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Fixed points of cyclic weakly (ψ, φ, L, A, B)-contractive mappings in ordered b-metric spaces with applications

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Abstract

We introduce the notion of ordered cyclic weakly (ψ, φ, L, A, B)-contractive mappings, and we establish some fixed and common fixed point results for this class of mappings in complete ordered b-metric spaces. Our results extend several known results from the context of ordered metric spaces to the setting of ordered b-metric spaces. They are also cyclic variants of some very recent results in ordered b-metric spaces with even weaker contractive conditions. Some examples support our results and show that the obtained extensions are proper. Moreover, an application to integral equations is given here to illustrate the usability of the obtained results.

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1 Introduction and preliminaries

The Banach contraction principle is a very popular tool for solving problems in nonlinear analysis. One of the interesting generalizations of this basic principle was given by Kirk *et al.* [1] in 2003 by introducing the following notion of cyclic representation.

Definition 1 [1] Let A and B be non-empty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$. Then T is called a cyclic map if $T(A) \subseteq B$ and $T(B) \subseteq A$.

The following interesting theorem for a cyclic map was given in [1].

Theorem 1 *Let A and B be nonempty closed subsets of a complete metric space (X, d). Suppose that $T : A \cup B \to A \cup B$ is a cyclic map such that*

$$d(Tx, Ty) \le kd(x, y)$$

for all $x \in A$ and $y \in B$, where $k \in [0, 1)$ is a constant. Then T has a unique fixed point u and $u \in A \cap B$.

It should be noted that cyclic contractions (unlike Banach-type contractions) need not be continuous, which is an important gain of this approach. Following the work of Kirk *et* *al.*, several authors proved many fixed point results for cyclic mappings, satisfying various (nonlinear) contractive conditions.

Berinde initiated in [2] the concept of almost contractions and obtained several interesting fixed point theorems. This has been a subject of intense study since then, see, *e.g.*, [3–7]. Some authors used related notions as 'condition (B)' (Babu *et al.* [8]) and 'almost generalized contractive condition' for two maps (Ćirić *et al.* [9]), and for four maps (Aghajani *et al.* [10]). See also a note by Pacurar [11]. Here, we recall one of the respective definitions.

Definition 2 [9] Let f and g be two self-mappings on a metric space (X,d). They are said to satisfy almost generalized contractive condition, if there exist a constant $\delta \in (0,1)$ and some $L \geq 0$ such that

$$d(fx,gy) \leq \delta \max\left\{d(x,y), d(x,fx), d(y,gy), \frac{d(x,gy)+d(y,fx)}{2}\right\} + L\min\bigl\{d(x,fx), d(y,gy), d(x,gy), d(y,fx)\bigr\}$$

for all $x, y \in X$.

Khan *et al.* [12] introduced the concept of an altering distance function as follows.

Definition 3 [12] A function $\varphi : [0,+\infty) \to [0,+\infty)$ is called an altering distance function if the following properties hold:

1. φ is continuous and non-decreasing.
2. $\varphi(t) = 0$ if and only if $t = 0$.

So far, many authors have studied fixed point theorems, which are based on altering distance functions.

The concept of a b-metric space was introduced by Bakhtin in [13], and later used by Czerwik in [14, 15]. After that, several interesting results about the existence of fixed points for single-valued and multi-valued operators in b-metric spaces have been obtained (see, *e.g.*, [16–28]). Recently, Hussain and Shah [29] obtained some results on KKM mappings in cone b-metric spaces.

Consistent with [15] and [28], the following definitions and results will be needed in the sequel.

Definition 4 [15] Let X be a (nonempty) set, and let $s \geq 1$ be a given real number. A function $d : X \times X \to R^+$ is a b-metric if for all $x, y, z \in X$, the following conditions hold:

(b_1) $d(x,y) = 0$ iff $x = y$,
(b_2) $d(x,y) = d(y,x)$,
(b_3) $d(x,z) \leq s[d(x,y) + d(y,z)]$.

In this case, the pair (X,d) is called a b-metric space.

It should be noted that the class of b-metric spaces is effectively larger than the class of metric spaces, since a b-metric is a metric if (and only if) $s = 1$. Here, we present an easy example to show that in general, a b-metric need not necessarily be a metric (see also [28, p.264]).

Example 1 Let (X, ρ) be a metric space and $d(x,y) = (\rho(x,y))^p$, where $p > 1$ is a real number. Then d is a b-metric with $s = 2^{p-1}$. Condition (b_3) follows easily from the convexity of the function $f(x) = x^p$ $(x > 0)$.

The notions of b-convergent and b-Cauchy sequences, as well as of b-complete b-metric spaces are introduced in an obvious way (see, *e.g.*, [18]).

It should be noted that in general, a b-metric function $d(x,y)$ for $s > 1$ need not be jointly continuous in both variables. The following example (corrected from [22]) illustrates this fact.

Example 2 Let $X = \mathbb{N} \cup \{\infty\}$, and let $d : X \times X \to \mathbb{R}$ be defined by

$$d(m,n) = \begin{cases} 0, & \text{if } m = n, \\ |\frac{1}{m} - \frac{1}{n}|, & \text{if one of } m, n \text{ is even and the other is even or } \infty, \\ 5, & \text{if one of } m, n \text{ is odd and the other is odd (and } m \neq n) \text{ or } \infty, \\ 2, & \text{otherwise.} \end{cases}$$

Then considering all possible cases, it can be checked that for all $m, n, p \in X$, we have

$$d(m,p) \leq \frac{5}{2}\bigl(d(m,n) + d(n,p)\bigr).$$

Thus, (X, d) is a b-metric space (with $s = 5/2$). Let $x_n = 2n$ for each $n \in \mathbb{N}$. Then

$$d(2n, \infty) = \frac{1}{2n} \to 0 \quad \text{as } n \to \infty,$$

that is, $x_n \to \infty$, but $d(x_n, 1) = 2 \not\to 5 = d(\infty, 1)$ as $n \to \infty$.

Aghajani *et al.* [16] proved the following simple lemma about the b-convergent sequences.

Lemma 1 *Let (X,d) be a b-metric space with $s \geq 1$, and suppose that $\{x_n\}$ and $\{y_n\}$ b-converge to x, y, respectively. Then we have*

$$\frac{1}{s^2} d(x,y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x,y).$$

In particular, if $x = y$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have

$$\frac{1}{s} d(x,z) \leq \liminf_{n\to\infty} d(x_n, z) \leq \limsup_{n\to\infty} d(x_n, z) \leq s d(x,z).$$

The existence of fixed points for mappings in partially ordered metric spaces was first investigated in 2004 by Ran and Reurings [30], and then by Nieto and Lopez [31]. Afterwards, this area was a field of intensive study of many authors.

Shatanawi and Postolache proved in [32] the following common fixed point results for cyclic contractions in the framework of ordered metric spaces.

Theorem 2 [32] *Let $(X,\preceq,d)$ be a complete ordered metric space, and let A, B be closed nonempty subsets of X with $X = A\cup B$. Let $f,g : X\to X$ be two mappings, which are (A,B)-weakly increasing (see further Definition 6). Assume that*

(a) *$A\cup B$ is a cyclic representation of X w.r.t. the pair (f,g), i.e., $f(A)\subset B$ and $g(B)\subset A$;*
(b) *there exist $0<\delta<1$ and an altering distance function ψ such that for any two comparable elements $x,y\in X$ with $x\in A$ and $y\in B$, we have*

$$\psi\big(d(fx,gy)\big)\leq\delta\psi\left(\max\left\{d(x,y),d(x,fx),d(y,gy),\frac{1}{2}\big(d(x,gy)+d(y,fx)\big)\right\}\right);$$

(c) *f or g is continuous, or*
(c′) *the space $(X,\preceq,d)$ is regular.*

Then f and g have a common fixed point.

Here, the ordered metric space $(X,\preceq,d)$ is called regular if for any non-decreasing sequence $\{x_n\}$ in X such that $x_n\to x\in X$, as $n\to\infty$, one has $x_n\preceq x$ for all $n\in\mathbb{N}$.

By an ordered b-metric space, we mean a triple $(X,\preceq,d)$, where $(X,\preceq)$ is a partially ordered set, and (X,d) is a b-metric space. Fixed points in such spaces were studied, *e.g.*, by Aghajani *et al.* [16] and Roshan *et al.* [27]. In the last mentioned paper, the following common fixed point results for contractions in ordered b-metric spaces were proved.

Theorem 3 [27] *Let $(X,\preceq,d)$ be a complete ordered b-metric space, and let $f,g : X\to X$ be two weakly increasing mappings. Suppose that there exist two altering distance functions ψ, φ and a constant $L\geq 0$ such that the inequality*

$$\psi\big(s^4d(fx,gy)\big)\leq\psi\big(M_s(x,y)\big)-\varphi\big(M_s(x,y)\big)+L\psi\big(N(x,y)\big)$$

holds for all comparable $x,y\in X$, where

$$M_s(x,y)=\max\left\{d(x,y),d(x,fx),d(y,gy),\frac{d(x,gy)+d(y,fx)}{2s}\right\}$$

and

$$N(x,y)=\min\big\{d(y,gy),d(x,gy),d(y,fx)\big\}.$$

If either [f or g is continuous], or the space $(X,\preceq,d)$ is regular, then f and g have a common fixed point.

In this paper, we introduce the notion of ordered cyclic weakly (ψ,φ,L,A,B)-contractions and then derive fixed point and common fixed point theorems for these cyclic contractions in the setup of complete ordered b-metric spaces. Our results extend some fixed point theorems from the framework of ordered metric spaces, in particular Theorem 2. On the other hand, they are cyclic variants of Theorem 3 with even weaker contractive conditions.

We show by examples that the obtained extensions are proper. Moreover, an application to integral equations is given here to illustrate the usability of the obtained results.

2 Common fixed point results

In this section, we introduce the notion of ordered cyclic weakly (ψ,φ,L,A,B)-contractive pair of self-mappings and prove our main results.

Definition 5 Let $(X,\preceq,d)$ be an ordered b-metric space, let $f,g:X\to X$ be two mappings, and let A and B be nonempty closed subsets of X. The pair (f,g) is called an ordered cyclic weakly (ψ,φ,L,A,B)-contraction if

(1) $X=A\cup B$ is a cyclic representation of X w.r.t. the pair (f,g); that is, $fA\subseteq B$ and $gB\subseteq A$;

(2) there exist two altering distance functions ψ, φ and a constant $L\geq 0$, such that for arbitrary comparable elements $x,y\in X$ with $x\in A$ and $y\in B$, we have

$$\psi\bigl(s^2d(fx,gy)\bigr)\leq\psi\bigl(M_s(x,y)\bigr)-\varphi\bigl(M_s(x,y)\bigr)+L\psi\bigl(N(x,y)\bigr), \tag{2.1}$$

where

$$M_s(x,y)=\max\left\{d(x,y),d(x,fx),d(y,gy),\frac{d(x,gy)+d(y,fx)}{2s}\right\} \tag{2.2}$$

and

$$N(x,y)=\min\bigl\{d(y,gy),d(x,gy),d(y,fx)\bigr\}. \tag{2.3}$$

Definition 6 [32] Let $(X,\preceq)$ be a partially ordered set, and let A and B be closed subsets of X with $X=A\cup B$. Let $f,g:X\to X$ be two mappings. The pair (f,g) is said to be (A,B)-weakly increasing if $fx\preceq gfx$ for all $x\in A$ and $gy\preceq fgy$ for all $y\in B$.

Theorem 4 *Let $(X,\preceq,d)$ be a complete ordered b-metric space, and let A and B be closed subsets of X. Let $f,g:X\to X$ be two (A,B)-weakly increasing mappings with respect to $\preceq$. Suppose that*

(a) *the pair (f,g) is an ordered cyclic weakly (ψ,φ,L,A,B)-contraction*;

(b) *f or g is continuous.*

Then f and g have a common fixed point $u\in A\cap B$.

Proof Let us divide the proof into two parts.

First part. We prove that $u\in A\cap B$ is a fixed point of f if and only if u is a fixed point of g. Suppose that u is a fixed point of f. As $u\preceq u$ and $u\in A\cap B$, by (2.1), we have

$$\begin{aligned}
\psi\bigl(s^2d(u,gu)\bigr)&=\psi\bigl(s^2d(fu,gu)\bigr)\\
&\leq\psi\left(\max\left\{d(u,fu),d(u,gu),\frac{1}{2s}\bigl(d(u,gu)+d(u,fu)\bigr)\right\}\right)\\
&\quad-\varphi\left(\max\left\{d(u,fu),d(u,gu),\frac{1}{2s}\bigl(d(u,gu)+d(u,fu)\bigr)\right\}\right)\\
&\quad+L\min\bigl\{d(u,gu),d(u,fu)\bigr\}\\
&=\psi\bigl(d(u,gu)\bigr)-\varphi\bigl(d(u,gu)\bigr)\\
&\leq\psi\bigl(s^2d(u,gu)\bigr)-\varphi\bigl(d(u,gu)\bigr).
\end{aligned}$$

It follows that $\varphi(d(u,gu)) = 0$. Therefore, $d(u,gu) = 0$, and hence $gu = u$. Similarly, we can show that if u is a fixed point of g, then u is a fixed point of f.

Second part (construction of a sequence by iterative technique).

Let $x_0 \in A$, and let $x_1 = fx_0$. Since $fA \subseteq B$, we have $x_1 \in B$. Also, let $x_2 = gx_1$. Since $gB \subseteq A$, we have $x_2 \in A$. Continuing this process, we can construct a sequence $\{x_n\}$ in X such that $x_{2n+1} = fx_{2n}, x_{2n+2} = gx_{2n+1}, x_{2n} \in A$ and $x_{2n+1} \in B$. Since f and g are (A,B)-weakly increasing, we have

$$\begin{aligned} x_1 = fx_0 \preceq gfx_0 = x_2 = gx_1 \preceq fgx_1 = x_3 \preceq \cdots \\ \preceq x_{2n+1} = fx_{2n} \preceq gfx_{2n} = x_{2n+2} \preceq \cdots . \end{aligned}$$

If $x_{2n} = x_{2n+1}$, for some $n \in \mathbb{N}$, then $x_{2n} = fx_{2n}$. Thus, x_{2n} is a fixed point of f. By the first part of proof, we conclude that x_{2n} is also a fixed point of g. Similarly, if $x_{2n+1} = x_{2n+2}$, for some $n \in \mathbb{N}$, then $x_{2n+1} = gx_{2n+1}$. Thus, x_{2n+1} is a fixed point of g. By the first part of proof, we conclude that x_{2n+1} is also a fixed point of f. Therefore, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Now, we complete the proof in the following steps.

Step 1. We will prove that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

As x_{2n} and x_{2n+1} are comparable and $x_{2n} \in A$ and $x_{2n+1} \in B$, by (2.1), we have

$$\begin{aligned} \psi\bigl(d(x_{2n+1},x_{2n+2})\bigr) &\le \psi\bigl(s^2 d(x_{2n+1},x_{2n+2})\bigr) \\ &= \psi\bigl(s^2 d(fx_{2n},gx_{2n+1})\bigr) \\ &\le \psi\bigl(M_s(x_{2n},x_{2n+1})\bigr) - \varphi\bigl(M_s(x_{2n},x_{2n+1})\bigr) + L\psi\bigl(N(x_{2n},x_{2n+1})\bigr), \end{aligned}$$

where

$$\begin{aligned} M_s(x_{2n},x_{2n+1}) &= \max\Biggl\{d(x_{2n},x_{2n+1}), d(x_{2n},fx_{2n}), d(x_{2n+1},gx_{2n+1}), \\ &\qquad \frac{d(fx_{2n},x_{2n+1}) + d(x_{2n},gx_{2n+1})}{2s}\Biggr\} \\ &= \max\biggl\{d(x_{2n},x_{2n+1}), d(x_{2n+1},x_{2n+2}), \frac{d(x_{2n},x_{2n+2})}{2s}\biggr\} \\ &\le \max\Biggl\{d(x_{2n},x_{2n+1}), d(x_{2n+1},x_{2n+2}), \\ &\qquad \frac{s[d(x_{2n},x_{2n+1}) + d(x_{2n+1},x_{2n+2})]}{2s}\Biggr\} \\ &= \max\bigl\{d(x_{2n},x_{2n+1}), d(x_{2n+1},x_{2n+2})\bigr\}, \end{aligned}$$

and

$$\begin{aligned} N(x_{2n},x_{2n+1}) &= \min\bigl\{d(x_{2n+1},gx_{2n+1}), d(x_{2n+1},fx_{2n}), d(x_{2n},gx_{2n+1})\bigr\} \\ &= \min\bigl\{d(x_{2n+1},x_{2n+2}), d(x_{2n+1},x_{2n+1}), d(x_{2n},x_{2n+2})\bigr\} = 0. \end{aligned}$$

Hence, we have

$$\begin{aligned}\psi\bigl(d(x_{2n+1},x_{2n+2})\bigr) &\le \psi\bigl(\max\bigl\{d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2})\bigr\}\bigr)\\ &\quad-\varphi\bigl(\max\bigl\{d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2})\bigr\}\bigr).\end{aligned} \tag{2.4}$$

If

$$\max\bigl\{d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2})\bigr\}=d(x_{2n+1},x_{2n+2}),$$

then (2.4) becomes

$$\begin{aligned}\psi\bigl(d(x_{2n+1},x_{2n+2})\bigr) &\le \psi\bigl(d(x_{2n+1},x_{2n+2})\bigr)-\varphi\bigl(d(x_{2n+1},x_{2n+2})\bigr)\\ &< \psi\bigl(d(x_{2n+1},x_{2n+2})\bigr),\end{aligned}$$

which gives a contradiction. So,

$$\max\bigl\{d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2})\bigr\}=d(x_{2n},x_{2n+1}),$$

and hence, (2.4) becomes

$$\begin{aligned}\psi\bigl(d(x_{2n+1},x_{2n+2})\bigr) &\le \psi\bigl(d(x_{2n},x_{2n+1})\bigr)-\varphi\bigl(d(x_{2n},x_{2n+1})\bigr)\\ &< \psi\bigl(d(x_{2n},x_{2n+1})\bigr).\end{aligned} \tag{2.5}$$

Similarly, we can show that

$$\psi\bigl(d(x_{2n+1},x_{2n})\bigr) < \psi\bigl(d(x_{2n},x_{2n-1})\bigr). \tag{2.6}$$

By (2.5) and (2.6), we get that $\{d(x_n,x_{n+1}) : n\in\mathbb{N}\}$ is a non-increasing sequence of positive numbers. Hence, there is $r\ge 0$ such that

$$\lim_{n\to\infty} d(x_n,x_{n+1})=r.$$

Letting $n\to\infty$ in (2.5), we get

$$\psi(r)\le\psi(r)-\varphi(r),$$

which implies that $\varphi(r)=0$, and hence $r=0$. So, we have

$$\lim_{n\to\infty} d(x_n,x_{n+1})=0. \tag{2.7}$$

Step 2. We will prove that $\{x_n\}$ is a b-Cauchy sequence. Because of (2.7), it is sufficient to show that $\{x_{2n}\}$ is a b-Cauchy sequence. Suppose on the contrary, *i.e.*, that $\{x_{2n}\}$ is not a b-Cauchy sequence. Then there exists $\varepsilon>0$, for which we can find two subsequences $\{x_{2m_i}\}$ and $\{x_{2n_i}\}$ of $\{x_{2n}\}$ such that n_i is the smallest index, for which

$$n_i>m_i>i,\quad d(x_{2m_i},x_{2n_i})\ge\varepsilon. \tag{2.8}$$

This means that

$$d(x_{2m_i}, x_{2n_i-2}) < \varepsilon. \tag{2.9}$$

From (2.8) and using the triangular inequality, we get

$$\varepsilon \leq d(x_{2m_i}, x_{2n_i}) \leq sd(x_{2m_i}, x_{2m_i+1}) + sd(x_{2m_i+1}, x_{2n_i}).$$

Using (2.7) and taking the upper limit as $i \to \infty$, we get

$$\frac{\varepsilon}{s} \leq \limsup_{i\to\infty} d(x_{2m_i+1}, x_{2n_i}). \tag{2.10}$$

On the other hand, we have

$$d(x_{2m_i}, x_{2n_i-1}) \leq sd(x_{2m_i}, x_{2n_i-2}) + sd(x_{2n_i-2}, x_{2n_i-1}).$$

Using (2.7), (2.9) and taking the upper limit as $i \to \infty$, we get

$$\limsup_{i\to\infty} d(x_{2m_i}, x_{2n_i-1}) \leq \varepsilon s. \tag{2.11}$$

Again, using the triangular inequality, we have

$$\begin{aligned} d(x_{2m_i}, x_{2n_i}) &\leq sd(x_{2m_i}, x_{2n_i-2}) + sd(x_{2n_i-2}, x_{2n_i}) \\ &\leq sd(x_{2m_i}, x_{2n_i-2}) + s^2 d(x_{2n_i-2}, x_{2n_i-1}) + s^2 d(x_{2n_i-1}, x_{2n_i}) \end{aligned}$$

and

$$d(x_{2m_i+1}, x_{2n_i-1}) \leq sd(x_{2m_i+1}, x_{2m_i}) + sd(x_{2m_i}, x_{2n_i-1}).$$

Taking the upper limit as $i \to \infty$ in the above inequalities, and using (2.7), (2.9) and (2.11), we get

$$\limsup_{i\to\infty} d(x_{2m_i}, x_{2n_i}) \leq \varepsilon s \tag{2.12}$$

and

$$\limsup_{i\to\infty} d(x_{2m_i+1}, x_{2n_i-1}) \leq \varepsilon s^2. \tag{2.13}$$

Since x_{2m_i} and x_{2n_i-1} are comparable and $x_{2m_i} \in A$ and $x_{2n_i-1} \in B$, using (2.1) we have

$$\begin{aligned} \psi\big(s^2 d(x_{2m_i+1}, x_{2n_i})\big) &= \psi\big(s^2 d(fx_{2m_i}, gx_{2n_i-1})\big) \\ &\leq \psi\big(M_s(x_{2m_i}, x_{2n_i-1})\big) - \varphi\big(M_s(x_{2m_i}, x_{2n_i-1})\big) \\ &\quad + L\psi\big(N(x_{2m_i}, x_{2n_i-1})\big), \end{aligned} \tag{2.14}$$

where

$$M_s(x_{2m_i}, x_{2n_i-1}) = \max\left\{d(x_{2m_i}, x_{2n_i-1}), d(x_{2m_i}, x_{2m_i+1}), d(x_{2n_i-1}, x_{2n_i}), \frac{d(x_{2m_i}, x_{2n_i}) + d(x_{2m_i+1}, x_{2n_i-1})}{2s}\right\} \tag{2.15}$$

and

$$N(x_{2m_i}, x_{2n_i-1}) = \min\left\{d(x_{2n_i-1}, x_{2n_i}), d(x_{2m_i}, x_{2n_i}), d(x_{2n_i-1}, x_{2m_i+1})\right\}. \tag{2.16}$$

Taking the upper limit in (2.15) and using (2.7) and (2.11)-(2.13), we get

$$\begin{aligned}\limsup_{i\to\infty} M_s(x_{2m_i}, x_{2n_i-1}) &= \max\left\{\limsup_{i\to\infty} d(x_{2m_i}, x_{2n_i-1}), 0, 0, \frac{\limsup_{i\to\infty} d(x_{2m_i}, x_{2n_i}) + \limsup_{i\to\infty} d(x_{2m_i+1}, x_{2n_i-1})}{2s}\right\} \\ &\leq \max\left\{\varepsilon s, \frac{\varepsilon s + \varepsilon s^2}{2s}\right\} = \varepsilon s.\end{aligned}$$

Hence, we have

$$\limsup_{i\to\infty} M_s(x_{2m_i}, x_{2n_i-1}) \leq \varepsilon s, \tag{2.17}$$

and, from (2.16),

$$\limsup_{i\to\infty} N(x_{2m_i}, x_{2n_i-1}) = 0. \tag{2.18}$$

Now, taking the upper limit as $i \to \infty$ in (2.14) and using (2.10), (2.17) and (2.18), we have

$$\begin{aligned}\psi(\varepsilon s) = \psi\left(s^2 \frac{\varepsilon}{s}\right) &\leq \psi\left(s^2 \limsup_{i\to\infty} d(x_{2m_i+1}, x_{2n_i})\right) \\ &\leq \psi\left(\limsup_{i\to\infty} M_s(x_{2m_i}, x_{2n_i-1})\right) - \varphi\left(\liminf_{i\to\infty} M_s(x_{2m_i}, x_{2n_i-1})\right) \\ &\leq \psi(\varepsilon s) - \varphi\left(\liminf_{i\to\infty} M_s(x_{2m_i}, x_{2n_i-1})\right),\end{aligned}$$

which implies that $\varphi(\liminf_{i\to\infty} M_s(x_{2m_i}, x_{2n_i-1})) = 0$. By (2.15), it follows that $\liminf_{i\to\infty} d(x_{2m_i}, x_{2n_i}) = 0$, which is in contradiction with (2.8). Hence $\{x_n\}$ is a b-Cauchy sequence in X.

Step 3 (existence of a common fixed point).

As $\{x_n\}$ is a b-Cauchy sequence in X which is a b-complete b-metric space, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$, and

$$\lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} fx_{2n} = u.$$

Now, without loss of generality, we may assume that f is continuous. Using the triangular inequality, we get

$$d(u,fu) \leq sd(u,fx_{2n}) + sd(fx_{2n},fu).$$

Letting $n \to \infty$, we get

$$d(u,fu) \leq s \lim_{n\to\infty} d(u,fx_{2n}) + s \lim_{n\to\infty} d(fx_{2n},fu) = 0.$$

Hence, we have $fu = u$. Thus, u is a fixed point of f and, since A and B are closed subsets of X, $u \in A \cap B$. By the first part of proof, we conclude that u is also a fixed point of g. □

The assumption of continuity of one of the mappings f or g in Theorem 4 can be replaced by another condition, which is often used in similar situations. Namely, we shall use the notion of a regular ordered b-metric space, which is defined analogously to the case of the standard metric (see the paragraph following Theorem 2).

Theorem 5 *Let the hypotheses of Theorem* 4 *be satisfied, except that condition* (b) *is replaced by the assumption*

(b′) *the space* $(X, \preceq, d)$ *is regular.*

Then f and g have a common fixed point in X.

Proof Repeating the proof of Theorem 4, we construct an increasing sequence $\{x_n\}$ in X such that $x_n \to u$ for some $u \in X$. As A and B are closed subsets of X, we have $u \in A \cap B$. Using the assumption (b′) on X, we have $x_n \preceq u$ for all $n \in \mathbb{N}$. Now, we show that $fu = gu = u$. By (2.1), we have

$$\begin{aligned}\psi\big(s^2 d(x_{2n+1},gu)\big) &= \psi\big(s^2 d(fx_{2n},gu)\big)\\ &\leq \psi\big(M_s(x_{2n},u)\big) - \varphi\big(M_s(x_{2n},u)\big) + L\psi\big(N(x_{2n},u)\big),\end{aligned} \tag{2.19}$$

where

$$\begin{aligned}M_s(x_{2n},u) &= \max\left\{d(x_{2n},u), d(x_{2n},fx_{2n}), d(u,gu), \frac{d(x_{2n},gu)+d(fx_{2n},u)}{2s}\right\}\\ &= \max\left\{d(x_{2n},u), d(x_{2n},x_{2n+1}), d(u,gu), \frac{d(x_{2n},gu)+d(x_{2n+1},u)}{2s}\right\}\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}N(x_{2n},u) &= \min\big\{d(u,gu), d(u,fx_{2n}), d(x_{2n},gu)\big\}\\ &= \min\big\{d(u,gu), d(u,x_{2n+1}), d(x_{2n},gu)\big\}.\end{aligned} \tag{2.21}$$

Letting $n \to \infty$ in (2.20) and (2.21) and using Lemma 1, we get

$$\limsup_{i\to\infty} M_s(x_{2n},u) \leq \max\left\{d(u,gu), \frac{sd(u,gu)}{2s}\right\} = d(u,gu), \tag{2.22}$$

and $N(x_{2n}, u) \to 0$. Now, taking the upper limit as $n \to \infty$ in (2.19) and using Lemma 1 and (2.22), we get

$$\begin{aligned}\psi\big(sd(u,gu)\big) = \psi\left(s^2\frac{1}{s}d(u,gu)\right) &\le \psi\Big(s^2 \limsup_{n\to\infty} d(x_{2n+1},gu)\Big)\\ &\le \psi\Big(\limsup_{n\to\infty} M_s(x_{2n},u)\Big) - \varphi\Big(\liminf_{n\to\infty} M_s(x_{2n},u)\Big)\\ &\le \psi\big(sd(u,gu)\big) - \varphi\Big(\liminf_{n\to\infty} M_s(x_{2n},u)\Big).\end{aligned}$$

It follows that $\varphi(\liminf_{n\to\infty} M_s(x_{2n}, u)) = 0$, and hence, by (2.20), that $d(u,gu) = 0$. Thus, u is a fixed point of g. On the other hand, similar to the first part of the proof of Theorem 4, we can show that $fu = u$. Hence, u is a common fixed point of f and g. □

3 Consequences and examples

As consequences, we have the following results.

By putting $A = B = X$ in Theorems 4 and 5, we obtain improvements of the main results (Theorems 5 and 6) of Roshan *et al.* [27], *i.e.*, of Theorem 3 of the present paper (note that we have s^2 instead of s^4 in the contractive condition).

Taking $\varphi = (1 - \delta)\psi$, $0 < \delta < 1$ in Theorems 4 and 5, we get the following.

Corollary 1 *Let $(X, \preceq, d)$ be a complete ordered b-metric space, and let A and B be closed subsets of X. Let $f, g : X \to X$ be two (A,B)-weakly increasing mappings with respect to $\preceq$. Suppose that*

(a) *$X = A \cup B$ is a cyclic representation of X w.r.t. the pair (f,g);*
(b) *there exist $0 < \delta < 1$, $L \ge 0$ and an altering distance function ψ such that for any comparable elements $x, y \in X$ with $x \in A$ and $y \in B$, we have*

$$\psi\big(s^2 d(fx,gy)\big) \le \delta\psi\big(M_s(x,y)\big) + L\psi\big(N(x,y)\big), \tag{3.1}$$

where $M_s(x,y)$ and $N(x,y)$ are given by (2.2) and (2.3), respectively;
(c) *f or g is continuous, or*
(c′) *the space $(X, \preceq, d)$ is regular.*

Then f and g have a common fixed point $u \in A \cap B$.

Taking $s = 1$ and $L = 0$ in Corollary 1, we obtain Theorems 2.1 and 2.2 of Shatanawi and Postolache [32] (Theorem 2 in this paper).

Taking $\psi(t) = t$ for $t \in [0, +\infty)$ in Corollary 1, we get the following.

Corollary 2 *Let $(X, \preceq, d)$ be a complete ordered b-metric space. Let A and B be nonempty closed subsets of X, and let $f, g : X \to X$ be two (A,B)-weakly increasing mappings with respect to $\preceq$ such that $f(A) \subseteq B$ and $g(B) \subseteq A$. Suppose that there exist $\delta \in (0,1)$ and $L \ge 0$ such that*

$$\begin{aligned}d(fx,gy) \le{}& \frac{\delta}{s^2}\max\left\{d(x,y), d(x,fx), d(y,gy), \frac{d(x,gy)+d(fx,y)}{2s}\right\}\\ &+ \frac{L}{s^2}\min\big\{d(y,gy), d(x,gy), d(y,fx)\big\}\end{aligned}$$

for all comparable elements $x, y \in X$ with $x \in A$ and $y \in B$. If either f or g is continuous, or the space $(X, \preceq, d)$ is regular, then f and g have a common fixed point.

Putting $f = g$ in Theorems 4 and 5, the following corollary is obtained which extends and improves Theorems 3 and 4 in [27].

Corollary 3 *Let $(X, \preceq, d)$ be a complete ordered b-metric space, and let A and B be closed subsets of X. Let $f : X \to X$ be a mapping such that f is non-decreasing with respect to $\preceq$. Assume the following*:

(a) *$A \cup B$ is a cyclic representation of X w.r.t. f, that is, $fA \subseteq B$, $fB \subseteq A$;*
(b) *there exist two altering distance functions ψ, φ, and $L \geq 0$ such that*

$$\psi\left(s^2 d(fx, fy)\right) \leq \psi\left(M_s(x, y)\right) - \varphi\left(M_s(x, y)\right) + L\psi\left(N(x, y)\right) \tag{3.2}$$

for all comparable $x, y \in X$ with $x \in A$ and $y \in B$, where

$$M_s(x, y) = \max\left\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2s}\right\}$$

and

$$N(x, y) = \min\left\{d(x, fx), d(x, fy), d(y, fx)\right\}.$$

(c) *f is continuous, or*
(c′) *the space $(X, \preceq, d)$ is regular.*

If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Again, taking $\varphi = (1 - \delta)\psi$, $0 < \delta < 1$ in Corollary 3, we get the following.

Corollary 4 *Let $(X, \preceq, d)$ be a complete ordered b-metric space, let and A and B be closed subsets of X. Let $f : X \to X$ be a non-decreasing map with respect to $\preceq$. Suppose that*

(a) *$X = A \cup B$ is a cyclic representation of X w.r.t. f;*
(b) *there exist $0 < \delta < 1$, $L \geq 0$ and an altering distance function ψ such that for any comparable elements $x, y \in X$ with $x \in A$ and $y \in B$, we have*

$$\psi\left(s^2 d(fx, fy)\right) \leq \delta\psi\left(M_s(x, y)\right) + L\psi\left(N(x, y)\right), \tag{3.3}$$

where $M_s(x, y)$ and $N(x, y)$ are given in Corollary 3;
(c) *f is continuous, or*
(c′) *the space $(X, \preceq, d)$ is regular.*

Then f has a fixed point $u \in A \cap B$.

Remark 1 (Common) fixed points of the given mappings in Theorems 4 and 5 and Corollaries 3 and 4 need not be unique (see further Example 4). However, it is easy to show that they must be unique in the case that the respective sets of (common) fixed points are well ordered (recall that a subset W of a partially ordered set is said to be well ordered if every two elements of W are comparable).

We illustrate our results with the following two examples.

Example 3 Consider the b-metric space (X,d) given in Example 2, ordered by natural ordering and a mapping $f : X \to X$ given as

$$fn = \begin{cases} 8n, & \text{if } n \in \mathbb{N}, \\ \infty, & \text{if } n = \infty. \end{cases}$$

If $A = \{n : n \in \mathbb{N}\} \cup \{\infty\}$ and $B = \{8n : n \in \mathbb{N}\} \cup \{\infty\}$, then $A \cup B$ is a cyclic representation of X with respect to f. Take $\psi : [0,+\infty) \to [0,+\infty)$ given as $\psi(t) = \sqrt{t}$, $\delta = 5/4\sqrt{2}$ (< 1) and $L \geq 0$ arbitrary. In order to check the contractive condition (3.3), consider the following cases.

If $x, y \in \mathbb{N}$, then

$$\begin{aligned} \psi\bigl(s^2 d(fx,fy)\bigr) &= \psi\left(\left(\frac{5}{2}\right)^2 d(8x,8y)\right) = \sqrt{\frac{5^2}{2^2 \cdot 8}\left|\frac{1}{x} - \frac{1}{y}\right|} \\ &\leq \frac{5}{4\sqrt{2}}\psi\bigl(d(x,y)\bigr) \leq \delta\psi\bigl(M_s(x,y)\bigr) + L\psi\bigl(N(x,y)\bigr) \end{aligned}$$

and (3.3) holds. If $x = \infty$ and y is an even integer, then

$$\begin{aligned} \psi\bigl(s^2 d(fx,fy)\bigr) &= \psi\left(\left(\frac{5}{2}\right)^2 d(\infty,8y)\right) = \sqrt{\frac{5^2}{2^2 \cdot 8} \cdot \frac{1}{y}} \\ &\leq \frac{5}{4\sqrt{2}}\psi\bigl(d(x,y)\bigr) \leq \delta\psi\bigl(M_s(x,y)\bigr) + L\psi\bigl(N(x,y)\bigr). \end{aligned}$$

Finally, if $x = \infty$ and y is an odd integer, then $d(x,y) = 5$ and (3.3) trivially holds.

Hence, all the conditions of Corollary 4 are satisfied. Obviously, f has a (unique) fixed point ∞, belonging to $A \cap B$.

We now present an example showing that there are situations where our results can be used to conclude about the existence of (common) fixed points, while some other known results cannot be applied.

Example 4 Let $X = \{0,1,2,3,4\}$ be equipped with the following partial order:

$$\preceq := \bigl\{(0,0),(1,1),(1,2),(2,2),(3,2),(3,3),(4,2),(4,4)\bigr\}.$$

Define a b-metric $d : X \times X \to \mathbb{R}^+$ by

$$d(x,y) = \begin{cases} 0, & \text{if } x = y, \\ (x+y)^2, & \text{if } x \neq y. \end{cases}$$

It is easy to see that (X,d) is a b-complete b-metric space with $s = 49/25$. Set $A = \{0,1,2,3,4\}$ and $B = \{0,2\}$, and define self-maps f and g by

$$f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 2 & 2 \end{pmatrix}, \qquad g = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 2 & 4 & 3 \end{pmatrix}.$$

It is easy to see that f and g are (A,B)-weakly increasing mappings with respect to $\preceq$, and that f and g are continuous. Also, $A \cup B = X$, $f(A) \subseteq B$ and $g(B) \subseteq A$.

Define $\psi : [0,\infty) \to [0,\infty)$ by $\psi(t) = \sqrt{t}$. One can easily check that the pair (f,g) satisfies the requirements of Corollary 1, with any δ and $L \geq 0$, as the left-hand side of the contractive condition (3.1) is equal to 0 for all comparable x, y such that $x \in A$ and $y \in B$. Hence, f and g have a common fixed point. Indeed, 0 and 2 are two common fixed points of f and g. (Note that the ordered set $(\{0,2\},\preceq)$ is not well ordered).

However, take $x = 1 \in A$ and $y = 0 \in B$ (which are not comparable). Then

$$\begin{aligned}\psi\bigl(s^2 d(f1,g0)\bigr) &= \sqrt{s^2(2+0)^2} = 2s > 3 \\ &> 3\delta + L\cdot 0 = \delta\psi\bigl(M_s(1,0)\bigr) + L\psi\bigl(N(1,0)\bigr),\end{aligned}$$

where $0 < \delta < 1$ and $L \geq 0$ are arbitrary, since

$$M_s(1,0) = \max\left\{d(1,0), d(1,2), d(0,0), \frac{d(1,0)+d(0,2)}{2s}\right\} = 3^2$$

and

$$N(1,0) = \min\bigl\{d(0,0), d(1,0), d(0,2)\bigr\} = 0.$$

Hence, this result cannot be applied in the context of b-metric spaces without order.

4 Application to existence of solutions of integral equations

Integral equations like (4.1) have been studied in many papers (see, *e.g.*, [22, 33] and the references therein). In this section, we look for a nonnegative solution to (4.1) in $X = C([0,T],\mathbb{R})$.

Consider the integral equation

$$u(t) = \int_0^T G(t,s) f\bigl(s,u(s)\bigr)\,ds \quad \text{for all } t \in [0,T], \tag{4.1}$$

where $T > 0$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $G : [0,T] \times [0,T] \to [0,\infty)$ are continuous functions.

Let $X = C([0,T])$ be the set of real continuous functions on $[0,T]$. We endow X with the b-metric

$$\mathcal{D}(u,v) = \max_{t\in[0,T]} \bigl(u(t) - v(t)\bigr)^2 \quad \text{for all } u,v \in X.$$

Clearly, $(X,\mathcal{D})$ is a complete b-metric space (with the parameter $s' = 2$). We endow X with the partial order $\preceq$ given by

$$x \preceq y \quad\Longleftrightarrow\quad x(t) \leq y(t) \quad \text{for all } t \in [0,T].$$

Clearly, the space $(X,\preceq,\mathcal{D})$ is regular.

Let $\alpha, \beta \in X$ and $\alpha_0, \beta_0 \in \mathbb{R}$ such that

$$\alpha_0 \leq \alpha(t) \leq \beta(t) \leq \beta_0 \quad \text{for all } t \in [0,T]. \tag{4.2}$$

Assume that for all $t \in [0, T]$, we have

$$\alpha(t) \le \int_0^T G(t,s) f\bigl(s, \beta(s)\bigr)\, ds \tag{4.3}$$

and

$$\beta(t) \ge \int_0^T G(t,s) f\bigl(s, \alpha(s)\bigr)\, ds. \tag{4.4}$$

Let for all $s \in [0, T]$, $f(s, \cdot)$ be a decreasing function, that is,

$$x, y \in \mathbb{R}, \quad x \ge y \quad \Longrightarrow \quad f(s,x) \le f(s,y). \tag{4.5}$$

Assume that $\gamma > 0$ is such that

$$4\gamma \left(\max_{t \in [0,T]} \int_0^T G(t,s)\, ds \right)^2 < 1. \tag{4.6}$$

Define a mapping $\mathcal{T} : X \to X$ by

$$\mathcal{T}u(t) = \int_0^T G(t,s) f\bigl(s, u(s)\bigr)\, ds \quad \text{for all } t \in [0, T].$$

Suppose that for all $s \in [0, T]$ and for all comparable $x, y \in X$ with ($x(s) \le \beta_0$ and $y(s) \ge \alpha_0$) or ($x(s) \ge \alpha_0$ and $y(s) \le \beta_0$),

$$\begin{aligned} 0 &\le f\bigl(s, x(s)\bigr) - f\bigl(s, y(s)\bigr) \\ &\le \biggl(\gamma \max \biggl\{ \bigl| x(s) - y(s) \bigr|^2, \bigl| x(s) - \mathcal{T}x(s) \bigr|^2, \bigl| y(s) - \mathcal{T}y(s) \bigr|^2, \\ &\quad \frac{|x(s) - \mathcal{T}y(s)|^2 + |y(s) - \mathcal{T}x(s)|^2}{4} \biggr\} \biggr)^{\frac{1}{2}}. \end{aligned} \tag{4.7}$$

Theorem 6 *Under the assumptions* (4.2)-(4.7), *the integral equation* (4.1) *has a solution in the set* $\{u \in C([0, T]) : \alpha \preceq u \preceq \beta\}$.

Proof Define closed subsets of X, A_1 and A_2 by

$$A_1 = \{u \in X : u \preceq \beta\} \quad \text{and} \quad A_2 = \{u \in X : u \succeq \alpha\}.$$

Consider the mapping $\mathcal{T} : X \to X$ defined above. We will prove that

$$\mathcal{T}(A_1) \subseteq A_2 \quad \text{and} \quad \mathcal{T}(A_2) \subseteq A_1. \tag{4.8}$$

Suppose that $u \in A_1$, that is,

$$u(s) \preceq \beta(s) \quad \text{for all } s \in [0, T].$$

Applying, condition (4.5), since $G(t,s) \geq 0$ for all $t, s \in [0, T]$, we obtain that

$$G(t,s)f\bigl(s,u(s)\bigr) \geq G(t,s)f\bigl(s,\beta(s)\bigr) \quad \text{for all } t,s \in [0,T].$$

The above inequality with condition (4.3) implies that

$$\int_0^T G(t,s)f\bigl(s,u(s)\bigr)\,ds \geq \int_0^T G(t,s)f\bigl(s,\beta(s)\bigr)\,ds \geq \alpha(t)$$

for all $t \in [0, T]$. Thus, we have $\mathcal{T}u \in A_2$.

Similarly, let $u \in A_2$, that is,

$$u(s) \geq \alpha(s) \quad \text{for all } s \in [0,T].$$

Using condition (4.5), since $G(t,s) \geq 0$ for all $t, s \in [0, T]$, we obtain that

$$G(t,s)f\bigl(s,u(s)\bigr) \leq G(t,s)f\bigl(s,\alpha(s)\bigr) \quad \text{for all } t,s \in [0,T].$$

The above inequality with condition (4.4) implies that

$$\int_0^T G(t,s)f\bigl(s,u(s)\bigr)\,ds \leq \int_0^T G(t,s)f\bigl(s,\alpha(s)\bigr)\,ds \leq \beta(t)$$

for all $t \in [0, T]$. Hence, we have $\mathcal{T}u \in A_1$. Thus, (4.8) holds.

Now, let $(u, v) \in A_1 \times A_2$, that is, for all $t \in [0, T]$,

$$u(t) \leq \beta(t), \qquad v(t) \geq \alpha(t).$$

This implies from condition (4.2) that for all $t \in [0, T]$,

$$u(t) \leq \beta_0, \qquad v(t) \geq \alpha_0.$$

Also, if $x \preceq y$, then by (4.7), we have

$$\mathcal{T}y(t) - \mathcal{T}x(t) = \int_0^T G(t,s)\bigl[f\bigl(s,y(s)\bigr) - f\bigl(s,x(s)\bigr)\bigr]\,ds \geq 0$$

for all $t \in [0, T]$. That is, $\mathcal{T}x \preceq \mathcal{T}y$. Hence, $\mathcal{T}$ is increasing.

Now, by the conditions (4.6) and (4.7), we have for all $t \in [0, T]$ and for all comparable $x \in A_1$ and $y \in A_2$,

$$\begin{aligned}
&\bigl(\mathcal{T}x(t) - \mathcal{T}y(t)\bigr)^2 \\
&\quad = \left(\int_0^T G(t,s)\bigl[f\bigl(s,x(s)\bigr) - f\bigl(s,y(s)\bigr)\bigr]\,ds\right)^2 \\
&\quad \leq \left(\int_0^T G(t,s)\bigl[f\bigl(s,x(s)\bigr) - f\bigl(s,y(s)\bigr)\bigr]\,ds\right)^2 \\
&\quad \leq \left(\int_0^T G(t,s)\left(\gamma \max\left\{\bigl|x(s) - y(s)\bigr|^2, \bigl|x(s) - \mathcal{T}x(s)\bigr|^2, \bigl|y(s) - \mathcal{T}y(s)\bigr|^2,\right.\right.\right.
\end{aligned}$$

$$\frac{|x(s)-\mathcal{T}y(s)|^2+|y(s)-\mathcal{T}x(s)|^2}{4}\bigg\}\bigg)^{\frac{1}{2}}\,ds\bigg)^2$$

$$\begin{aligned}
&\le \left(\int_0^T G(t,s)\left(\gamma \max\left\{\max_{s\in[0,T]}\left|x(s)-y(s)\right|^2, \max_{s\in[0,T]}\left|x(s)-\mathcal{T}x(s)\right|^2,\right.\right.\right.\\
&\quad \max_{s\in[0,T]}\left|y(s)-\mathcal{T}y(s)\right|^2,\\
&\quad \left.\left.\left.\frac{\max_{s\in[0,T]}\left|x(s)-\mathcal{T}y(s)\right|^2+\max_{s\in[0,T]}\left|y(s)-\mathcal{T}x(s)\right|^2}{4}\right\}\right)^{\frac{1}{2}}ds\right)^2\\
&= \gamma\left(\int_0^T G(t,s)\,ds\right)^2 \max\left\{\mathcal{D}(x,y),\mathcal{D}(x,\mathcal{T}x),\mathcal{D}(y,\mathcal{T}y),\frac{\mathcal{D}(x,\mathcal{T}y)+\mathcal{D}(y,\mathcal{T}x)}{2s'}\right\},
\end{aligned}$$

which implies that

$$\mathcal{D}(\mathcal{T}x,\mathcal{T}y)\le \frac{\delta}{2s'}\max\left\{\mathcal{D}(x,y),\mathcal{D}(x,\mathcal{T}x),\mathcal{D}(y,\mathcal{T}y),\frac{\mathcal{D}(x,\mathcal{T}y)+\mathcal{D}(y,\mathcal{T}x)}{2s'}\right\}$$

with $\delta = 4\gamma(\max_{t\in[0,T]}\int_0^T G(t,s)\,ds)^2 < 1$.

Now, all the conditions of Corollary 2 (with $\mathcal{T}=g=f$ and $L=0$) hold, and $\mathcal{T}$ has a fixed point z in

$$A_1\cap A_2=\left\{u\in C\big([0,T]\big):\alpha(t)\le u(t)\le \beta(t),\ \text{for all } t\in[0,T]\right\}.$$

That is, $z\in A_1\cap A_2$ is the solution to (4.1). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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