of a sequence $\{x_n\}$ is defined by

$$\lim_{n\to+\infty} d(x_n, x) = 0 \quad \text{if and only if} \quad \lim_{n\to+\infty} x_n = x.$$

Also, a sequence $\{x_n\} \subset X$ is said to be a Cauchy sequence if, for every given $\varepsilon > 0$, there exists a positive integer $n(\varepsilon)$ such that $d(x_m, x_n) < \varepsilon$ for all $m, n \geq n(\varepsilon)$. A symmetric space (X, d) is said to be complete if and only if each of its Cauchy sequences converges to some $x \in X$.

Definition 2.2 Let X be a nonempty set and let $K \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}_+$ is said to be of metric type if and only if for all $x, y, z \in X$ the following conditions are satisfied:

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, z) \leq K[d(x, y) + d(y, z)]$.

A triplet (X, d, K) is called a metric-type space.

We observe that a metric-type space is included in the class of symmetric spaces. So the notions of convergent sequence, Cauchy sequence and complete space are defined as in symmetric spaces.

Next, we give some examples of metric-type spaces.

Example 2.3 [4] The space l_p with $(0 < p < 1)$

$$l_p = \left\{ \{x_n\} \subset \mathbb{R} : \sum_{n=1}^{+\infty} |x_n|^p < +\infty \right\},$$

together with the function $d : l_p \times l_p \to \mathbb{R}$, defined by

$$d(x, y) = \left(\sum_{n=1}^{+\infty} |x_n - y_n|^p \right)^{1/p},$$

where $x = \{x_n\}, y = \{y_n\} \in l_p$, is a metric-type space. By an elementary calculation, we obtain $d(x,y) \leq 2^{1/p}[d(x,y) + d(y,z)]$.

Example 2.4 [4] The space L^p $(0 < p < 1)$ of all real functions $x : [0,1] \to \mathbb{R}$ such that

$$\int_0^1 |x(t)|^p \, dt < +\infty$$

is a metric-type space if we take

$$d(x,y) = \left(\int_0^1 |x(t) - y(t)|^p \, dt \right)^{1/p}$$

for each $x, y \in L^p$. The constant K is again equal to $2^{1/p}$.

For more examples of metric-type (or b-metric) spaces, we refer to [8, 10, 11]. We recall that a quasi-norm $\| \cdot \|$ defined on a real vector space X is a mapping $X \to \mathbb{R}_+$ such that:
(1) $\|x\| > 0$ if and only if $x \neq 0$;
(2) $\|\lambda x\| = |\lambda| \|x\|$ for $\lambda \in \mathbb{R}$ and $x \in X$;
(3) $\|x + y\| \leq K[\|x\| + \|y\|]$ for all $x, y \in X$, where $K \geq 1$ is a constant independent of x, y.
A triplet $(X, \| \cdot \|, K)$ is called a quasi-Banach space.

What makes quasi-Banach spaces different from the more classical Banach spaces is the triangle inequality. In quasi-Banach spaces, the triangle inequality is allowed to hold approximately: $\|x + y\| \leq K(\|x\| + \|y\|)$ for some constant $K \geq 1$. This relaxation leads to a broad class of spaces. Lebesgue spaces L^p are Banach spaces for $1 \leq p \leq +\infty$ and only quasi-Banach spaces for $0 < p < 1$.

Remark 2.5 Let $(X, \| \cdot \|, K)$ be a quasi-Banach space, then the mapping $d : X \times X \to \mathbb{R}_+$ defined by $d(x,y) = \|x - y\|$ for all $x, y \in X$ is a metric-type (b-metric). In general, a metric-type (b-metric) function d is not continuous (see [23, 26]).

The following result is useful for some of the proofs in the paper.

Lemma 2.6 *Let (X, d, K) be a metric-type space and let $\{x_k\}_{k=0}^n \subset X$. Then*

$$d(x_n, x_0) \leq K d(x_0, x_1) + \cdots + K^{n-1} d(x_{n-2}, x_{n-1}) + K^{n-1} d(x_{n-1}, x_n).$$

From Lemma 2.6, we deduce the following lemma.

Lemma 2.7 *Let $\{y_n\}$ be a sequence in a metric-type space (X, d, K) such that*

$$d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \leq \lambda \bigl[d(x_{n-1}, x_n) + d(y_{n-1}, y_n) \bigr]$$

for some λ, $0 < \lambda < 1/K$, and each $n \in \mathbb{N}$. Then $\{x_n\}$ and $\{y_n\}$ are two Cauchy sequences in X.

Definition 2.8 (Mixed monotone property) Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$. We say that the mapping F has the mixed monotone property if F is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument. That is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad x_1 \preceq x_2 \quad \Rightarrow \quad F(x_1, y) \preceq F(x_2, y) \tag{2.1}$$

and

$$y_1, y_2 \in X, \quad y_1 \preceq y_2 \quad \Rightarrow \quad F(x, y_1) \succeq F(x, y_2). \tag{2.2}$$

Definition 2.9 [13] Let $F : X \times X \to X$. We say that $(x, y) \in X \times X$ is a coupled fixed point of F if $F(x, y) = x$ and $F(y, x) = y$.

Lakshmikantham and Ćirić [13] introduced the following concept of a mixed g-monotone mapping.

Definition 2.10 [13] Let $(X, \preceq)$ be a partially ordered set, $F : X \times X \to X$ and $g : X \to X$. We say that F has the mixed g-monotone property if F is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad g(x_1) \le g(x_2) \quad \text{implies} \quad F(x_1, y) \le F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad g(y_1) \le g(y_2) \quad \text{implies} \quad F(x, y_1) \ge F(x, y_2).$$

Note that if g is the identity mapping, then this definition reduces to Definition 2.8.

Definition 2.11 [13] An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if

$$F(x, y) = g(x), \qquad F(y, x) = g(y).$$

3 Main results

Let $(C, \preceq)$ be an ordered subset of a quasi-Banach space $(X, \|\cdot\|, K)$. Throughout this paper, we assume that the partial order $\preceq$ have the following properties:

(A) If $x \preceq y$ and $\lambda \in \mathbb{R}_+$, then $\lambda x \preceq \lambda y$;

(B) If $x \preceq y$ and $z \in C$, then $x + z \preceq y + z$.

The following theorem is our first main result.

Theorem 3.1 *Let $(C, \preceq)$ be an ordered closed and convex subset of a quasi-Banach space $(X, \|\cdot\|, K)$ where $1 \le K < \sqrt{2}$ and $d : X \times X \to \mathbb{R}_+$ is such that $d(x, y) = \|x - y\|$. Assume that $F : C \times C \to C$ is a mapping with the mixed monotone property on C and suppose that there exist non-negative real numbers α, β and γ with $0 \le \gamma + 2\alpha + 2\beta + 1 < 2/K^2$ such that*

$$d\big(F(x, y), F(u, v)\big) \le \alpha d\big(x, F(x, y)\big) + \beta d\big(y, F(y, x)\big) + \frac{\gamma}{2}\big[d(x, u) + d(y, v)\big] \tag{3.1}$$

for all $x, y, u, v \in C$, for which $u \preceq x$ and $y \preceq v$. Also suppose that either

(a) *F is continuous, or*
(b) *C has the following property*:

$$\text{if a non-decreasing sequence } \{x_n\} \to x, \text{ then } x_n \preceq x \text{ for all } n \geq 0,$$
$$\text{if a non-increasing sequence } \{y_n\} \to y, \text{ then } y \preceq y_n \text{ for all } n \geq 0.$$

If there exist $x_0, y_0 \in C$ such that $x_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq y_0$, then there exist $x, y \in C$ such that $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point in C.

Proof Let $x_0, y_0 \in C$ be such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$. Then

$$x_0 = \frac{x_0 + x_0}{2} \preceq \frac{x_0 + F(x_0, y_0)}{2} \quad \text{and} \quad y_0 = \frac{y_0 + y_0}{2} \succeq \frac{y_0 + F(y_0, x_0)}{2}.$$

Define $x_1, y_1 \in X$ such that $x_1 = \frac{x_0+F(x_0,y_0)}{2}$ and $y_1 = \frac{y_0+F(y_0,x_0)}{2}$. Similarly, $x_2 = \frac{x_1+F(x_1,y_1)}{2}$ and $y_2 = \frac{y_1+F(y_1,x_1)}{2}$. We construct two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$y_{n+1} = \frac{y_n + F(y_n, x_n)}{2} \quad \text{for all } n \geq 0, \tag{3.2}$$

and

$$x_{n+1} = \frac{x_n + F(x_n, y_n)}{2} \quad \text{for all } n \geq 0. \tag{3.3}$$

Let us prove that

$$x_n \preceq x_{n+1} \quad \text{and} \quad y_n \succeq y_{n+1} \quad \text{for all } n \geq 0. \tag{3.4}$$

Since

$$x_0 \preceq \frac{x_0 + F(x_0, y_0)}{2} = x_1 \quad \text{and} \quad y_0 \succeq \frac{y_0 + F(y_0, x_0)}{2} = y_1,$$

then (3.4) hold for $n = 0$. Suppose that (3.4) hold for $n \geq 1$. Since F has the mixed monotone property, so we have

$$\begin{aligned} x_{n+1} &= \frac{x_n + F(x_n, y_n)}{2} \preceq \frac{x_n + F(x_{n+1}, y_n)}{2} \\ &\preceq \frac{x_{n+1} + F(x_{n+1}, y_n)}{2} \preceq \frac{x_{n+1} + F(x_{n+1}, y_{n+1})}{2} = x_{n+2} \end{aligned}$$

and

$$\begin{aligned} y_{n+1} &= \frac{y_n + F(y_n, x_n)}{2} \succeq \frac{y_{n+1} + F(y_n, x_n)}{2} \\ &\succeq \frac{y_{n+1} + F(y_{n+1}, x_n)}{2} \succeq \frac{y_{n+1} + F(y_{n+1}, x_{n+1})}{2} = y_{n+2}. \end{aligned}$$

Then, by mathematical induction, it follows that (3.4) hold for all $n \geq 0$.

By (3.2) and (3.3) we have

$$x_{n+1} - x_n = \frac{x_n - x_{n-1} + [F(x_n, y_n) - F(x_{n-1}, y_{n-1})]}{2}$$

and

$$y_{n+1} - y_n = \frac{y_n - y_{n-1} + [F(y_n, x_n) - F(y_{n-1}, x_{n-1})]}{2}.$$

Thus

$$\|x_{n+1} - x_n\| \le \frac{K\|x_n - x_{n-1}\| + K\|F(x_n, y_n) - F(x_{n-1}, y_{n-1})\|}{2}$$

and

$$\|y_{n+1} - y_n\| \le \frac{K\|y_n - y_{n-1}\| + K\|F(y_n, x_n) - F(y_{n-1}, x_{n-1})\|}{2}.$$

Therefore

$$\frac{2}{K} d(x_{n+1}, x_n) - d(x_n, x_{n-1}) \le d\big(F(x_n, y_n), F(x_{n-1}, y_{n-1})\big) \tag{3.5}$$

and

$$\frac{2}{K} d(y_{n+1}, y_n) - d(y_n, y_{n-1}) \le d\big(F(y_n, x_n), F(y_{n-1}, x_{n-1})\big). \tag{3.6}$$

Also,

$$x_{n-1} - F(x_{n-1}, y_{n-1}) = 2\left(x_{n-1} - \frac{x_{n-1} + F(x_{n-1}, y_{n-1})}{2}\right) = 2(x_{n-1} - x_n).$$

Then

$$d\big(x_{n-1}, F(x_{n-1}, y_{n-1})\big) = 2d(x_{n-1}, x_n). \tag{3.7}$$

Similarly,

$$d\big(y_{n-1}, F(y_{n-1}, x_{n-1})\big) = 2d(y_{n-1}, y_n). \tag{3.8}$$

On the other hand, by (3.1) and (3.4), we have

$$\begin{aligned} d\big(F(x_{n-1}, y_{n-1}), F(x_n, y_n)\big) &\le \alpha d\big(x_{n-1}, F(x_{n-1}, y_{n-1})\big) + \beta d\big(y_{n-1}, F(y_{n-1}, x_{n-1})\big) \\ &\quad + \frac{\gamma}{2}\big[d(x_{n-1}, x_n) + d(y_{n-1}, y_n)\big] \end{aligned}$$

and

$$\begin{aligned} d\big(F(y_{n-1}, x_{n-1}), F(y_n, x_n)\big) &\le \alpha d\big(y_{n-1}, F(y_{n-1}, x_{n-1})\big) + \beta d\big(x_{n-1}, F(x_{n-1}, y_{n-1})\big) \\ &\quad + \frac{\gamma}{2}\big[d(y_{n-1}, y_n) + d(x_{n-1}, x_n)\big]. \end{aligned}$$

Hence, by (3.5), (3.6), (3.7) and (3.8), we have

$$\begin{aligned}\frac{2}{K}d(x_{n+1},x_n)-d(x_n,x_{n-1}) &\leq 2\alpha d(x_{n-1},x_n)+2\beta d(y_{n-1},y_n)\\&\quad+\frac{\gamma}{2}\bigl[\bigl(d(x_{n-1},x_n)+d(y_{n-1},y_n)\bigr)\bigr]\end{aligned}$$

and

$$\begin{aligned}\frac{2}{K}d(y_{n+1},y_n)-d(y_n,y_{n-1}) &\leq 2\alpha d(y_{n-1},y_n)+2\beta d(x_{n-1},x_n)\\&\quad+\frac{\gamma}{2}\bigl[\bigl(d(y_{n-1},y_n)+d(x_{n-1},x_n)\bigr)\bigr].\end{aligned}$$

Thus

$$d(x_{n+1},x_n)+d(y_{n+1},y_n)\leq\frac{K}{2}(\gamma+2\alpha+2\beta+1)\bigl(d(x_{n-1},x_n)+d(y_{n-1},y_n)\bigr).$$

By Lemma 2.7 we conclude that $\{x_n\}$ and $\{y_n\}$ are two Cauchy sequences. Then there exist $x^*,y^*\in C$ such that $x_n\to x^*$ and $y_n\to y^*$.

At first, we assume that F is continuous. Hence

$$x^*=\lim_{n\to\infty}x_n=\lim_{n\to\infty}F(x_{n-1},y_{n-1})=F\bigl(x^*,y^*\bigr).$$

Similarly,

$$y^*=F\bigl(y^*,x^*\bigr).$$

That is, F has a coupled fixed point.

Now we assume that (b) holds. Since $x_n\to x^*$ and $y_n\to y^*$ as $n\to\infty$, then (b) implies that $x_n\preceq x^*$ and $y^*\preceq y_n$ for all $n\geq 0$. Now, by (3.1) with $x=x_n$, $y=y_n$, $u=x^*$, $v=y^*$, we have

$$d\bigl(F(x_n,y_n),F\bigl(x^*,y^*\bigr)\bigr)\leq\alpha d\bigl(x_n,F(x_n,y_n)\bigr)+\beta d\bigl(y_n,F(y_n,x_n)\bigr)+\frac{\gamma}{2}\bigl[d\bigl(x_n,x^*\bigr)+d\bigl(y_n,y^*\bigr)\bigr],$$

which implies

$$d\bigl(F(x_n,y_n),F\bigl(x^*,y^*\bigr)\bigr)\leq 2\alpha d(x_n,x_{n+1})+2\beta d(y_n,y_{n+1})+\frac{\gamma}{2}\bigl[d\bigl(x_n,x^*\bigr)+d\bigl(y_n,y^*\bigr)\bigr].$$

Taking the limit as $n\to\infty$ in the above inequality, we have

$$\lim_{n\to\infty}d\bigl(F(x_n,y_n),F\bigl(x^*,y^*\bigr)\bigr)=0.$$

Also, by (3.3) we get $\lim_{n\to\infty}F(x_n,y_n)=x^*$. Now we can write

$$d\bigl(x^*,F\bigl(x^*,y^*\bigr)\bigr)\leq K\lim_{n\to\infty}d\bigl(x^*,F(x_n,y_n)\bigr)+K\lim_{n\to\infty}d\bigl(F(x_n,y_n),F\bigl(x^*,y^*\bigr)\bigr),$$

and hence $d(x^*,F(x^*,y^*))=0$. That is, $x^*=F(x^*,y^*)$. Similarly, $y^*=F(y^*,x^*)$ as required. □

If in Theorem 3.1 we take $\alpha=\beta=0$, we obtain following result.

Corollary 3.2 *Let $(C, \preceq)$ be an ordered closed and convex subset of a quasi-Banach space $(X, \|\cdot\|, K)$, where $1 \leq K < \sqrt{2}$, and let $d : X \times X \to \mathbb{R}_+$ be such that $d(x,y) = \|x-y\|$. Assume that $F : C \times C \to C$ is a mapping such that F has the mixed monotone property on X and there exists a non-negative real number γ with $0 \leq \gamma + 1 < 2/K^2$ such that*

$$d\big(F(x,y), F(u,v)\big) \leq \frac{\gamma}{2}\big[d(x,u) + d(y,v)\big]$$

for all $x, y, u, v \in X$, for which $u \preceq x$ and $y \preceq v$. Also suppose that either

(a) *F is continuous, or*

(b) *C has the following property:*

$$\text{if a non-decreasing sequence } \{x_n\} \to x, \text{ then } x_n \preceq x \text{ for all } n \geq 0,$$

$$\text{if a non-increasing sequence } \{y_n\} \to y, \text{ then } y \preceq y_n \text{ for all } n \geq 0.$$

If there exist $x_0, y_0 \in C$ such that $x_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq y_0$, then there exist $x, y \in C$ such that $x = F(x,y)$ and $y = F(y,x)$, that is, F has a coupled fixed point.

The following theorem is our second main result.

Theorem 3.3 *Let $(C, \preceq)$ be an ordered closed and convex subset of a quasi-Banach space $(X, \|\cdot\|, K)$, where $1 \leq K < \frac{1+\sqrt{17}}{4}$, and let $d : X \times X \to \mathbb{R}_+$ be such that $d(x,y) = \|x-y\|$. Assume that $F : C \times C \to C$ is a mapping such that F has the mixed monotone property on X and there exists a non-negative real number α with $0 \leq \alpha + 2K - 1 < 2/K$ such that*

$$d\big(F(x,y), F(u,v)\big) \leq \frac{\alpha}{2}\sqrt{d(x,u)^2 + d(y,v)^2 + \frac{1}{2K^2} d\big(x, F(x,y)\big) d\big(y, F(y,x)\big)} \tag{3.9}$$

for all $x, y, u, v \in C$, for which $u \preceq x$ and $y \preceq v$. Also suppose that either

(a) *F is continuous, or*

(b) *C has the following property:*

$$\text{if a non-decreasing sequence } \{x_n\} \to x, \text{ then } x_n \preceq x \text{ for all } n \geq 0,$$

$$\text{if a non-increasing sequence } \{y_n\} \to y, \text{ then } y \preceq y_n \text{ for all } n \geq 0.$$

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq y_0$, then F has a coupled fixed point.

Proof Let $x_0, y_0 \in X$ be such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$. Then

$$x_0 = \frac{(2K-1)x_0 + x_0}{2K} \preceq \frac{(2K-1)x_0 + F(x_0, y_0)}{2K}$$

and

$$y_0 = \frac{(2K-1)y_0 + y_0}{2K} \succeq \frac{(2K-1)y_0 + F(y_0, x_0)}{2K}.$$

Define $x_1, y_1 \in X$ such that $x_1 = \frac{(2K-1)x_0+F(x_0,y_0)}{2}$ and $y_1 = \frac{(2K-1)y_0+F(y_0,x_0)}{2K}$. Similarly, $x_2 = \frac{x_1+F(x_1,y_1)}{2}$ and $y_2 = \frac{(2K-1)y_1+F(y_1,x_1)}{2K}$. We construct two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$y_{n+1} = \frac{(2K-1)y_n + F(y_n,x_n)}{2K} \quad \text{for all } n \geq 0 \tag{3.10}$$

and

$$x_{n+1} = \frac{(2K-1)x_n + F(x_n,y_n)}{2K} \quad \text{for all } n \geq 0. \tag{3.11}$$

Let us prove that

$$x_n \preceq x_{n+1} \quad \text{and} \quad y_n \succeq y_{n+1} \quad \text{for all } n \geq 0. \tag{3.12}$$

As

$$x_0 \preceq \frac{(2K-1)x_0 + F(x_0,y_0)}{2K} = x_1 \quad \text{and} \quad y_0 \succeq \frac{(2K-1)y_0 + F(y_0,x_0)}{2K} = y_1,$$

so (3.12) hold for $n = 0$. Suppose that (3.12) hold for $n \geq 1$. As F has the mixed monotone property, so

$$\begin{aligned} x_{n+1} = \frac{(2K-1)x_n + F(x_n,y_n)}{2K} &\preceq \frac{(2K-1)x_n + F(x_{n+1},y_n)}{2K} \\ &\preceq \frac{(2K-1)x_{n+1} + F(x_{n+1},y_n)}{2K} \\ &\preceq \frac{(2K-1)x_{n+1} + F(x_{n+1},y_{n+1})}{2K} \\ &= x_{n+2} \end{aligned}$$

and

$$\begin{aligned} y_{n+2} = \frac{(2K-1)y_{n+1} + F(y_{n+1},x_{n+1})}{2K} &\preceq \frac{(2K-1)y_{n+1} + F(y_{n+1},x_n)}{2K} \\ &\preceq \frac{(2K-1)y_{n+1} + F(y_n,x_n)}{2K} \\ &\preceq \frac{(2K-1)y_n + F(y_n,x_n)}{2K} \\ &= y_{n+1}. \end{aligned}$$

Then, by mathematical induction, it follows that (3.12) holds for all $n \geq 0$.

By (3.10) and (3.11), we have

$$x_{n+1} - x_n = \frac{(2K-1)(x_n - x_{n-1}) + [F(x_n,y_n) - F(x_{n-1},y_{n-1})]}{2K}$$

and

$$y_{n+1} - y_n = \frac{(2K-1)(y_n - y_{n-1}) + [F(y_n,x_n) - F(y_{n-1},x_{n-1})]}{2K}.$$

Thus

$$\|x_{n+1}-x_n\| \le \frac{(2K-1)\|x_n-x_{n-1}\| + \|F(x_n,y_n)-F(x_{n-1},y_{n-1})\|}{2}$$

and

$$\|y_{n+1}-y_n\| \le \frac{(2K-1)\|y_n-y_{n-1}\| + \|F(y_n,x_n)-F(y_{n-1},x_{n-1})\|}{2}.$$

Therefore

$$2d(x_{n+1},x_n)-(2K-1)d(x_n,x_{n-1}) \le d\bigl(F(x_n,y_n),F(x_{n-1},y_{n-1})\bigr) \tag{3.13}$$

and

$$2d(y_{n+1},y_n)-(2K-1)d(y_n,y_{n-1}) \le d\bigl(F(y_n,x_n),F(y_{n-1},x_{n-1})\bigr). \tag{3.14}$$

Also, we have

$$x_{n-1}-F(x_{n-1},y_{n-1}) = 2K\left(x_{n-1}-\frac{(2K-1)x_{n-1}+F(x_{n-1},y_{n-1})}{2K}\right) = 2K(x_{n-1}-x_n),$$

which implies

$$d\bigl(x_{n-1},F(x_{n-1},y_{n-1})\bigr) = 2Kd(x_{n-1},x_n). \tag{3.15}$$

Similarly, we have

$$d\bigl(y_{n-1},F(y_{n-1},x_{n-1})\bigr) = 2Kd(y_{n-1},y_n). \tag{3.16}$$

Now, by (3.9), (3.13) and (3.15), (3.16), we have

$$\begin{aligned}
&2d(x_{n+1},x_n)-(2K-1)d(x_n,x_{n-1}) \\
&\quad \le \frac{\alpha}{2}\sqrt{d(x_{n-1},x_n)^2+d(y_{n-1},y_n)^2+2d(x_{n-1},x_n)d(y_{n-1},y_n)} \\
&\quad = \frac{\alpha}{2}\bigl[\bigl(d(x_{n-1},x_n)+d(y_{n-1},y_n)\bigr)\bigr].
\end{aligned}$$

Similarly,

$$2d(y_{n+1},y_n)-(2K-1)d(y_n,y_{n-1}) \le \frac{\alpha}{2}\bigl[(d(y_{n-1},y_n)+d(x_{n-1},x_n)\bigr].$$

Thus

$$d(x_{n+1},x_n)+d(y_{n+1},y_n) \le \frac{1}{2}(\alpha+2K-1)\bigl(d(x_{n-1},x_n)+d(y_{n-1},y_n)\bigr).$$

By Lemma 2.7, we conclude that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Thus, there exist $x^*, y^* \in C$ such that $x_n \to x^*$ and $y_n \to y^*$.

Now, proceeding as in the proof of Theorem 3.1, we can prove that (x^*, y^*) is a coupled fixed point of F. □

Since

$$\begin{aligned}
&\frac{\alpha}{2}\sqrt{\frac{1}{2K^2}d\big(x,F(x,y)\big)d\big(y,F(y,x)\big)} \\
&\quad \leq \frac{\alpha}{2}\sqrt{d(x,u)^2 + d(y,v)^2 + \frac{1}{2K^2}d\big(x,F(x,y)\big)d\big(y,F(y,x)\big)},
\end{aligned}$$

so by Theorem 3.3 we obtain the following result.

Corollary 3.4 *Let $(C,\preceq)$ be an ordered closed and convex subset of a quasi-Banach space $(X,\|\cdot\|,K)$, where $1 \leq K < \frac{1+\sqrt{17}}{4}$, and let $d : X \times X \to \mathbb{R}_+$ be such that $d(x,y) = \|x-y\|$. Assume that $F : C \times C \to C$, F has the mixed monotone property on C and for a non-negative real number α with $0 \leq \alpha + 2K - 1 < 2/K$, F satisfies following inequality*:

$$d\big(F(x,y),F(u,v)\big) \leq \frac{\alpha\sqrt{2}}{4K}\sqrt{d\big(x,F(x,y)\big)d\big(y,F(y,x)\big)}$$

for all $x,y,u,v \in X$, for which $u \preceq x$ and $y \preceq v$. Also suppose that either

(a) *F is continuous, or*

(b) *C has the following property*:

$$\begin{aligned}
&\textit{if a non-decreasing sequence } \{x_n\} \to x, \textit{ then } x_n \preceq x \textit{ for all } n \geq 0, \\
&\textit{if a non-increasing sequence } \{y_n\} \to y, \textit{ then } y \preceq y_n \textit{ for all } n \geq 0.
\end{aligned}$$

If there exist $x_0, y_0 \in C$ such that $x_0 \preceq F(x_0,y_0)$ and $F(y_0,x_0) \preceq y_0$, then F has a coupled fixed point.

By taking $K = 1$ in the above proved results, we can obtain the following couple fixed results in Banach spaces.

Corollary 3.5 *Let $(C,\preceq)$ be an ordered closed and convex subset of a Banach space $(X,\|\cdot\|)$, and let $F : C \times C \to C$ be a mapping such that F has the mixed monotone property on C. Suppose that there exist non-negative real numbers α, β and γ with $0 \leq \gamma + 2\alpha + 2\beta < 1$ such that*

$$d\big(F(x,y),F(u,v)\big) \leq \alpha d\big(x,F(x,y)\big) + \beta d\big(y,F(y,x)\big) + \frac{\gamma}{2}\big[d(x,u) + d(y,v)\big]$$

for all $x,y,u,v \in C$ with $u \preceq x$ and $y \preceq v$. Also suppose that either

(a) *F is continuous, or*

(b) *C has the following property*:

$$\begin{aligned}
&\textit{if a non-decreasing sequence } \{x_n\} \to x, \textit{ then } x_n \preceq x \textit{ for all } n \geq 0, \\
&\textit{if a non-increasing sequence } \{y_n\} \to y, \textit{ then } y \preceq y_n \textit{ for all } n \geq 0.
\end{aligned}$$

If there exist $x_0, y_0 \in C$ such that $x_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq y_0$, then F has a coupled fixed point.

Corollary 3.6 *Let $(C, \preceq)$ be an ordered closed and convex subset of a Banach space $(X, \|\cdot\|)$, and let $F : C \times C \to C$ be a mapping such that F has the mixed monotone property on C. Suppose that there exists a non-negative real number γ with $0 \leq \gamma < 1$ such that*

$$d\big(F(x,y), F(u,v)\big) \leq \frac{\gamma}{2}\big[d(x,u) + d(y,v)\big]$$

for all $x, y, u, v \in C$ with $u \preceq x$ and $y \preceq v$. Also suppose that either

(a) *F is continuous, or*

(b) *C has the following property:*

$$\text{if a non-decreasing sequence } \{x_n\} \to x, \text{ then } x_n \preceq x \text{ for all } n \geq 0,$$

$$\text{if a non-increasing sequence } \{y_n\} \to y, \text{ then } y \preceq y_n \text{ for all } n \geq 0.$$

If there exist $x_0, y_0 \in C$ such that $x_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq y_0$, then F has a coupled fixed point.

Corollary 3.7 *Let $(C, \preceq)$ be an ordered closed and convex subset of a Banach space $(X, \|\cdot\|)$, and let $F : C \times C \to C$ be a mapping such that F has the mixed monotone property on C. Suppose that there exists a non-negative real number α with $0 \leq \alpha < 1$ such that*

$$d\big(F(x,y), F(u,v)\big) \leq \frac{\alpha}{2}\sqrt{d(x,u)^2 + d(y,v)^2 + \frac{1}{2}d\big(x, F(x,y)\big)d\big(y, F(y,x)\big)}$$

for all $x, y, u, v \in C$ with $u \preceq x$ and $y \preceq v$. Also suppose that either

(a) *F is continuous, or*

(b) *C has the following property:*

$$\text{if a non-decreasing sequence } \{x_n\} \to x, \text{ then } x_n \preceq x \text{ for all } n \geq 0, \tag{3.17}$$

$$\text{if a non-increasing sequence } \{y_n\} \to y, \text{ then } y \preceq y_n \text{ for all } n \geq 0. \tag{3.18}$$

If there exist $x_0, y_0 \in C$ such that $x_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq y_0$, then F has a coupled fixed point.

Corollary 3.8 *Let $(C, \preceq)$ be an ordered closed and convex subset of a Banach space $(X, \|\cdot\|)$, and let $F : C \times C \to C$ be a mapping such that F has the mixed monotone property on C. Suppose that there exists a non-negative real number α with $0 \leq \alpha < 1$ such that*

$$d\big(F(x,y), F(u,v)\big) \leq \frac{\alpha\sqrt{2}}{4}\sqrt{d\big(x, F(x,y)\big)d\big(y, F(y,x)\big)}$$

for all $x, y, u, v \in C$ with $u \preceq x$ and $y \preceq v$. Also suppose that either

(a) *F is continuous, or*

(b) *C has the following property*:

$$\text{if a non-decreasing sequence } \{x_n\} \to x, \text{ then } x_n \preceq x \text{ for all } n \geq 0,$$
$$\text{if a non-increasing sequence } \{y_n\} \to y, \text{ then } y \preceq y_n \text{ for all } n \geq 0.$$

If there exist $x_0, y_0 \in C$ such that $x_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq y_0$, then F has a coupled fixed point.

The following lemma is an easy consequence of the axiom of choice (see p.5 [25], AC5: For every function $f : X \to X$, there is a function g such that $D(g) = R(f)$ and for every $x \in D(g)$, $f(gx) = x$).

Lemma 3.9 *Let X be a nonempty set and $g : X \to X$ be a mapping. Then there exists a subset $E \subseteq X$ such that $g(E) = g(X)$ and $g : E \to X$ is one-to-one.*

As an application of Theorem 3.1, we now establish a coupled coincidence point result.

Theorem 3.10 *Let $(C, \preceq)$ be a nonempty ordered subset of a quasi-Banach space $(X, \|\cdot\|, K)$, where $1 \leq K < \sqrt{2}$, and let $d : X \times X \to \mathbb{R}_+$ be such that $d(x, y) = \|x - y\|$. Assume that $g : C \to C$ and $F : C \times C \to C$ are mappings where F has the mixed g-monotone property on C, $g(C)$ is closed and convex and $F(C \times C) \subseteq g(C)$. Suppose that there exist non-negative real numbers α, β and a real number γ with $0 \leq \gamma + 2\alpha + 2\beta + 1 < 2/K^2$ such that*

$$d\big(F(x, y), F(u, v)\big) \leq \alpha d\big(gx, F(x, y)\big) + \beta d\big(gy, F(y, x)\big) + \frac{\gamma}{2}\big[d(gx, gu) + d(gy, gv)\big] \tag{3.19}$$

for all $x, y, u, v \in C$, for which $gu \preceq gx$ and $gy \preceq gv$. Also suppose that either
(a) *F is continuous, or*
(b) *C has the following property*:

$$\text{if a non-decreasing sequence } \{gx_n\} \to gx, \text{ then } gx_n \preceq gx \text{ for all } n \geq 0,$$
$$\text{if a non-increasing sequence } \{gy_n\} \to gy, \text{ then } gy \preceq gy_n \text{ for all } n \geq 0.$$

If there exist $x_0, y_0 \in C$ such that $g(x_0) \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq g(y_0)$, then there exist $x, y \in C$ such that $gx = F(x, y)$ and $gy = F(y, x)$, that is, F and g have a coupled coincidence point in C.

Proof Using Lemma 3.9, there exists $E \subseteq C$ such that $g(E) = g(C)$ and $g : E \to C$ is one-to-one. We define a mapping $G : g(E) \times g(E) \to g(E)$ by

$$G(gx, gy) = F(x, y), \tag{3.20}$$

for all $gx, gy \in g(E)$. As g is one-to-one on $g(E)$ and $F(C \times C) \subseteq g(C)$, so G is well defined. Thus, it follows from (3.19) and (3.20) that

$$\begin{aligned} &d\big(G(gx, gy), F(gu, gv)\big) + \alpha d\big(gx, G(gx, gy)\big) + \beta d\big(gy, G(gy, gx)\big) \\ &\quad \leq \frac{\gamma}{2}\big[d(gx, gu) + d(gy, gv)\big] \end{aligned} \tag{3.21}$$

for all $gx, gy, gu, gv \in g(C)$, for which $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$. Since F has the mixed g-monotone property, for all $gx, gy \in g(C)$,

$$gx_1, gx_2 \in g(C), \quad g(x_1) \preceq g(x_2) \quad \text{implies} \quad G(gx_1, gy) \preceq G(gx_2, gy) \tag{3.22}$$

and

$$gy_1, gy_2 \in g(C), \quad g(y_1) \preceq g(y_2) \quad \text{implies} \quad G(gx, gy_1) \succeq G(gx, gy_2), \tag{3.23}$$

which imply that G has the mixed monotone property. Also, there exist $x_0, y_0 \in C$ such that

$$g(x_0) \preceq F(x_0, y_0) \quad \text{and} \quad g(y_0) \succeq F(y_0, x_0).$$

This implies that there exist $gx_0, gy_0 \in g(C)$ such that

$$g(x_0) \preceq G(gx_0, gy_0) \quad \text{and} \quad g(y_0) \succeq G(gy_0, gx_0).$$

Suppose that assumption (a) holds. Since F is continuous, G is also continuous. Using Theorem 3.1 to the mapping G, it follows that G has a coupled fixed point $(u, v) \in g(C) \times g(C)$.

Suppose that assumption (b) holds. We conclude similarly that the mapping G has a coupled fixed point $(u, v) \in g(C) \times g(C)$. Finally, we prove that F and g have a coupled coincidence point. Since (u, v) is a coupled fixed point of G, we get

$$u = G(u, v) \quad \text{and} \quad v = G(v, u). \tag{3.24}$$

Since $(u, v) \in g(C) \times g(C)$, there exists a point $(u_0, v_0) \in C \times C$ such that

$$u = gu_0 \quad \text{and} \quad v = gv_0. \tag{3.25}$$

It follows from (3.24) and (3.25) that

$$gu_0 = G(gu_0, gv_0) \quad \text{and} \quad gv_0 = G(gv_0, gu_0). \tag{3.26}$$

Combining (3.20) and (3.26), we get

$$gu_0 = F(u_0, v_0) \quad \text{and} \quad gv_0 = F(v_0, u_0).$$

Thus, (u_0, v_0) is a required coupled coincidence point of F and g. This completes the proof. □

Similarly, as an application of Theorem 3.3, we can prove the following coupled coincidence point result.

Theorem 3.11 *Let $(C,\preceq)$ be a nonempty ordered subset of a quasi-Banach space $(X,\|\cdot\|,K)$, where $1\leq K<\frac{1+\sqrt{17}}{4}$, and let $d:X\times X\to\mathbb{R}_+$ be such that $d(x,y)=\|x-y\|$. Assume that $g:C\to C$ and $F:C\times C\to C$ are mappings where F has the mixed g-monotone property on C, $g(C)$ is closed and convex and $F(C\times C)\subseteq g(C)$. Suppose that there exists a real number α with $0\leq\alpha+2K-1<2/K$ such that*

$$\begin{aligned} &d\bigl(F(x,y),F(u,v)\bigr)\\ &\quad\leq\frac{\alpha}{2}\sqrt{d(gx,gu)^2+d(gy,gv)^2+\frac{1}{2K^2}d\bigl(gx,F(x,y)\bigr)d\bigl(gy,F(y,x)\bigr)} \end{aligned}\tag{3.27}$$

for all $x,y,u,v\in C$, for which $gu\preceq gx$ and $gy\preceq gv$. Also suppose that either

(a) *F is continuous, or*

(b) *C has the following property:*

$$\begin{aligned} &\textit{if a non-decreasing sequence } \{gx_n\}\to gx,\ \textit{then } gx_n\preceq gx \textit{ for all } n\geq 0,\\ &\textit{if a non-increasing sequence } \{gy_n\}\to gy,\ \textit{then } gy\preceq gy_n \textit{ for all } n\geq 0. \end{aligned}$$

If there exist $x_0,y_0\in C$ such that $g(x_0)\preceq F(x_0,y_0)$ and $F(y_0,x_0)\preceq g(y_0)$, then F and g have a coupled coincidence point in C.

4 Existence of a solution for a system of integral equations

We consider the space $\mathcal{X}=C([0,T],\mathbb{R})$ of continuous functions defined on $I=[0,T]$ endowed with the structure $(\mathcal{X},\|\cdot\|)$ given by

$$\|u\|=\sup_{t\in[0,T]}\bigl|u(t)\bigr|$$

for all $u\in\mathcal{X}$. We endow $\mathcal{X}$ with the partial order $\preceq$ given by

$$x\preceq y\quad\Longleftrightarrow\quad x(t)\leq y(t)\quad\text{for all } t\in[0,T].$$

Clearly, the partial order $\preceq$ satisfies conditions A and B. Further, it is known that $(\mathcal{X},d,\preceq)$ is regular [24], that is,

$$\begin{aligned} &\text{if a non-decreasing sequence } \{x_n\}\to x,\ \text{then } x_n\preceq x \text{ for all } n\geq 0,\\ &\text{if a non-increasing sequence } \{y_n\}\to y, \text{then } y\preceq y_n \text{ for all } n\geq 0. \end{aligned}$$

Motivated by the work in [1, 17, 18, 26], we study the existence of solutions for a system of nonlinear integral equations using the results proved in the previous section.

Consider the integral equations in the following system.

$$\begin{aligned} x(t)&=P(t)+\int_0^T S(t,r)\bigl[f\bigl(r,x(r)\bigr)+k\bigl(r,y(r)\bigr)\bigr]\,dr,\\ y(t)&=P(t)+\int_0^T S(t,r)\bigl[f\bigl(r,y(r)\bigr)+k\bigl(r,x(r)\bigr)\bigr]\,dr. \end{aligned}\tag{4.1}$$

We will consider system (4.1) under the following assumptions:

(i) $f, k : [0, T] \times \mathbb{R} \to \mathbb{R}$ are continuous;
(ii) $P : [0, T] \to \mathbb{R}$ is continuous;
(iii) $S : [0, T] \times \mathbb{R} \to [0, \infty)$ is continuous;
(iv) there exist $a, b, c > 0$ with $0 \leq 2a + 2b + c < 1$ such that for all $r \in [0, T]$ and all $x(r), y(r), u(r), v(r) \in \mathcal{X}$ with $u(r) \preceq x(r) \preceq y(r) \preceq v(r)$, we have

$$\begin{aligned}
0 &\leq f\bigl(r, y(r)\bigr) - f\bigl(r, x(r)\bigr) \leq a\bigl|x(r) - F\bigl(x(r), y(r)\bigr)\bigr| + b\bigl|y(r) - F\bigl(y(r), x(r)\bigr)\bigr| \\
&\quad + \frac{c}{2}\bigl[\bigl(x(r) - u(r)\bigr) + \bigl(y(r) - v(r)\bigr)\bigr], \\
0 &\leq k\bigl(r, x(r)\bigr) - k\bigl(r, y(r)\bigr) \leq a\bigl|x(r) - F\bigl(x(r), y(r)\bigr)\bigr| + b\bigl|y(r) - F\bigl(y(r), x(r)\bigr)\bigr| \\
&\quad + \frac{c}{2}\bigl[\bigl(x(r) - u(r)\bigr) + \bigl(y(r) - v(r)\bigr)\bigr],
\end{aligned}$$

where

$$F(x, y)(t) = P(t) + \int_0^T S(t, r)\bigl[f\bigl(r, x(r)\bigr) + k\bigl(r, y(r)\bigr)\bigr]\,dr;$$

(v) there exist continuous functions $\alpha, \gamma : [0, T] \to \mathbb{R}$ such that

$$\begin{aligned}
\alpha(t) &\leq P(t) + \int_0^T S(t, r)\bigl[f\bigl(r, \alpha(r)\bigr) + k\bigl(r, \gamma(r)\bigr)\bigr]\,dr, \\
\gamma(t) &\geq P(t) + \int_0^T S(t, r)\bigl[f\bigl(r, \gamma(r)\bigr) + k\bigl(r, \alpha(r)\bigr)\bigr]\,dr;
\end{aligned}$$

(vi) assume that

$$\sup_{t \in [0,T]} \int_0^T S(t, r)\,dr \leq 1/2.$$

Theorem 4.1 *Under assumptions* (i)-(vi), *system* (4.1) *has a solution in* $\mathcal{X}^2$, *where* $\mathcal{X} = (C([0, T], \mathbb{R}))$ *is defined above.*

Proof We consider the operator $F : \mathcal{X}^2 \to \mathcal{X}$ defined by

$$F(x_1, x_2)(t) = P(t) + \int_0^T S(t, r)\bigl[f\bigl(r, x_1(r)\bigr) + k\bigl(r, x_2(r)\bigr)\bigr]\,dr$$

for all $t, r \in [0, T]$, $x_1, x_2 \in \mathcal{X}$.
Clearly, F has the mixed monotone property [26].
Let $x, y, u, v \in \mathcal{X}$ with $u \preceq x \preceq y \preceq v$. Since F has the mixed monotone property, we have

$$F(u, v) \preceq F(x, y).$$

Notice that

$$\begin{aligned}
&\bigl|F(x, y)(t) - F(u, v)(t)\bigr| \\
&\quad = \biggl|\int_0^T S(t, r)\bigl[f\bigl(r, x(r)\bigr) - f\bigl(r, u(r)\bigr)\bigr]\,dr + \int_0^T S(t, r)\bigl[k\bigl(r, y(r)\bigr) - k\bigl(r, v(r)\bigr)\bigr]\,dr\biggr|
\end{aligned}$$

$$\begin{aligned}
&\leq \int_0^T S(t,r)\big[\big|f\big(r,x(r)\big)-f\big(r,u(r)\big)\big|\big]\,dr + \int_0^T S(t,r)\big[\big|k\big(r,y(r)\big)-k\big(r,v(r)\big)\big|\big]\,dr \\
&\leq \int_0^T S(t,r)\bigg[a\big|x(r)-F\big(x(r),y(r)\big)\big| + b\big|y(r)-F\big(y(r),x(r)\big)\big| \\
&\quad + \frac{c}{2}\big[\big|x(r)-u(r)\big| + \big|y(r)-v(r)\big|\big]\bigg]\,dr \\
&\quad + \int_0^T S(t,r)\bigg[a\big|x(r)-F\big(x(r),y(r)\big)\big| + b\big|y(r)-F\big(y(r),x(r)\big)\big| \\
&\quad + \frac{c}{2}\big[\big|x(r)-u(r)\big| + \big|y(r)-v(r)\big|\big]\bigg]\,dr \\
&= 2\int_0^T S(t,r)\bigg[a\big|x(r)-F\big(x(r),y(r)\big)\big| + b\big|y(r)-F\big(y(r),x(r)\big)\big| \\
&\quad + \frac{c}{2}\big[\big|x(r)-u(r)\big| + \big|y(r)-v(r)\big|\big]\bigg]\,dr \\
&\leq 2\sup_{t\in[0,T]}\int_0^T S(t,r)\,dr\bigg[a\big\|x-F(x,y)\big\| + b\big\|y-F(y,x)\big\| + \frac{c}{2}\big(\|x-u\|+\|y-v\|\big)\bigg] \\
&\leq a\big\|x-F(x,y)\big\| + b\big\|y-F(y,x)\big\| + \frac{c}{2}\big(\|x-u\|+\|y-v\|\big).
\end{aligned}$$

Thus,

$$\big\|F(x,y)-F(u,v)\big\| \leq a\big\|x-F(x,y)\big\| + b\big\|y-F(y,x)\big\| + \frac{c}{2}\big(\|x-u\|+\|y-v\|\big).$$

Further, by (v), we get

$$\alpha \leq F(\alpha,\gamma), \qquad \gamma \geq F(\gamma,\alpha).$$

All of the conditions of Corollary 3.5 are satisfied, so we deduce the existence of $x_1, x_2 \in \mathcal{X}$ such that $x_1 = F(x_1,x_2)$ and $x_2 = F(x_2,x_1)$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia. [2]Young Researchers and Elite Club, Rasht Branch, Islamic Azad University, Rasht, Iran.

Acknowledgements
This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. Therefore, the first and third authors acknowledge with thanks DSR, KAU for financial support. The second author is thankful for the support of Rasht Branch, Islamic Azad University. The authors would like to express their thanks to the referees for their helpful comments and suggestions.

Received: 4 June 2013 Accepted: 27 August 2013 Published: 07 Nov 2013

References
1. Agarwal, RP, Hussain, N, Taoudi, MA: Fixed point theorems in ordered Banach spaces and applications to nonlinear integral equations. Abstr. Appl. Anal. **2012**, Article ID 245872 (2012)
2. Akbar, F, Khan, AR: Common fixed point and approximation results for noncommuting maps on locally convex spaces. Fixed Point Theory Appl. **2009**, Article ID 207503 (2009)

3. Alsulami, S, Hussain, N, Alotaibi, A: Coupled fixed and coincidence points for monotone operators in partial metric spaces. Fixed Point Theory Appl. **2012**, Article ID 173 (2012)
4. Berinde, V: Generalized contractions in quasi-metric spaces. In: Seminar on Fixed Point Theory. Preprint, vol. 93, pp. 3-9 (1993)
5. Berinde, V, Borcut, M: Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces. Nonlinear Anal. **74**, 4889-4897 (2011)
6. Ćirić, L, Hussain, N, Cakic, N: Common fixed points for Ciric type f-weak contraction with applications. Publ. Math. (Debr.) **76**(1-2), 31-49 (2010)
7. Ćirić, L, Abbas, M, Saadati, R, Hussain, N: Common fixed points of almost generalized contractive mappings in ordered metric spaces. Appl. Math. Comput. **217**, 5784-5789 (2011)
8. Czerwik, S: Contraction mappings in b-metric spaces. Acta Math. Inform. Univ. Ostrav. **1**, 5-11 (1993)
9. Khamsi, MA: Remarks on cone metric spaces and fixed point theorems of contractive mappings. Fixed Point Theory Appl. **2010**, Article ID 315398 (2010)
10. Khamsi, MA, Hussain, N: KKM mappings in metric type spaces. Nonlinear Anal. **73**, 3123-3129 (2010)
11. Jovanović, M, Kadelburg, Z, Radenović, S: Common fixed point results in metric type spaces. Fixed Point Theory Appl. **2010**, Article ID 978121 (2010)
12. Bhaskar, TG, Lakshmikantham, V: Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Anal. **65**, 1379-1393 (2006)
13. Lakshmikantham, V, Ćirić, L: Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. Nonlinear Anal. **70**, 4341-4349 (2009)
14. Choudhury, BS, Kundu, A: A coupled coincidence point result in partially ordered metric spaces for compatible mappings. Nonlinear Anal. **73**, 2524-2531 (2010)
15. Cho, YJ, Shah, MH, Hussain, N: Coupled fixed points of weakly F-contractive mappings in topological spaces. Appl. Math. Lett. **24**, 1185-1190 (2011)
16. Guo, D, Lakshmikantham, V: Coupled fixed points of nonlinear operators with applications. Nonlinear Anal. **11**, 623-632 (1987)
17. Luong, NV, Thuan, NX: Coupled fixed point in partially ordered metric spaces and applications. Nonlinear Anal. **74**, 983-992 (2011)
18. Hussain, N, Khan, AR, Agarwal, RP: Krasnosel'skii and Ky Fan type fixed point theorems in ordered Banach spaces. J. Nonlinear Convex Anal. **11**(3), 475-489 (2010)
19. Hussain, N, Đorić, D, Kadelburg, Z, Radenović, S: Suzuki-type fixed point results in metric type spaces. Fixed Point Theory Appl. **2012**, Article ID 126 (2012)
20. Hussain, N, Shah, MH, Kutbi, MA: Coupled coincidence point theorems for nonlinear contractions in partially ordered quasi-metric spaces with a Q-function. Fixed Point Theory Appl. **2011**, Article ID 703938 (2011)
21. Hussain, N, Alotaibi, A: Coupled coincidences for multi-valued nonlinear contractions in partially ordered metric spaces. Fixed Point Theory Appl. **2011**, Article ID 82 (2011)
22. Hussain, N, Latif, A, Shah, MH: Coupled and tripled coincidence point results without compatibility. Fixed Point Theory Appl. **2011**, Article ID 77 (2012)
23. Hussain, N, Shah, MH: KKM mappings in cone b-metric spaces. Comput. Math. Appl. **62**, 1677-1684 (2011)
24. Nieto, JJ, Rodriguez-López, R: Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations. Order **22**(3), 223-239 (2005)
25. Herman, R, Jean, ER: Equivalents of the Axiom of Choice. North-Holland, Amsterdam (1970)
26. Parvaneh, V, Roshan, JR, Radenović, S: Existence of tripled coincidence point in ordered b-metric spaces and application to a system of integral equations. Fixed Point Theory Appl. **2013**, Article ID 130 (2013)
27. Salimi, P, Vetro, P: Common fixed point results on quasi-Banach spaces and integral equations. Georgian Math. J. (in press)
28. Sintunavarat, W, Cho, YJ, Kumam, P: Coupled coincidence point theorems for contractions without commutative condition in intuitionistic fuzzy normed spaces. Fixed Point Theory Appl. **2011**, Article ID 81 (2011)
29. Suzuki, T: A generalized Banach contraction principle that characterizes metric completeness. Proc. Am. Math. Soc. **136**, 1861-1869 (2008)
30. Takahashi, W, Lin, L-J, Wang, SY: Fixed point theorems for contractively generalized hybrid mappings in complete metric spaces. J. Nonlinear Convex Anal. **13**(2), 195-206 (2012)

10.1186/1687-1812-2013-261
Cite this article as: Hussain et al.: Coupled fixed point results on quasi-Banach spaces with application to a system of integral equations. *Fixed Point Theory and Applications* 2013, 2013:261