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# Hybrid viscosity approximation methods for general systems of variational inequalities in Banach spaces

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**Abstract**
Let $X$ be a uniformly convex and 2-uniformly smooth Banach space. In this paper, we propose an implicit iterative method and an explicit iterative method for solving a general system of variational inequalities (in short, GSVI) in $X$ based on Korpelevich's extragradient method and viscosity approximation method. We show that the proposed algorithms converge strongly to some solutions of the GSVI under consideration. When $X$ is a 2-uniformly smooth Banach space with weakly sequentially continuous duality mapping, we also propose two methods, which were inspired and motivated by Korpelevich's extragradient method and Mann's iterative method. Furthermore, it is also proven that the proposed algorithms converge strongly to some solutions of the considered GSVI.
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## 1 Introduction

Let $X$ be a real Banach space whose dual space is denoted by $X^*$. Let $U = \{x \in X : \|x\| = 1\}$. A Banach space $X$ is said to be uniformly convex if for each $\epsilon \in (0,2]$, there exists $\delta > 0$ such that for all $x, y \in U$,

$$\|x - y\| \geq \epsilon \quad \Rightarrow \quad \|x + y\|/2 \leq 1 - \delta.$$

It is known that a uniformly convex Banach space is reflexive and strict convex. A Banach space $X$ is said to be smooth if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in U$. $X$ is said to be uniformly smooth if this limit is attained uniformly for $x, y \in U$. The norm of $X$ is said to be the Frechet differential if for each $x \in U$, this limit is attained uniformly for $y \in U$. Also, we define a function $\rho : [0,\infty) \to [0,\infty)$ called the modulus of smoothness of $X$ as follows:

$$\rho(\tau) = \sup\left\{\frac{1}{2}\bigl(\|x + y\| + \|x - y\|\bigr) - 1 : x, y \in X, \|x\| = 1, \|y\| = \tau\right\}.$$

It is known that $X$ is uniformly smooth if and only if $\lim_{\tau\to 0}\rho(\tau)/\tau = 0$. Let $q$ be a fixed real number with $1 < q \le 2$. Then a Banach space $X$ is said to be $q$-uniformly smooth if there exists a constant $c > 0$ such that $\rho(\tau) \le c\tau^q$ for all $\tau > 0$.

Let $X^*$ be the dual of $X$. The normalized duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in X,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is an immediate consequence of the Hahn-Banach theorem that $J(x)$ is nonempty for each $x \in X$. Moreover, it is known that $J$ is single-valued if and only if $X$ is smooth, whereas if $X$ is uniformly smooth, then the mapping $J$ is uniformly continuous on bounded subsets of $X$. Let $C$ be a nonempty closed convex subset of a real Banach space $X$. A mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

We use the notation $\rightharpoonup$ to indicate the weak convergence and the one $\to$ to indicate the strong convergence.

**Definition 1.1** Let $A : C \to X$ be a mapping of $C$ into $X$. Then $A$ is said to be

(i) accretive if for each $x, y \in C$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \ge 0,$$

where $J$ is the normalized duality mapping;

(ii) $\alpha$-strongly accretive if for each $x, y \in C$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \ge \alpha\|x - y\|^2$$

for some $\alpha \in (0, 1)$;

(iii) $\beta$-inverse-strongly accretive if for each $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \ge \beta\|Ax - Ay\|^2$$

for some $\beta > 0$;

(iv) $\lambda$-strictly pseudocontractive if for each $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \le \|x - y\|^2 - \lambda\|x - y - (Ax - Ay)\|^2$$

for some $\lambda \in (0, 1)$.

Very recently, Yao *et al.* [1] studied the following general system of variational inequalities (GSVI) in a real smooth Banach space $X$, which is to find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B_1 y^* + x^* - y^*, J(x - x^*)\rangle \ge 0, & \forall x \in C, \\ \langle \mu_2 B_2 x^* + y^* - x^*, J(x - y^*)\rangle \ge 0, & \forall x \in C, \end{cases} \tag{1.1}$$

where $C$ is a nonempty, closed and convex subset of $X$, $B_1, B_2 : C \to X$ are two nonlinear mappings, and $\mu_1$ and $\mu_2$ are two positive constants. The set of solutions of GSVI (1.1) is denoted by $\mathrm{GSVI}(C, B_1, B_2)$. In particular, if $X = H$, a real Hilbert space, then GSVI (1.1) reduces to the following GSVI of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \mu_2 B_2 x^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.2}$$

which was considered by Ceng *et al.* [2]. The set of solutions of problem (1.2) is also denoted by $\mathrm{GSVI}(C, B_1, B_2)$. In [2], problem (1.2) was transformed into a fixed point problem in the following way.

**Lemma 1.1** (See [2]) *For given $\bar{x}, \bar{y} \in C$, $(\bar{x}, \bar{y})$ is a solution of problem* (1.2) *if and only if $\bar{x}$ is a fixed point of the mapping $G : C \to C$ defined by*

$$G(x) = P_C\big[P_C(x - \mu_2 B_2 x) - \mu_1 B_1 P_C(x - \mu_2 B_2 x)\big], \quad \forall x \in C, \tag{1.3}$$

*where $\bar{y} = P_C(\bar{x} - \mu_2 B_2 \bar{x})$.*

In this paper, we continue to study problem GSVI (1.1). We propose implicit and explicit algorithms based on Korpelevich's extragradient method [3], viscosity approximation method [4] and Mann's iterative method [5] to find approximate solutions of GSVI (1.1). Strong convergence results of these methods will be established under very mild conditions. We observe that some recent results in this direction have been obtained in, *e.g.*, [6–10] and the references therein.

## 2 Preliminaries

We need the following lemmas that will be used in the sequel.

**Lemma 2.1** (See [11]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n) s_n + \alpha_n \beta_n + \gamma_n, \quad \forall n \geq 0,$$

*where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the conditions*:

(i) $\{\alpha_n\} \subset [0,1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\limsup_{n\to\infty} \beta_n \leq 0$;
(iii) $\gamma_n \geq 0$ $(\forall n \geq 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$.

*Then* $\limsup_{n\to\infty} s_n = 0$.

**Lemma 2.2** (See [11]) *In a smooth Banach space $X$, the following inequality holds*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\big\langle y, J(x + y)\big\rangle, \quad \forall x, y \in X.$$

Let LIM be a continuous linear functional on $l^\infty$ and $(a_0, a_1, \ldots) \in l^\infty$. We write $\mathrm{LIM}\, a_n$ instead of $\mathrm{LIM}((a_0, a_1, \ldots))$. LIM is said to be a Banach limit if LIM satisfies $\|\mathrm{LIM}\| = \mathrm{LIM}\, 1 = 1$, and $\mathrm{LIM}\, a_{n+1} = \mathrm{LIM}\, a_n$ for all $(a_0, a_1, \ldots) \in l^\infty$. It is well known that for the Banach limit LIM, the following hold:

(i) for all $n \geq 1$, $a_n \leq c_n$ implies that $\operatorname{LIM} a_n \leq \operatorname{LIM} c_n$;
(ii) $\operatorname{LIM} a_{n+N} = \operatorname{LIM} a_n$ for any fixed positive integer $N$;
(iii) $\liminf_{n\to\infty} a_n \leq \operatorname{LIM} a_n \leq \limsup_{n\to\infty} a_n$ for all $(a_0, a_1, \ldots) \in l^\infty$.

**Lemma 2.3** (See [4]) *Let $(a_0, a_1, \ldots) \in l^\infty$. If $\operatorname{LIM} a_n = 0$, then there exists a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that $a_{n_k} \to 0$ as $k \to \infty$.*

We also need the following lemmas for the proofs of our main results.

**Lemma 2.4** (See [12]) *Let $q$ be a given real number with $1 < q \leq 2$, and let $X$ be a $q$-*uniformly *smooth Banach space. Then*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + 2\|\kappa y\|^q, \quad \forall x, y \in X,$$

*where $\kappa$ is the $q$-uniformly smooth constant of $X$, and $J_q$ is the generalized duality mapping from $X$ into $2^{X^*}$ defined by*

$$J_q(x) = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\}, \quad \forall x \in X.$$

Let $D$ be a subset of $C$, and let $\Pi$ be a mapping of $C$ into $D$. Then $\Pi$ is said to be sunny if

$$\Pi\big[\Pi(x) + t\big(x - \Pi(x)\big)\big] = \Pi(x),$$

whenever $\Pi(x) + t(x - \Pi(x)) \in C$ for $x \in C$ and $t \geq 0$. A mapping $\Pi$ of $C$ into itself is called a retraction if $\Pi^2 = \Pi$. If a mapping $\Pi$ of $C$ into itself is a retraction, then $\Pi(z) = z$ for every $z \in R(\Pi)$, where $R(\Pi)$ is the range of $\Pi$. A subset $D$ of $C$ is called a sunny nonexpansive retract of $C$ if there exists a sunny nonexpansive retraction from $C$ onto $D$. The following lemma concerns the sunny nonexpansive retraction.

**Lemma 2.5** (See [13]) *Let $C$ be a nonempty closed convex subset of a real smooth Banach space $X$, let $D$ be a nonempty subset of $C$, and let $\Pi$ be a retraction from $C$ onto $D$. Then $\Pi$ is sunny and nonexpansive if and only if*

$$\langle x - \Pi(x), J\big(y - \Pi(x)\big)\rangle \leq 0$$

*for all $x \in C$ and $y \in D$.*

It is well known that if $X = H$ a Hilbert space, then a sunny nonexpansive retraction $\Pi_C$ is coincident with the metric projection from $X$ onto $C$; that is, $\Pi_C = P_C$. Let $C$ be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space $X$, and let $T : C \to C$ be a nonexpansive mapping with the fixed point set $\operatorname{Fix}(T) \neq \emptyset$. Then the set $\operatorname{Fix}(T)$ is a sunny nonexpansive retract of $C$.

**Lemma 2.6** (Demiclosedness principle; see [14]) *Let $X$ be a uniformly convex Banach space or a reflexive Banach space satisfying Opial's condition, let $C$ be a nonempty closed convex subset of $X$, and let $T : C \to C$ be a nonexpansive mapping. Then the mapping $I - T$*

*is demiclosed on $C$, where $I$ is the identity mapping; that is, if $\{x_n\}$ is a sequence of $C$ such that $x_n \rightharpoonup x$ and $(I-T)x_n \to y$, then $(I-T)x = y$.*

**Lemma 2.7** (See [15]) *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space $X$, and let $\{\alpha_n\}$ be a sequence in $[0,1]$, which satisfies the following condition*

$$0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1.$$

*Suppose that $x_{n+1} = \alpha_n x_n + (1-\alpha_n)z_n$, $\forall n \ge 0$ and $\limsup_{n\to\infty}(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|) \le 0$. Then $\lim_{n\to\infty}\|z_n - x_n\| = 0$.*

**Lemma 2.8** (See [1]) *Let $C$ be a nonempty closed convex subset of a real smooth Banach space $X$. Assume that the mapping $F : C \to X$ is accretive and weakly continuous along segments (i.e., $F(x+ty) \rightharpoonup F(x)$ as $t \to 0$). Then the variational inequality*

$$\text{find } \tilde{x} \in C: \quad \langle F(\tilde{x}), J(x-\tilde{x})\rangle \ge 0, \quad \forall x \in C,$$

*is equivalent to the dual variational inequality*

$$\text{find } \tilde{x} \in C: \quad \langle F(x), J(x-\tilde{x})\rangle \ge 0, \quad \forall x \in C.$$

**Lemma 2.9** (See [1]) *Let $C$ be a nonempty closed convex subset of a 2-uniformly smooth Banach space $X$. Let the mapping $B_i : C \to X$ be $\alpha_i$-inverse-strongly accretive for $i = 1,2$. Then we have*

$$\|(I-\mu_i B_i)x - (I-\mu_i B_i)y\|^2 \le \|x-y\|^2 + 2\lambda_i(\kappa^2\lambda_i - \alpha_i)\|B_i x - B_i y\|^2, \quad \forall x,y \in C$$

*for $i = 1,2$. In particular, if $0 \le \mu_i \le \frac{\alpha_i}{\kappa^2}$, then $I - \mu_i B_i$ is nonexpansive for $i = 1,2$.*

**Lemma 2.10** (See [1]) *Let $C$ be a nonempty closed convex subset of a real 2-uniformly smooth Banach space $X$. Let $\Pi_C$ be a sunny nonexpansive retraction from $X$ onto $C$. Let the mapping $B_i : C \to X$ be $\alpha_i$-inverse-strongly accretive for $i = 1,2$. Let $G : C \to C$ be the mapping defined by (1.3). If $0 < \mu_i \le \frac{\alpha_i}{\kappa^2}$, then $G : C \to C$ is nonexpansive for $i = 1,2$.*

## 3 Implicit iterative schemes

In this section, we propose implicit iterative schemes and show the strong convergence theorems. First, we state the following obvious proposition.

**Proposition 3.1** *Let $C$ be a nonempty closed convex subset of a real smooth Banach space $X$, and let $F : C \to X$ be a mapping.*

(i) *If $F : C \to X$ is $\alpha$-strongly accretive and $\lambda$-strictly pseudocontractive with $\alpha + \lambda \ge 1$, then $I - F$ is nonexpansive, and $F$ is Lipschitz-continuous with constant $1 + \frac{1}{\lambda}$;*
(ii) *If $F : C \to X$ is $\alpha$-strongly accretive and $\lambda$-strictly pseudocontractive with $\alpha + \lambda \ge 1$, then for any fixed $\tau \in (0,1)$, $I - \tau F$ is a contraction with coefficient $1 - \tau(1 - \sqrt{\frac{1-\alpha}{\lambda}})$.*

**Lemma 3.1** *Let $C$ be a nonempty closed convex subset of a real smooth Banach space $X$. Let $\Pi_C$ be a sunny nonexpansive retraction from $X$ onto $C$, and let the mapping $B_i : C \to X$ be $\alpha_i$-inverse-strongly accretive for $i = 1, 2$. For given $x^*, y^* \in C$, $(x^*, y^*)$ is a solution of GSVI* (1.1) *if and only if $x^* = \Pi_C(y^* - \mu_1 B_1 y^*)$, where $y^* = \Pi_C(x^* - \mu_2 B_2 x^*)$.*

*Proof* Rewriting GSVI (1.1) as

$$\begin{cases} \langle x^* - (y^* - \mu_1 B_1 y^*), J(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle y^* - (x^* - \mu_2 B_2 x^*), J(x - y^*)\rangle \geq 0, & \forall x \in C, \end{cases} \tag{3.1}$$

the proof then follows from Lemma 2.4. □

By Lemma 3.1, we observe that

$$x^* = \Pi_C\bigl[\Pi_C\bigl(x^* - \mu_2 B_2 x^*\bigr) - \mu_1 B_1 \Pi_C\bigl(x^* - \mu_2 B_2 x^*\bigr)\bigr],$$

which implies that $x^*$ is a fixed point of the mapping $G$. Throughout this paper, the set of fixed points of the mapping $G$ is denoted by $\Omega$.

To solve GSVI (1.1), we first propose an implicit algorithm as follows. Let $C$ be a nonempty closed convex subset of a real 2-uniformly smooth Banach space $X$. Let $\Pi_C$ be a sunny nonexpansive retraction from $X$ onto $C$. As previously, let $\Xi_C$ be the set of all contractions on $C$. Let the mapping $B_i : C \to X$ be $\alpha_i$-inverse-strongly accretive for $i = 1, 2$. Let $f \in \Xi_C$ with coefficient $\rho \in (0, 1)$ and $F : C \to X$ be $\alpha$-strongly accretive and $\lambda$-strictly pseudocontractive with $\alpha + \lambda \geq 1$. In what follows, we assume that $0 < \mu_i \leq \frac{\alpha_i}{\kappa^2}$ for $i = 1, 2$. For any given $t \in (0, 1)$, we define a mapping $T_t : C \to C$ by

$$T_t x = tf(x_t) + (1 - t)\Pi_C(I - \theta_t F)\Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)x, \quad \forall x \in C, \tag{3.2}$$

where $\theta_t \in [0, 1)$, $\forall t \in (0, 1)$.

Define another mapping $S_t$:

$$\begin{aligned} S_t x &= \Pi_C(I - \theta_t F)\Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)x \\ &= \Pi_C\bigl[(1 - \theta_t)I + \theta_t(I - F)\bigr]\Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)x, \quad \forall x \in C. \end{aligned} \tag{3.3}$$

Then $T_t$ is rewritten as

$$T_t x = tf(x_t) + (1 - t)S_t x, \quad \forall x \in C. \tag{3.4}$$

Let us show that $S_t : C \to C$ is nonexpansive. As a matter of fact, since $\alpha + \lambda \geq 1$, by Proposition 3.1 we know that $I - F$ is nonexpansive; that is,

$$\bigl\|(I - F)x - (I - F)y\bigr\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Hence, $I - \theta_t F = (1 - \theta_t)I + \theta_t(I - F)$ is nonexpansive. So, $\Pi_C(I - \theta_t F)$ is nonexpansive. We note that by Lemma 2.9, $\Pi_C(I - \mu_i B_i)$ is nonexpansive for $i = 1, 2$. Thus, it follows from

(3.3) that $S_t : C \to C$ is nonexpansive. This together with (3.4) implies that for all $x, y \in C$,

$$\begin{aligned}\|T_t x - T_t y\| &= \big\|t\big(f(x) - f(y)\big) + (1-t)(S_t x - S_t y)\big\| \\ &\le t\big\|f(x) - f(y)\big\| + (1-t)\|S_t x - S_t y\| \\ &\le t\rho\|x - y\| + (1-t)\|x - y\| \\ &= \big(1 - t(1-\rho)\big)\|x - y\|.\end{aligned}$$

So, $T_t : C \to C$ is a contraction. Therefore, the Banach contraction principle guarantees that $T_t$ has a unique fixed point in $C$, which we denote by $x_t$; that is,

$$x_t = tf(x_t) + (1-t)S_t x_t = tf(x_t) + (1-t)\Pi_C(I - \theta_t F)\Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)x_t. \quad (3.5)$$

We now state and prove our first main result.

**Theorem 3.1** *Let $C$ be a nonempty closed convex subset of a real 2-uniformly smooth Banach space $X$. Let $\Pi_C$ be a sunny nonexpansive retraction from $X$ onto $C$. Let the mapping $B_i : C \to X$ be $\alpha_i$-inverse-strongly accretive for $i = 1, 2$. Let $f \in \Xi_C$ with coefficient $\rho \in (0,1)$, and let $F : C \to X$ be $\alpha$-strongly accretive and $\lambda$-strictly pseudocontractive with $\alpha + \lambda \ge 1$. Assume that $0 < \mu_i \le \frac{\alpha_i}{\kappa^2}$ for $i = 1, 2$. Let $x_t \in C$ be the unique solution in $C$ to Equation (3.5), where $\theta_t \in [0,1)$, $\forall t \in (0,1)$ and $\lim_{t\to 0^+} \theta_t/t = 0$. Then $\Omega \neq \emptyset$ if and only if*

$$\limsup_{t\to 0^+} \|x_t\| < \infty, \quad (3.6)$$

*and in this case, $\{x_t\}$ converges as $t \to 0^+$ strongly to an element of $\Omega$. In addition, if we define $Q : \Xi_C \to \Omega$ by*

$$Q(f) := s - \lim_{t\to 0} x_t, \quad \forall f \in \Xi_C, \quad (3.7)$$

*then $Q(f)$ solves the variational inequality problem (VIP)*

$$\big\langle (I - f)Q(f), J\big(Q(f) - p\big)\big\rangle \le 0, \quad \forall f \in \Xi_C, p \in \Omega.$$

*In particular, if $f = u \in C$ is a constant, then (3.7) reduces to the sunny nonexpansive retraction of Reich from $C$ onto $\Omega$,*

$$\big\langle Q(u) - u, J\big(Q(u) - p\big)\big\rangle \le 0, \quad \forall u \in C, p \in \Omega.$$

*Proof* If $\Omega \neq \emptyset$, we can take $p \in \Omega$ to derive from (3.5) that for $t \in (0,1)$,

$$\begin{aligned}\|x_t - p\| &\le t\big\|f(x_t) - p\big\| + (1-t)\|S_t x_t - p\| \\ &\le t\big(\big\|f(x_t) - f(p)\big\| + \big\|f(p) - p\big\|\big) + (1-t)\big(\|S_t x_t - S_t p\| + \|S_t p - p\|\big) \\ &= t\big(\big\|f(x_t) - f(p)\big\| + \big\|f(p) - p\big\|\big) \\ &\quad + (1-t)\big(\|S_t x_t - S_t p\| + \big\|\Pi_C(I - \theta_t F)p - \Pi_C p\big\|\big)\end{aligned}$$

$$\begin{aligned}
&\le t\rho\|x_t - p\| + t\big\|f(p) - p\big\| + (1-t)\big(\|x_t - p\| + \theta_t\big\|F(p)\big\|\big) \\
&= \big(1 - t(1-\rho)\big)\|x_t - p\| + t\big\|f(p) - p\big\| + (1-t)\theta_t\big\|F(p)\big\| \\
&\le t\big\|f(p) - p\big\| + \big(1 - t(1-\rho)\big)\|x_t - p\| + \theta_t\big\|F(p)\big\|,
\end{aligned}$$

which implies that

$$\|x_t - p\| \le \frac{1}{1-\rho}\big\|f(p) - p\big\| + \frac{\theta_t}{t}\cdot\frac{\|F(p)\|}{1-\rho}.$$

Because $\lim_{t\to 0^+}\theta_t/t = 0$, we deduce that

$$\limsup_{t\to 0^+}\|x_t\| \le \|p\| + \frac{1}{1-\rho}\big\|f(p) - p\big\| < \infty, \tag{3.8}$$

and hence, (3.6) holds.

Conversely, assume (3.6); that is, $\{x_t\}$ remains bounded when $t \to 0^+$; hence, $f(x_t)$ and $F(G(x_t))$ are bounded when $t \to 0^+$, where $G$ is defined by (1.3). Because in terms of (3.5),

$$x_t - G(x_t) = \frac{t}{1-t}\big(f(x_t) - x_t\big) + S_t x_t - G(x_t), \tag{3.9}$$

we obtain

$$\begin{aligned}
\big\|x_t - G(x_t)\big\| &\le \frac{t}{1-t}\big\|f(x_t) - x_t\big\| + \big\|S_t x_t - G(x_t)\big\| \\
&= \frac{t}{1-t}\big\|f(x_t) - x_t\big\| + \big\|\Pi_C\big(G(x_t) - \theta_t F\big(G(x_t)\big)\big) - \Pi_C G(x_t)\big\| \\
&\le \frac{t}{1-t}\big\|f(x_t) - x_t\big\| + \theta_t\big\|F\big(G(x_t)\big)\big\|,
\end{aligned}$$

which hence yields

$$\lim_{t\to 0^+}\big\|x_t - G(x_t)\big\| = 0. \tag{3.10}$$

Now, assume that $t_n \to 0^+$. Since $\{x_t\}$ remains bounded as $t \to 0^+$. Set $x_n := x_{t_n}$. Then $\{x_n\}$ is bounded. Now, define $g : C \to [0,\infty)$ by

$$g(x) = \mathrm{LIM}\,\|x_n - x\|^2, \quad \forall x \in C,$$

where LIM is a Banach limit on $l^\infty$. Let

$$K = \Big\{x \in C : g(x) = \min_{y\in C}\mathrm{LIM}\,\|x_n - y\|^2\Big\}.$$

It is easily seen that $K$ is nonempty closed convex bounded subset of $X$. Since (note that $\|x_n - G(x_n)\| \to 0$)

$$\begin{aligned}
g\big(G(x)\big) &= \mathrm{LIM}\big\|x_n - G(x)\big\|^2 \\
&= \mathrm{LIM}\big\|G(x_n) - G(x_n)\big\|^2 \\
&\le \mathrm{LIM}\,\|x_n - x\|^2 = g(x),
\end{aligned}$$

it follows that $G(K) \subset K$; that is, $K$ is invariant under $G$. Since a uniformly smooth Banach space has the fixed point property for nonexpansive mappings, $G$ has a fixed point, say $z$, in $K$. Since $z$ is also a minimizer of $g$ over $C$, it follows that, for $t \in (0,1)$ and $x \in C$,

$$0 \leq \frac{g(z+t(x-z))-g(z)}{t} = \operatorname{LIM} \frac{\|(x_n-z)+t(z-x)\|^2-\|x_n-z\|^2}{t}.$$

The uniform smoothness of $X$ implies that the duality map $J$ is norm-to-norm uniformly continuous on bounded sets of $X$. Letting $t \to 0$, we find that the two limits above can be interchanged and obtain

$$\operatorname{LIM}\bigl\langle x-z, J(x_n-z)\bigr\rangle \leq 0, \quad \forall x \in C. \tag{3.11}$$

Since

$$\begin{aligned}
& x_t - z = t\bigl(f(x_t)-z\bigr) + (1-t)(S_t x_t - z), \\
& \|x_t - z\|^2 = t\bigl\langle f(x_t)-z, J(x_t-z)\bigr\rangle + (1-t)\bigl\langle S_t x_t - z, J(x_t-z)\bigr\rangle \\
&\qquad \leq t\bigl\langle f(x_t)-z, J(x_t-z)\bigr\rangle + (1-t)\|S_t x_t - z\|\,\|x_t-z\| \\
&\qquad \leq t\bigl\langle f(x_t)-z, J(x_t-z)\bigr\rangle + (1-t)\bigl(\|S_t x_t - S_t z\| + \|S_t z - z\|\bigr)\|x_t-z\| \\
&\qquad = t\bigl\langle f(x_t)-z, J(x_t-z)\bigr\rangle \\
&\qquad\quad + (1-t)\bigl(\|S_t x_t - S_t z\| + \bigl\|\Pi_C(I-\theta_t F)z - \Pi_C z\bigr\|\bigr)\|x_t-z\| \\
&\qquad \leq t\bigl\langle f(x_t)-z, J(x_t-z)\bigr\rangle + (1-t)\bigl(\|x_t-z\| + \theta_t\bigl\|F(z)\bigr\|\bigr)\|x_t-z\| \\
&\qquad \leq t\bigl\langle f(x_t)-z, J(x_t-z)\bigr\rangle + (1-t)\|x_t-z\|^2 + \theta_t\bigl\|F(z)\bigr\|\,\|x_t-z\|.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|x_t - z\|^2 &\leq \bigl\langle f(x_t)-z, J(x_t-z)\bigr\rangle + \frac{\theta_t}{t}\bigl\|F(z)\bigr\|\,\|x_t-z\| \\
&= \bigl\langle f(x_t)-x, J(x_t-z)\bigr\rangle + \bigl\langle x-z, J(x_t-z)\bigr\rangle + \frac{\theta_t}{t}\bigl\|F(z)\bigr\|\,\|x_t-z\|.
\end{aligned} \tag{3.12}$$

So by (3.11), for $x \in C$,

$$\begin{aligned}
\operatorname{LIM}\|x_n - z\|^2 &\leq \operatorname{LIM}\bigl\langle f(x_n)-x, J(x_n-z)\bigr\rangle + \operatorname{LIM}\bigl\langle x-z, J(x_n-z)\bigr\rangle \\
&\leq \operatorname{LIM}\bigl\langle f(x_n)-x, J(x_n-z)\bigr\rangle \\
&\leq \operatorname{LIM}\bigl\|f(x_n)-x\bigr\|\,\|x_n-z\|.
\end{aligned}$$

In particular,

$$\operatorname{LIM}\|x_n - z\|^2 \leq \operatorname{LIM}\bigl\|f(x_n)-f(z)\bigr\|\,\|x_n-z\| \leq \rho\|x_n-z\|^2.$$

Thus,

$$\operatorname{LIM}\|x_n - z\|^2 = 0,$$

and there exists a subsequence which is still denoted by $\{x_n\}$ such that $x_n \to z$.

Now, assume that there exists another subsequence $\{x_m\}$ of $\{x_t\}$ such that $x_m \to \bar{z} \in \Omega$. It follows from (3.12) that

$$\|\bar{z} - z\|^2 \le \langle f(\bar{z}) - z, J(\bar{z} - z)\rangle. \tag{3.13}$$

Interchange $\bar{q}$ and $q$ to obtain

$$\|z - \bar{z}\|^2 \le \langle f(z) - \bar{z}, J(z - \bar{z})\rangle. \tag{3.14}$$

Adding up (3.13) and (3.14) yields

$$\begin{aligned} 2\|\bar{z} - z\|^2 &\le \langle f(\bar{z}) - f(z), J(\bar{z} - z)\rangle + \langle \bar{z} - z, J(\bar{z} - z)\rangle \\ &\le (1 + \rho)\|\bar{z} - z\|^2. \end{aligned}$$

Since $\rho \in (0,1)$, this implies that $\bar{z} = z$. Therefore, $x_t \to z$ as $t \to 0$.

Define $Q : \Xi_C \to \Omega$ by

$$Q(f) := s - \lim_{t\to 0} x_t. \tag{3.15}$$

Since $x_t = tf(x_t) + (1 - t)S_t x_t$, we have

$$(I - f)x_t = -\frac{1-t}{t}\big(x_t - G(x_t) + G(x_t) - S_t x_t\big). \tag{3.16}$$

Hence, for $p \in \Omega$,

$$\begin{aligned} \langle (I - f)x_t, J(x_t - p)\rangle &= -\frac{1-t}{t}\langle (I - G)x_t - (I - G)p, J(x_t - p)\rangle \\ &\quad - \frac{1-t}{t}\langle G(x_t) - S_t x_t, J(x_t - p)\rangle \\ &\le \frac{1-t}{t}\|G(x_t) - S_t x_t\|\varphi\big(\|x_t - p\|\big) \\ &= \frac{1-t}{t}\|\Pi_C G(x_t) - \Pi_C(I - \theta_t F)G(x_t)\|\varphi\big(\|x_t - p\|\big) \\ &\le \frac{\theta_t}{t}\|F\big(G(x_t)\big)\|\|x_t - p\|. \end{aligned} \tag{3.17}$$

Because $\theta_t/t \to 0$ and $x_t \to Q(f)$ as $t \to 0^+$, taking the limit as $t \to 0^+$ in (3.17), we obtain that

$$\langle (I - f)Q(f), J\big(Q(f) - p\big)\rangle \le 0. \tag{3.18}$$

If $f(x) = u$ ($\forall x \in C$) is a constant, then

$$\langle Q(u) - u, J\big(Q(u) - p\big)\rangle \le 0. \tag{3.19}$$

Hence, $Q$ reduces to the sunny nonexpansive retraction from $C$ to $\Omega$. □

**Theorem 3.2** *Let $C$ be a nonempty closed convex subset of a 2-uniformly smooth Banach space $X$ with weakly sequentially continuous duality mapping $J$. Let $\Pi_C$ be a sunny nonexpansive retraction from $X$ onto $C$. Let the mapping $B_i : C \to X$ be $\alpha_i$-inverse-strongly accretive with $0 \le \mu_i \le \frac{\alpha_i}{\kappa^2}$ for $i = 1, 2$. Let $F : C \to X$ be $\alpha$-strongly accretive and $\lambda$-strictly pseudocontractive with $\alpha + \lambda > 1$. Assume that $\Omega \neq \emptyset$. Let the net $\{x_t\}$ be defined by the implicit scheme*

$$x_t = \Pi_C(I - tF)\Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)x_t, \quad \forall t \in (0,1). \tag{3.20}$$

*Then $\{x_t\}$ converges in norm, as $t \to 0^+$, to the unique solution of the VIP*

$$\textit{find } \tilde{x} \in \Omega: \quad \langle F(\tilde{x}), J(x - \tilde{x}) \rangle \ge 0, \quad \forall x \in \Omega. \tag{3.21}$$

*Proof* For any given $t \in (0,1)$, consider the following mapping

$$W_t x := \Pi_C(I - tF)\Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)x, \quad \forall x \in C. \tag{3.22}$$

By Proposition 3.1(ii) and Lemma 2.9, we know that $\Pi_C(I - \mu_i B_i)$ is nonexpansive for $i = 1, 2$, and $I - tF$ is contractive with coefficient $1 - t(1 - \sqrt{\frac{1-\alpha}{\lambda}})$. Hence,

$$\|W_t x - W_t y\| \le \left(1 - t\left(1 - \sqrt{\frac{1-\alpha}{\lambda}}\right)\right)\|x - y\|, \quad \forall x, y \in C.$$

This means that $W_t$ is a contraction. Therefore, the Banach contraction principle guarantees that $W_t$ has a unique fixed point in $C$, which we denote by $x_t$. This shows that the implicit scheme (3.20) is well defined.

Now, we show that $\{x_t\}$ is bounded. As a matter of fact, take $p \in \Omega$ arbitrarily. Then it follows from (3.20) and Lemma 3.1 that

$$\begin{aligned}
\|x_t - p\| &= \left\|\Pi_C(I - tF)G(x_t) - \Pi_C(I - tF)G(p) + \Pi_C(I - tF)G(p) - p\right\| \\
&\le \left\|\Pi_C(I - tF)G(x_t) - \Pi_C(I - tF)G(p)\right\| + \left\|\Pi_C(I - tF)p - \Pi_C p\right\| \\
&= \|W_t x_t - W_t p\| + \left\|\Pi_C(I - tF)p - \Pi_C p\right\| \\
&\le \left(1 - t\left(1 - \sqrt{\frac{1-\alpha}{\lambda}}\right)\right)\|x_t - p\| + t\left\|F(p)\right\| \\
&= (1 - t\bar{\gamma})\|x_t - p\| + t\left\|F(p)\right\|,
\end{aligned}$$

where $\bar{\gamma} = 1 - \sqrt{\frac{1-\alpha}{\lambda}}$. Thus, it immediately follows that

$$\|x_t - p\| \le \frac{1}{\bar{\gamma}}\left\|F(p)\right\|.$$

Therefore, $\{x_t\}$ is bounded and so are the nets $\{G(x_t)\}$, $\{F(G(x_t))\}$. Furthermore, by Lemma 2.10, we know that $G : C \to C$ is nonexpansive. Thus,

$$\left\|x_t - G(x_t)\right\| = \left\|\Pi_C(I - tF)G(x_t) - \Pi_C G(x_t)\right\| \le t\left\|F\left(G(x_t)\right)\right\| \to 0$$

as $t \to 0$. That is,

$$\lim_{t\to 0}\|x_t - G(x_t)\| = 0.$$

Furthermore, we show that $\{x_t\}$ is relatively norm-compact as $t \to 0^+$. Assume that $\{t_n\} \subset (0,1)$ is such that $t_n \to 0^+$ as $n \to \infty$. Put $x_n := x_{t_n}$. Then it is clear that

$$\|x_n - G(x_n)\| \to 0 \quad \text{as } n \to \infty. \tag{3.23}$$

We can rewrite (3.20) as

$$x_t = \Pi_C(I - tF)G(x_t) - (I - tF)G(x_t) + (I - tF)G(x_t).$$

For any $p \in \Omega \subset C$, by Lemma 2.5, we have

$$\begin{aligned}
&\langle x_t - (I - tF)G(x_t), J(x_t - p)\rangle \\
&\quad = \langle \Pi_C(I - tF)G(x_t) - (I - tF)G(x_t), J(\Pi_C(I - tF)G(x_t) - p)\rangle \le 0.
\end{aligned}$$

According to this fact, we deduce that

$$\begin{aligned}
\|x_t - p\|^2 &= \langle x_t - p, J(x_t - p)\rangle \\
&= \langle x_t - (I - tF)G(x_t), J(x_t - p)\rangle + \langle (I - tF)G(x_t) - p, J(x_t - p)\rangle \\
&\le \langle (I - tF)G(x_t) - p, J(x_t - p)\rangle \\
&= \langle (I - tF)(G(x_t) - p), J(x_t - p)\rangle - t\langle F(p), J(x_t - p)\rangle \\
&\le (1 - t\bar{\gamma})\|G(x_t) - p\|\|x_t - p\| - t\langle F(p), J(x_t - p)\rangle \\
&\le (1 - t\bar{\gamma})\|x_t - p\|^2 - t\langle F(p), J(x_t - p)\rangle.
\end{aligned}$$

It turns out that

$$\|x_t - p\|^2 \le \frac{1}{\bar{\gamma}}\langle F(p), J(p - x_t)\rangle, \quad \forall p \in \Omega. \tag{3.24}$$

In particular,

$$\|x_n - p\|^2 \le \frac{1}{\bar{\gamma}}\langle F(p), J(p - x_n)\rangle, \quad \forall p \in \Omega. \tag{3.25}$$

Since $\{x_n\}$ is bounded, we may assume, without loss of generality, that $\{x_n\}$ converges weakly to a point $\tilde{x} \in C$. Noticing (3.23), we can use Lemma 2.6 to get $\tilde{x} \in \Omega$. Therefore, we can substitute $\tilde{x}$ for $p$ in (3.25) to get

$$\|x_n - \tilde{x}\|^2 \le \frac{1}{\bar{\gamma}}\langle F(\tilde{x}), J(\tilde{x} - x_n)\rangle, \tag{3.26}$$

which together with the weakly sequential continuity of $J$ implies that

$$\lim_{n\to\infty}\|x_n - \tilde{x}\| = 0.$$

This has proven the relative norm compactness of the net $\{x_n\}$ as $t \to 0^+$.

We also show that $\tilde{x}$ solves the VIP (3.21). From (3.20), we have

$$\begin{aligned} x_t &= \Pi_C(I-tF)G(x_t) - (I-tF)G(x_t) + (I-tF)G(x_t) \\ \Rightarrow\quad x_t &= \Pi_C(I-tF)G(x_t) - (I-tF)G(x_t) \\ &\quad - \big((I-tF)x_t - (I-tF)G(x_t)\big) + x_t - tF(x_t) \\ \Rightarrow\quad F(x_t) &= \frac{1}{t}\big[\Pi_C(I-tF)G(x_t) - (I-tF)G(x_t) - \big((I-tF)x_t - (I-tF)G(x_t)\big)\big]. \end{aligned}$$

For any $z \in \Omega$, utilizing the nonexpansivity of $G$, we obtain that

$$\langle x_t - G(x_t), J(x_t - z)\rangle = \langle (I-G)x_t - (I-G)z, J(x_t - z)\rangle \geq 0,$$

and hence,

$$\begin{aligned} \langle F(x_t), J(x_t - z)\rangle &= \frac{1}{t}\langle \Pi_C(I-tF)G(x_t) - (I-tF)G(x_t), J(x_t - z)\rangle \\ &\quad - \frac{1}{t}\langle \big((I-tF)x_t - (I-tF)G(x_t)\big), J(x_t - z)\rangle \\ &\leq -\frac{1}{t}\langle \big((I-tF)x_t - (I-tF)G(x_t)\big), J(x_t - z)\rangle \\ &= -\frac{1}{t}\langle x_t - G(x_t), J(x_t - z)\rangle + \langle F(x_t) - F\big(G(x_t)\big), J(x_t - z)\rangle \\ &\leq \langle F(x_t) - F\big(G(x_t)\big), J(x_t - z)\rangle. \end{aligned}$$

Therefore,

$$\langle F(x_t), J(x_t - z)\rangle \leq \langle F(x_t) - F\big(G(x_t)\big), J(x_t - z)\rangle. \tag{3.27}$$

Since $F$ is $\alpha$-strongly accretive, we have

$$0 \leq \alpha\|x_t - z\|^2 \leq \langle F(x_t) - F(z), J(x_t - z)\rangle.$$

It follows that

$$\langle F(z), J(x_t - z)\rangle \leq \langle F(x_t), J(x_t - z)\rangle. \tag{3.28}$$

Combining (3.27) and (3.28), we get

$$\langle F(z), J(x_t - z)\rangle \leq \langle F(x_t) - F\big(G(x_t)\big), J(x_t - z)\rangle. \tag{3.29}$$

Now, replacing $t$ in (3.29) with $t_n$ and letting $n \to \infty$, noticing that $x_{t_n} \to \tilde{x}$ and $x_{t_n} - G(x_{t_n}) \to 0$ as $n \to \infty$, we derive

$$\langle F(z), J(\tilde{x} - z)\rangle \leq 0, \quad \forall z \in \Omega,$$

which is equivalent to its dual variational inequality (see Lemma 2.8)

$$\langle F(\tilde{x}), J(\tilde{x} - z)\rangle \leq 0, \quad \forall z \in \Omega. \tag{3.30}$$

That is, $\tilde{x} \in \Omega$ is a solution of VIP (3.21). Now, we show that the solution set of VIP (3.21) is a singleton. As a matter of fact, we assume that $\bar{x} \in \Omega$ is another solution of VIP (3.21). Then, we have

$$\langle F(\bar{x}), J(\bar{x} - \tilde{x})\rangle \leq 0.$$

From (3.30), we have

$$\langle F(\tilde{x}), J(\tilde{x} - \bar{x})\rangle \leq 0.$$

So,

$$\begin{aligned} &\langle F(\bar{x}), J(\bar{x} - \tilde{x})\rangle + \langle F(\tilde{x}), J(\tilde{x} - \bar{x})\rangle \leq 0 \\ &\quad \Rightarrow \quad \langle F(\bar{x}) - F(\tilde{x}), J(\bar{x} - \tilde{x})\rangle \leq 0 \\ &\quad \Rightarrow \quad \alpha\|\bar{x} - \tilde{x}\|^2 \leq 0. \end{aligned}$$

Therefore, $\bar{x} = \tilde{x}$. In summary, we have shown that each (strong) cluster point of the net $\{x_t\}$ (as $t \to 0$) equals to $\tilde{x}$. Therefore, $x_t \to \tilde{x}$ as $t \to 0$. This completes the proof. □

## 4 Explicit iterative schemes

In this section, we propose explicit iterative schemes which are the discretization of the implicit iterative schemes, and show the strong convergence theorems.

**Algorithm 4.1** Let $C$ be a nonempty closed convex subset of a real smooth Banach space $X$. Let $\Pi_C$ be a sunny nonexpansive retraction from $X$ onto $C$. Let $B_1, B_2 : C \to X$ be two nonlinear mappings. Let $f \in \Xi_C$ and $F : C \to X$ be $\alpha$-strongly accretive and $\lambda$-strictly pseudocontractive. For arbitrarily given $x_0 \in C$, let the sequence $\{x_n\}$ be generated iteratively by

$$x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)\Pi_C(I - \gamma_n F)\Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)x_n, \quad \forall n \geq 0, \tag{4.1}$$

where $\{\beta_n\} \subset (0,1)$, $\{\gamma_n\} \subset [0,1)$ and $\mu_1$, $\mu_2$ are two positive numbers.

In particular, if $B_1 = B_2 = A$, then (4.1) reduces to the following:

$$x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)\Pi_C(I - \gamma_n F)\Pi_C(I - \mu_1 A)\Pi_C(I - \mu_2 A)x_n, \quad \forall n \geq 0. \tag{4.2}$$

**Theorem 4.1** *Let $C$ be a nonempty closed convex subset of a real 2-uniformly smooth Banach space $X$. Let $\Pi_C$ be a sunny nonexpansive retraction from $X$ onto $C$. Let the mapping $B_i : C \to X$ be $\alpha_i$-inverse-strongly accretive for $i = 1, 2$. Let $f \in \Xi_C$ with coefficient $\rho \in (0,1)$, and let $F : C \to X$ be $\alpha$-strongly accretive and $\lambda$-strictly pseudocontractive with $\alpha + \lambda \geq 1$. Assume that $0 < \mu_i \leq \frac{\alpha_i}{\kappa^2}$ for $i = 1, 2$. Let $\Omega \neq \emptyset$, and assume that*

(i) $\beta_n \to 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$;
(ii) $\lim_{n\to\infty} \gamma_n/\beta_n = 0$;
(iii) $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \beta_{n-1}/\beta_n = 1$;
(iv) $\sum_{n=1}^{\infty} |\gamma_n - \gamma_{n-1}| < \infty$ *or* $\lim_{n\to\infty} |\gamma_n - \gamma_{n-1}|/\beta_n = 0$.

*Then the sequence* $\{x_n\}$ *generated by scheme* (4.1) *converges strongly to* $Q(f)$, *where* $Q : \Xi_C \to \Omega$ *is defined by* (3.15).

*Proof* For each $n \geq 0$, let $S_n$ be defined by

$$S_n x = \Pi_C(I - \gamma_n F)\Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)x, \quad \forall x \in C.$$

Then we know that

(i) the scheme (4.1) is rewritten as

$$x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)S_n x_n, \quad \forall n \geq 0; \tag{4.3}$$

(ii) $S_n$ is nonexpansive by the similar argument to that of the nonexpansivity of $S_t$ in (3.5);

(iii) $S_n p = \Pi_C(I - \gamma_n F)p$ for all $p \in \Omega$.

Thus, we deduce that for $p \in \Omega$,

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&\quad = \big\| \beta_n\big(f(x_n) - p\big) + (1 - \beta_n)(S_n x_n - p)\big\| \\
&\quad \leq \beta_n \big\|f(x_n) - p\big\| + (1 - \beta_n)\|S_n x_n - p\| \\
&\quad \leq \beta_n\big(\big\|f(x_n) - f(p)\big\| + \big\|f(p) - p\big\|\big) + (1 - \beta_n)\big(\|S_n x_n - S_n p\| + \|S_n p - p\|\big) \\
&\quad \leq \beta_n \rho\|x_n - p\| + \beta_n\big\|f(p) - p\big\| + (1 - \beta_n)\big(\|x_n - p\| + \big\|\Pi_C(I - \gamma_n F)p - \Pi_C p\big\|\big) \\
&\quad \leq \big(1 - \beta_n(1 - \rho)\big)\|x_n - p\| + \beta_n\big\|f(p) - p\big\| + \gamma_n\big\|F(p)\big\|.
\end{aligned} \tag{4.4}$$

Because $\lim_{n\to\infty} \gamma_n/\beta_n = 0$, we may assume without loss of generality that $\gamma_n \leq \beta_n$ for all $n \geq 0$. Hence, from (4.4), we get

$$\|x_{n+1} - p\| \leq \beta_n\big(\big\|f(p) - p\big\| + \big\|F(p)\big\|\big) + \big(1 - \beta_n(1 - \rho)\big)\|x_n - p\|, \quad \forall n \geq 0.$$

By induction, we conclude that

$$\|x_n - p\| \leq \max\left\{\frac{\|f(p) - p\| + \|F(p)\|}{1 - \rho}, \|x_0 - p\|\right\}, \quad \forall n \geq 0. \tag{4.5}$$

Therefore, $\{x_n\}$ is bounded, so are the sequences $\{f(x_n)\}$, $\{G(x_n)\}$, $\{S_n x_n\}$ and $\{F(G(x_n))\}$. Also, from (4.1), we have

$$\begin{aligned}
\big\|x_{n+1} - G(x_n)\big\| &\leq \beta_n\big\|f(x_n) - G(x_n)\big\| + (1 - \beta_n)\big\|S_n x_n - G(x_n)\big\| \\
&= \beta_n\big\|f(x_n) - G(x_n)\big\| + (1 - \beta_n)\big\|\Pi_C(I - \gamma_n F)G(x_n) - \Pi_C G(x_n)\big\| \\
&\leq \beta_n\big\|f(x_n) - G(x_n)\big\| + (1 - \beta_n)\gamma_n\big\|F\big(G(x_n)\big)\big\| \\
&\leq \beta_n\big\|f(x_n) - G(x_n)\big\| + \gamma_n\big\|F\big(G(x_n)\big)\big\|,
\end{aligned}$$

which together with $\beta_n \to 0$ and $\gamma_n \to 0$, implies that

$$\lim_{n\to\infty}\big\|x_{n+1} - G(x_n)\big\| = 0. \tag{4.6}$$

Now, we note that

$$\begin{aligned} x_{n+1} - x_n &= \beta_n f(x_n) + (1-\beta_n)S_n x_n - \beta_{n-1} f(x_n) - (1-\beta_{n-1})S_{n-1}x_{n-1} \\ &= (\beta_n - \beta_{n-1})\big(f(x_{n-1}) - S_{n-1}x_{n-1}\big) + \beta_n\big(f(x_n) - f(x_{n-1})\big) \\ &\quad + (1-\beta_n)(S_n x_n - S_n x_{n-1}) + (1-\beta_n)(S_n x_{n-1} - S_{n-1}x_{n-1}). \end{aligned}$$

Thus, it follows that

$$\begin{aligned} &\|x_{n+1} - x_n\| \\ &\quad \le |\beta_n - \beta_{n-1}| \big\| f(x_{n-1}) - S_{n-1}x_{n-1}\big\| + \beta_n \big\| f(x_n) - f(x_{n-1})\big\| + (1-\beta_n)\|S_n x_n - S_n x_{n-1}\| \\ &\qquad + (1-\beta_n)\|S_n x_{n-1} - S_{n-1}x_{n-1}\| \\ &\quad \le |\beta_n - \beta_{n-1}| \big\| f(x_{n-1}) - S_{n-1}x_{n-1}\big\| + \beta_n\rho\|x_n - x_{n-1}\| + (1-\beta_n)\|x_n - x_{n-1}\| \\ &\qquad + \big\| \Pi_C(I - \gamma_n F)G(x_{n-1}) - \Pi_C(I - \gamma_{n-1}F)G(x_{n-1})\big\| \\ &\quad \le M|\beta_n - \beta_{n-1}| + \beta_n\rho\|x_n - x_{n-1}\| + (1-\beta_n)\|x_n - x_{n-1}\| \\ &\qquad + |\gamma_n - \gamma_{n-1}| \big\| F\big(G(x_{n-1})\big)\big\| \\ &\quad = \big(1 - \beta_n(1-\rho)\big)\|x_n - x_{n-1}\| + M\big(|\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}|\big), \end{aligned}$$

where $\sup_{n\ge 0}\{\|f(x_n) - S_n x_n\| + \|F(G(x_n))\|\} \le M$ for some $M > 0$. So, utilizing Lemma 2.1, we obtain that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

This together with (4.6) implies that

$$\lim_{n\to\infty} \big\| x_n - G(x_n)\big\| = 0. \tag{4.7}$$

Let us show that

$$\limsup_{n\to\infty} \big\langle \tilde{x} - f(\tilde{x}), J(\tilde{x} - x_n)\big\rangle \le 0, \tag{4.8}$$

where $\tilde{x} = Q(f)$. Indeed we can write

$$x_t - x_n = t\big(f(x_t) - x_n\big) + (1-t)(S_t x_t - x_n).$$

Putting

$$a_n(t) = \big(\big\|G(x_n) - x_n\big\| + \theta_t \big\|F\big(G(x_n)\big)\big\|\big)\big[2\|x_t - x_n\| + \big\|G(x_n) - x_n\big\| + \theta_t\big\|F\big(G(x_n)\big)\big\|\big],$$

and using Lemma 2.2, we obtain

$$\begin{aligned} \|x_t - x_n\|^2 &\le (1-t)^2\|S_t x_t - x_n\|^2 + 2t\big\langle f(x_t) - x_n, J(x_t - x_n)\big\rangle \\ &\le (1-t)^2\big(\|S_t x_t - S_t x_n\| + \|S_t x_n - x_n\|\big)^2 \end{aligned}$$

$$\begin{aligned}
&\quad + 2t\big\langle f(x_t) - x_t, J(x_t - x_n)\big\rangle + 2t\|x_t - x_n\|^2 \\
&\le (1-t)^2\big(\|x_t - x_n\| + \big\|\Pi_C(I - \theta_t F)G(x_n) - \Pi_C x_n\big\|\big)^2 \\
&\quad + 2t\big\langle f(x_t) - x_t, J(x_t - x_n)\big\rangle + 2t\|x_t - x_n\|^2 \\
&\le (1-t)^2\big(\|x_t - x_n\| + \big\|G(x_n) - x_n\big\| + \theta_t\big\|F\big(G(x_n)\big)\big\|\big)^2 \\
&\quad + 2t\big\langle f(x_t) - x_t, J(x_t - x_n)\big\rangle + 2t\|x_t - x_n\|^2 \\
&\le (1-t)^2\|x_t - x_n\|^2 + a_n(t) \\
&\quad + 2t\big\langle f(x_t) - x_t, J(x_t - x_n)\big\rangle + 2t\|x_t - x_n\|^2.
\end{aligned}$$

The last inequality implies that

$$\big\langle x_t - f(x_t), J(x_t - x_n)\big\rangle \le \frac{t}{2}\|x_t - x_n\|^2 + \frac{1}{2t}a_n(t). \tag{4.9}$$

Note that

$$\begin{aligned}
\limsup_{n\to\infty}\frac{1}{2t}a_n(t) &= \limsup_{n\to\infty}\frac{1}{2t}\big(\big\|G(x_n) - x_n\big\| + \theta_t\big\|F\big(G(x_n)\big)\big\|\big) \\
&\quad \times \big[2\|x_t - x_n\| + \big\|G(x_n) - x_n\big\| + \theta_t\big\|F\big(G(x_n)\big)\big\|\big] \\
&= \frac{\theta_t}{2t}\limsup_{n\to\infty}\big\|F\big(G(x_n)\big)\big\|\big[2\|x_t - x_n\| + \theta_t\big\|F\big(G(x_n)\big)\big\|\big] \\
&\le \frac{\theta_t}{2t}\limsup_{n\to\infty}\big\|F\big(G(x_n)\big)\big\|\big[2\|x_t - x_n\| + 2\big\|F\big(G(x_n)\big)\big\|\big] \\
&\le \frac{\theta_t}{t}M_0^2,
\end{aligned}$$

where $M_0 > 0$ is a constant such that $M_0 \ge \|F(G(x_n))\| + \|x_t - x_n\|$ for all $n \ge 1$ and $t \in (0,1)$. It follows from (4.9) that

$$\limsup_{n\to\infty}\big\langle x_t - f(x_t), J(x_t - x_n)\big\rangle \le \frac{t}{2}M_0^2 + \frac{\theta_t}{t}M_0^2. \tag{4.10}$$

Taking the $\limsup$ as $t \to 0$ in (4.10), and noticing the fact that the two limits are interchangeable due to the fact that the duality map $J$ is norm-to-norm uniformly continuous on bounded sets of $X$, we obtain (4.8).

Finally, we show that $x_n \to \tilde{x}$. Write

$$x_{n+1} - \tilde{x} = \beta_n\big(f(x_n) - \tilde{x}\big) + (1-\beta_n)(S_n x_n - \tilde{x}),$$

and apply Lemma 2.2 to get

$$\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &\le (1-\beta_n)^2\|S_n x_n - \tilde{x}\|^2 + 2\beta_n\big\langle f(x_n) - \tilde{x}, J(x_{n+1} - \tilde{x})\big\rangle \\
&\le (1-\beta_n)^2\big(\|S_n x_n - S_n\tilde{x}\| + \|S_n\tilde{x} - \tilde{x}\|\big)^2 + 2\beta_n\big\langle f(x_n) - \tilde{x}, J(x_{n+1} - \tilde{x})\big\rangle \\
&\le (1-\beta_n)^2\big(\|x_n - \tilde{x}\| + \big\|\Pi_C(I - \gamma_n F)\tilde{x} - \tilde{x}\big\|\big)^2 + 2\beta_n\big\langle f(x_n) - \tilde{x}, J(x_{n+1} - \tilde{x})\big\rangle \\
&\le (1-\beta_n)^2\big(\|x_n - \tilde{x}\| + \gamma_n\big\|F(\tilde{x})\big\|\big)^2 + 2\beta_n\big\langle f(x_n) - f(\tilde{x}), J(x_{n+1} - \tilde{x})\big\rangle
\end{aligned}$$

$$
\begin{aligned}
&\quad + 2\beta_n\langle f(\tilde{x}) - \tilde{x}, J(x_{n+1} - \tilde{x})\rangle \\
&\leq (1-\beta_n)^2\|x_n - \tilde{x}\|^2 + \gamma_n\|F(\tilde{x})\|\bigl(2\|x_n - \tilde{x}\| + \gamma_n\|F(\tilde{x})\|\bigr) \\
&\quad + 2\beta_n\rho\|x_n - \tilde{x}\|\|x_{n+1} - \tilde{x}\| + 2\beta_n\langle f(\tilde{x}) - \tilde{x}, J(x_{n+1} - \tilde{x})\rangle \\
&\leq (1-\beta_n)^2\|x_n - \tilde{x}\|^2 + \gamma_n\|F(\tilde{x})\|\bigl(2\|x_n - \tilde{x}\| + \|F(\tilde{x})\|\bigr) \\
&\quad + \rho\beta_n\bigl(\|x_n - \tilde{x}\|^2 + \|x_{n+1} - \tilde{x}\|^2\bigr) + 2\beta_n\langle f(\tilde{x}) - \tilde{x}, J(x_{n+1} - \tilde{x})\rangle.
\end{aligned}
$$

It then follows that

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &\leq \frac{1-(2-\rho)\beta_n + \beta_n^2}{1-\rho\beta_n}\|x_n - \tilde{x}\|^2 \\
&\quad + \frac{\beta_n}{1-\rho\beta_n}\biggl[\frac{\gamma_n}{\beta_n}\|F(\tilde{x})\|\bigl(2\|x_n - \tilde{x}\| + \|F(\tilde{x})\|\bigr) + 2\langle f(\tilde{x}) - \tilde{x}, J(x_{n+1} - \tilde{x})\rangle\biggr] \\
&= \biggl(1 - \frac{2(1-\rho)\beta_n}{1-\rho\beta_n}\biggr)\|x_n - \tilde{x}\|^2 \\
&\quad + \frac{2(1-\rho)\beta_n}{1-\rho\beta_n}\cdot\frac{1}{2(1-\rho)}\biggl[\frac{\gamma_n}{\beta_n}\|F(\tilde{x})\|\bigl(2\|x_n - \tilde{x}\| + \|F(\tilde{x})\|\bigr) \\
&\quad + \beta_n\|x_n - \tilde{x}\|^2 + 2\langle f(\tilde{x}) - \tilde{x}, J(x_{n+1} - \tilde{x})\rangle\biggr].
\end{aligned}
$$

Put

$$
\tilde{\alpha}_n = \frac{2(1-\rho)\beta_n}{1-\rho\beta_n}
$$

and

$$
\tilde{\beta}_n = \frac{1}{2(1-\rho)}\biggl[\frac{\gamma_n}{\beta_n}\|F(\tilde{x})\|\bigl(2\|x_n - \tilde{x}\| + \|F(\tilde{x})\|\bigr) + \beta_n\|x_n - \tilde{x}\|^2 + 2\langle f(\tilde{x}) - \tilde{x}, J(x_{n+1} - \tilde{x})\rangle\biggr].
$$

It follows that

$$
\|x_{n+1} - \tilde{x}\|^2 \leq (1-\tilde{\alpha}_n)\|x_n - \tilde{x}\|^2 + \tilde{\alpha}_n\tilde{\beta}_n. \tag{4.11}
$$

Observe that

$$
\begin{aligned}
&\limsup_{n\to\infty}\langle f(\tilde{x}) - \tilde{x}, J(x_{n+1} - \tilde{x})\rangle \\
&\quad = \limsup_{n\to\infty}\bigl(\langle f(\tilde{x}) - \tilde{x}, J(x_n - \tilde{x})\rangle + \langle f(\tilde{x}) - \tilde{x}, J(x_{n+1} - \tilde{x}) - J(x_n - \tilde{x})\rangle\bigr) \\
&\quad = \limsup_{n\to\infty}\langle f(\tilde{x}) - \tilde{x}, J(x_n - \tilde{x})\rangle \leq 0
\end{aligned}
$$

due to (4.8). It is easily seen from conditions (i), (ii) that

$$
\tilde{\alpha}_n \to 0, \qquad \sum_{n=0}^{\infty}\tilde{\alpha}_n = \infty \quad \text{and} \quad \limsup_{n\to\infty}\tilde{\beta}_n \leq 0.
$$

Finally, apply Lemma 2.2 to (4.11) to conclude that $x_n \to \tilde{x}$. $\square$

**Algorithm 4.2** Let $C$ be a nonempty closed convex subset of a real smooth Banach space $X$. Let $\Pi_C$ be a sunny nonexpansive retraction from $X$ onto $C$. Let $B_1, B_2 : C \to X$ be two nonlinear mappings. Let $F : C \to X$ be $\alpha$-strongly accretive and $\lambda$-strictly pseudocontractive. For arbitrarily given $x_0 \in C$, let the sequence $\{x_n\}$ be generated iteratively by

$$x_{n+1} = \beta_n x_n + (1-\beta_n)\Pi_C(I-\alpha_n F)\Pi_C(I-\mu_1 B_1)\Pi_C(I-\mu_2 B_2)x_n, \quad \forall n \geq 0, \tag{4.12}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0,1]$ and $\mu_1$, $\mu_2$ are two positive numbers.

In particular, if $B_1 = B_2 = A$, then (4.12) reduces to the following:

$$x_{n+1} = \beta_n x_n + (1-\beta_n)\Pi_C(I-\alpha_n F)\Pi_C(I-\mu_1 A)\Pi_C(I-\mu_2 A)x_n, \quad \forall n \geq 0. \tag{4.13}$$

**Theorem 4.2** *Let $C$ be a nonempty closed convex subset of a 2-uniformly smooth Banach space $X$ with weakly sequentially continuous duality mapping $J$. Let $\Pi_C$ be a sunny nonexpansive retraction from $X$ onto $C$. Let the mapping $B_i : C \to X$ be $\alpha_i$-inverse-strongly accretive with $0 < \mu_i \leq \frac{\alpha_i}{\kappa^2}$ for $i = 1,2$. Let $F : C \to X$ be $\alpha$-strongly accretive and $\lambda$-strictly pseudocontractive with $\alpha + \lambda > 1$. Let $\Omega \neq \emptyset$, and let $\{x_n\}$ be the sequence generated by* (4.12). *Assume that the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions*:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

*Then the sequence $\{x_n\}$ converges strongly to the unique solution $\tilde{x} \in \Omega$ of VIP* (3.21).

*Proof* Take a fixed $p \in \Omega$ arbitrarily. Then $G(p) = p$ due to Lemma 1.1. By Lemma 2.10, we have

$$\|G(x_n) - p\| = \|G(x_n) - G(p)\| \leq \|x_n - p\|, \quad \forall n \geq 0.$$

Hence, it follows from Proposition 3.1(ii) that

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&\quad = \big\|\beta_n(x_n - p) + (1-\beta_n)\big(\Pi_C(I-\alpha_n F)G(x_n) - p\big)\big\| \\
&\quad \leq \beta_n\|x_n - p\| + (1-\beta_n)\big\|(I-\alpha_n F)G(x_n) - p\big\| \\
&\quad = \beta_n\|x_n - p\| + (1-\beta_n)\big\|(I-\alpha_n F)G(x_n) - (I-\alpha_n F)G(p) + (I-\alpha_n F)G(p) - p\big\| \\
&\quad \leq \beta_n\|x_n - p\| + (1-\beta_n)\big[\big\|(I-\alpha_n F)G(x_n) - (I-\alpha_n F)G(p)\big\| + \big\|(I-\alpha_n F)p - p\big\|\big] \\
&\quad \leq \beta_n\|x_n - p\| + (1-\beta_n)\big[(1-\alpha_n\bar{\gamma})\|x_n - p\| + \alpha_n\big\|F(p)\big\|\big] \\
&\quad = \big(1-(1-\beta_n)\alpha_n\bar{\gamma}\big)\|x_n - p\| + (1-\beta_n)\alpha_n\bar{\gamma}\frac{\|F(p)\|}{\bar{\gamma}},
\end{aligned}$$

where $\bar{\gamma} = 1 - \sqrt{\frac{1-\alpha}{\lambda}}$. By induction, we deduce that

$$\|x_{n+1} - p\| \leq \max\left\{\|x_0 - p\|, \frac{\|F(p)\|}{\bar{\gamma}}\right\}.$$

Therefore, $\{x_n\}$ is bounded. Hence, $\{G(x_n)\}$ and $\{F(G(x_n))\}$ are also bounded. Now, set $v_n = \Pi_C(I-\alpha_n F)G(x_n)$ for all $n \geq 0$. Then $x_{n+1} = \beta_n x_n + (1-\beta_n)v_n$ for $n \geq 0$. Hence, it follows that

$$\begin{aligned}
\|v_{n+1} - v_n\| &= \big\|\Pi_C(I - \alpha_{n+1}F)G(x_{n+1}) - \Pi_C(I - \alpha_n F)G(x_n)\big\| \\
&\le \big\|(I - \alpha_{n+1}F)G(x_{n+1}) - (I - \alpha_n F)G(x_n)\big\| \\
&= \big\|G(x_{n+1}) - G(x_n) - \alpha_{n+1}F\big(G(x_{n+1})\big) + \alpha_n F\big(G(x_n)\big)\big\| \\
&\le \big\|G(x_{n+1}) - G(x_n)\big\| + \alpha_{n+1}\big\|F\big(G(x_{n+1})\big)\big\| + \alpha_n\big\|F\big(G(x_n)\big)\big\| \\
&\le \|x_{n+1} - x_n\| + \alpha_{n+1}\big\|F\big(G(x_{n+1})\big)\big\| + \alpha_n\big\|F\big(G(x_n)\big)\big\|,
\end{aligned}$$

which together with $\alpha_n \to 0$ and the boundedness of $\{F(G(x_n))\}$ implies that

$$\limsup_{n\to\infty}\big(\|v_{n+1} - v_n\| - \|x_{n+1} - x_n\|\big) \le 0.$$

So, by Lemma 2.7 we get

$$\lim_{n\to\infty} \|v_n - x_n\| = 0.$$

Consequently,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1 - \beta_n)\|v_n - x_n\| = 0.$$

At the same time, we note that

$$\begin{aligned}
\big\|v_n - G(x_n)\big\| &= \big\|\Pi_C(I - \alpha_n F)G(x_n) - G(x_n)\big\| \\
&= \big\|\Pi_C(I - \alpha_n F)G(x_n) - \Pi_C G(x_n)\big\| \\
&\le \alpha_n\big\|F\big(G(x_n)\big)\big\| \\
&\to 0.
\end{aligned}$$

It follows from $\|v_n - x_n\| \to 0$ that

$$\lim_{n\to\infty} \big\|x_n - G(x_n)\big\| = 0.$$

Since $v_n = \Pi_C(I - \alpha_n F)G(x_n)$ for all $n \ge 0$, by Lemma 2.10 we have

$$\begin{aligned}
\big\|v_n - G(v_n)\big\| &\le \|v_n - x_n\| + \big\|x_n - G(x_n)\big\| + \big\|G(x_n) - G(v_n)\big\| \\
&\le \|v_n - x_n\| + \big\|x_n - G(x_n)\big\| + \|x_n - v_n\| \\
&\le 2\|v_n - x_n\| + \big\|x_n - G(x_n)\big\| \\
&\to 0.
\end{aligned} \tag{4.14}$$

Next, we show that

$$\limsup_{n\to\infty}\big\langle F(\tilde{x}), J(\tilde{x} - v_n)\big\rangle \le 0, \tag{4.15}$$

where $\tilde{x} \in \Omega$ is the unique solution of VIP (3.21).

To see this, we choose a subsequence $\{v_{n_j}\}$ of $\{v_n\}$ such that

$$\limsup_{n\to\infty}\langle F(\tilde{x}), J(\tilde{x}-v_n)\rangle = \lim_{j\to\infty}\langle F(\tilde{x}), J(\tilde{x}-v_{n_j})\rangle.$$

We may also assume that $v_{n_j} \rightharpoonup z \in C$. Note that $z \in \Omega$ in terms of Lemma 2.6 and (4.14). Therefore, it follows from VIP (3.21) and the weakly sequential continuity of $J$ that

$$\limsup_{n\to\infty}\langle F(\tilde{x}), J(\tilde{x}-v_n)\rangle = \lim_{j\to\infty}\langle F(\tilde{x}), J(\tilde{x}-v_{n_j})\rangle = \langle F(\tilde{x}), J(\tilde{x}-z)\rangle \le 0.$$

Since $v_n = \Pi_C(I-\alpha_n F)G(x_n)$, according to Lemma 2.5, we have

$$\langle (I-\alpha_n F)G(x_n) - \Pi_C(I-\alpha_n F)G(x_n), J(\tilde{x}-v_n)\rangle \le 0. \tag{4.16}$$

From (4.16), we have

$$\begin{aligned}
&\|v_n-\tilde{x}\|^2 \\
&\quad= \langle \Pi_C(I-\alpha_n F)G(x_n)-\tilde{x}, J(v_n-\tilde{x})\rangle \\
&\quad= \langle \Pi_C(I-\alpha_n F)G(x_n)-(I-\alpha_n F)G(x_n), J(v_n-\tilde{x})\rangle + \langle (I-\alpha_n F)G(x_n)-\tilde{x}, J(v_n-\tilde{x})\rangle \\
&\quad\le \langle (I-\alpha_n F)G(x_n)-\tilde{x}, J(v_n-\tilde{x})\rangle \\
&\quad= \langle (I-\alpha_n F)G(x_n)-(I-\alpha_n F)G(\tilde{x}), J(v_n-\tilde{x})\rangle + \langle (I-\alpha_n F)\tilde{x}-\tilde{x}, J(v_n-\tilde{x})\rangle \\
&\quad\le (1-\alpha_n\bar{\gamma})\|x_n-\tilde{x}\|\|v_n-\tilde{x}\| + \alpha_n\langle F(\tilde{x}), J(\tilde{x}-v_n)\rangle \\
&\quad\le \frac{(1-\alpha_n\bar{\gamma})^2}{2}\|x_n-\tilde{x}\|^2 + \frac{1}{2}\|v_n-\tilde{x}\|^2 + \alpha_n\langle F(\tilde{x}), J(\tilde{x}-v_n)\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|v_n-\tilde{x}\|^2 &\le (1-\alpha_n\bar{\gamma})^2\|x_n-\tilde{x}\|^2 + 2\alpha_n\langle F(\tilde{x}), J(\tilde{x}-v_n)\rangle \\
&\le (1-\alpha_n\bar{\gamma})\|x_n-\tilde{x}\|^2 + 2\alpha_n\langle F(\tilde{x}), J(\tilde{x}-v_n)\rangle.
\end{aligned} \tag{4.17}$$

Finally, we prove that $x_n \to \tilde{x}$ as $n\to\infty$. From (4.12) and (4.17),

$$\begin{aligned}
&\|x_{n+1}-\tilde{x}\|^2 \\
&\quad\le \beta_n\|x_n-\tilde{x}\|^2 + (1-\beta_n)\|v_n-\tilde{x}\| \\
&\quad\le \beta_n\|x_n-\tilde{x}\|^2 + (1-\beta_n)(1-\alpha_n\bar{\gamma})\|x_n-\tilde{x}\| + 2\alpha_n(1-\beta_n)\langle F(\tilde{x}), J(\tilde{x}-v_n)\rangle \\
&\quad= \bigl[1-\alpha_n(1-\beta_n)\bar{\gamma}\bigr]\|x_n-\tilde{x}\| + \alpha_n(1-\beta_n)\bar{\gamma}\left\{\frac{2}{\gamma}\langle F(\tilde{x}), J(\tilde{x}-v_n)\rangle\right\}.
\end{aligned} \tag{4.18}$$

We apply Lemma 2.1 to the relation (4.18) and conclude that $x_n \to \tilde{x}$ as $n\to\infty$. This completes the proof. □

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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