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Iterative approximation methods for mixed equilibrium problems for a countable family of quasi- ϕ -asymptotically nonexpansive multivalued mappings in Banach spaces

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Abstract

In this paper, we prove the existence of a solution of the mixed equilibrium problem $MEP(f, \varphi, C)$ by using the KKM mapping in a Banach space setting. Then, by virtue of this result, we introduce a hybrid iterative scheme for finding a common element of the set of solutions of $MEP(f, \varphi, C)$ and the set of common fixed points of a countable family of quasi- ϕ -asymptotically nonexpansive multivalued mappings. Furthermore, we prove that the sequences generated by the hybrid iterative scheme converge strongly to a common element of the set of solutions of $MEP(f, \varphi, C)$ and the set of common fixed points of a countable family of quasi- ϕ -asymptotically nonexpansive multivalued mappings.

Keywords: mixed equilibrium problems; shrinking projection method; KKM mappings; multivalued mappings

1 Introduction

Let E be a Banach space with the norm $\|\cdot\|$ and the dual E^* . We denote by \mathbb{N} and \mathbb{R} the sets of positive integers and real numbers, respectively. Also, we denote the normalized duality mapping from E to 2^{E^*} by J defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. Let C be a nonempty, closed and convex subset of a Banach space E . The bifunction $f : C \times C \rightarrow \mathbb{R}$ is said to be relaxed ξ -monotone if there exists a function $\xi : E \rightarrow \mathbb{R}$ positively homogeneous of degree p , that is, $\xi(tz) = t^p \xi(z)$ for all $t > 0$ and $z \in E$, where $p > 1$ is a constant such that

$$f(x, y) + f(y, x) \leq \xi(y - x) \quad \text{for all } x, y \in C. \quad (1.1)$$

If $\xi = 0$, then f is monotone, i.e.,

$$f(x, y) + f(y, x) \leq 0 \quad \text{for all } x, y \in C.$$

Example 1.1 Let $E = \mathbb{R}$ and $C = [0, 1]$. A bifunction $f : C \times C \rightarrow \mathbb{R}$ is defined by

$$f(x, y) = y(y - x), \quad \forall x, y \in [0, 1].$$

It is easy to see that $f(x, y) + f(y, x) = (x - y)^2 \geq 0$. If we put $\xi(x) = x^2$ for all $x \in \mathbb{R}$, then f is relaxed ξ -monotone, but not monotone.

Let us consider the bifunction $f : C \times C \rightarrow \mathbb{R}$ and a function $\varphi : C \rightarrow \mathbb{R}$. The mixed equilibrium problem (in short, $MEP(f, \varphi, C)$) is to find $\tilde{x} \in C$ such that

$$f(\tilde{x}, y) + \varphi(y) - \varphi(\tilde{x}) \geq 0, \quad \forall y \in C. \tag{1.2}$$

Problem (1.2) was studied by Ceng and Yao [1], Chulamjiak and Suantai [2] in Hilbert spaces and Banach spaces, resp. By using the well-known KKM technique, they gave the existence and uniqueness of solutions of the $MEP(f, \varphi, C)$ when f is monotone.

In the case of $f(x, y) = \langle Ax, \eta(y, x) \rangle$, where $A : C \rightarrow E^*$ and $\eta : C \times C \rightarrow E$, problem (1.2) is reduced to the following variational-like inequality problem (in short, $VLIP(A, \varphi, C)$), which is to find $\tilde{x} \in C$ such that

$$\langle A(\tilde{x}), \eta(y, \tilde{x}) \rangle + \varphi(y) - \varphi(\tilde{x}) \geq 0, \quad \forall y \in C. \tag{1.3}$$

Problem (1.3) was studied by Fang and Huang [3]. By using the KKM technique and η - ξ monotonicity of the mapping A , they obtained the existence of solutions of the variational-like inequality problem (1.3) in a real reflexive Banach space.

In particular, if $\varphi \equiv 0$, problem (1.2) is reduced to the well-known equilibrium problem (in short, $EP(f, C)$), which is to find $\tilde{x} \in C$ such that

$$f(\tilde{x}, y) \geq 0, \quad \forall y \in C. \tag{1.4}$$

The $EP(f, C)$ includes fixed point problems, optimization problems, variational inequality problems and Nash equilibrium problems as special cases (see also [4–6]).

Let S be a mapping from C into itself. A mapping S is said to be Lipschitz if there exists a constant $L > 0$ such that $\|Sx - Sy\| \leq L\|x - y\|$ for all $x, y \in C$. S is said to be nonexpansive, when $L = 1$. A point $p \in C$ is said to be a fixed point of S if $Sp = p$. A point $p \in C$ is said to be an asymptotic fixed point of S if there exists a sequence $\{x_n\} \subset C$ such that $x_n \rightarrow x \in E$ and $\lim_{n \rightarrow \infty} \|x_n - Sx_n\| = 0$. Denote the set of all fixed points of S and the set of all asymptotic fixed points of S by $F(S)$ and $\hat{F}(S)$, respectively.

Now, we assume that E is a smooth, strictly convex and reflexive Banach space. The Lyapunov function $\phi : E \times E \rightarrow \mathbb{R}^+$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E.$$

A mapping $S : C \rightarrow C$ is said to be relatively nonexpansive (see also [7–9]) if the following conditions are satisfied:

- (1) $F(S)$ is nonempty;
- (2) $\phi(p, Sx) \leq \phi(p, x)$ for all $x \in C, p \in F(S)$;
- (3) $\hat{F}(S) = F(S)$.

Lemma 1.2 [8] *Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E , and let T be a relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

In 2008, Takahashi and Zembayashi [10] introduced the following iterative scheme which is called the shrinking projection method:

$$\begin{cases} x_0 \in C, & C_0 = C, \\ y_n = J^{-1}[\alpha_n Jx_0 + (1 - \alpha_n)JSx_n], \\ u_n \in C \text{ such that} \\ f(u_n, y) + \frac{1}{r_n}(y - u_n, Ju_n - Jw_n) \geq 0, & \forall y \in C, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \phi(v, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, & n \geq 0, \end{cases} \quad (1.5)$$

where J is the duality mapping on E , Π_C is the generalized projection from E onto C and S is relatively nonexpansive single-valued mapping. They proved that the sequence $\{x_n\}$ converges strongly to a common element of the set of fixed points of a relatively nonexpansive single-valued mapping and the set of solutions of an equilibrium problem under appropriate conditions in a uniformly smooth and uniformly convex Banach space.

In the case of single-valued mapping, many authors such as in references [8, 9, 11–19] studied the problem of finding a common element of the set of fixed points of a nonexpansive or relatively nonexpansive mapping and the set of solutions of an equilibrium problem in the framework of Hilbert spaces and Banach spaces.

Let $N(C)$ and $CB(C)$ denote the family of nonempty subsets and nonempty closed bounded subsets of C , respectively. The Hausdorff metric on $CB(C)$ is defined by

$$H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\}$$

for all $A, B \in CB(C)$, where $d(x, D) = \inf\{\|x - y\|, y \in D\}$. Homaeipour and Razani [20] have defined the relatively nonexpansive multivalued mapping as follows.

A multivalued mapping $S : C \rightarrow N(C)$ is said to be relatively nonexpansive if the following conditions are satisfied:

- (1) $F(S)$ is nonempty;
- (2) $\phi(p, z) \leq \phi(p, x)$ for all $x \in C, p \in F(S), z \in Sx$;
- (3) $\hat{F}(S) = F(S)$.

It is obvious that the class of relatively nonexpansive multivalued mappings contains properly the class of relatively nonexpansive multivalued mappings (for example, see [20]). They introduced the following iterative sequence for finding a fixed point of the relatively nonexpansive multivalued mapping $S : C \rightarrow N(C)$:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \Pi_C(\alpha_n J(x_n) + (1 - \alpha_n)J(z_n)), & z_n \in Sx_n, \forall n \geq 1, \end{cases} \quad (1.6)$$

where $\{\alpha_n\} \subset [0, 1]$. Moreover, they proved weak and strong convergence theorems for a single relatively nonexpansive multivalued mapping in a uniformly convex and uniformly smooth Banach space E .

A multivalued mapping $S : C \rightarrow CB(C)$ is said to be quasi- ϕ -nonexpansive [21, 22] if $F(S) \neq \emptyset$ and

$$\phi(p, z) \leq \phi(p, x), \quad \forall x \in C, p \in F(S), z \in Sx.$$

$S : C \rightarrow CB(C)$ is said to be quasi- ϕ -asymptotically nonexpansive [21, 22] if $F(S) \neq \emptyset$ and there exists a real sequence $l_n \subset [1, +\infty)$ with $l_n \rightarrow 1$ such that

$$\phi(p, z_n) \leq l_n \phi(p, x), \quad \forall x \in C, p \in F(S), z_n \in S^n x.$$

We remark from the above definitions that the class of quasi- ϕ -asymptotically nonexpansive multivalued mappings contains properly the class of quasi- ϕ -nonexpansive multivalued mappings as a subclass, and the class of quasi- ϕ -nonexpansive multivalued mappings contains properly the class of relatively nonexpansive mappings as a subclass.

Recently, Yi [21] introduced the following modifying Halpern iterative sequence for a quasi- ϕ -asymptotically nonexpansive multivalued mapping S . Let $\{x_n\}$ be the sequence generated by

$$\begin{cases} x_0 \in C, & C_0 = C, \\ y_n = J^{-1}[\alpha_n Jx_0 + (1 - \alpha_n)Jz_n], & z_n \in S^n x_n, \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \alpha_n \phi(v, x_0) + (1 - \alpha_n)\phi(v, x_n) + \zeta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, & n \geq 0, \end{cases} \tag{1.7}$$

where $\zeta_n = (l_n - 1) \sup_{p \in F(S)} \phi(p, x_n)$ and $\{\alpha_n\} \subset (0, 1)$. Under suitable conditions, the author proved that $\{x_n\}$ converges strongly to $\Pi_{F(S)} x_0$.

Motivated and inspired by the above results, we investigate the problem of finding a common element of the set of solutions of $MEP(f, \phi, C)$ and the set of common fixed points of a countable family of quasi- ϕ -asymptotically nonexpansive multivalued mappings in Banach spaces.

The rest of the paper is organized as follows. In Section 2, we provide necessary concepts and lemmas. In Section 3, we derive the existence and uniqueness of solutions of the auxiliary problems for the $MEP(f, \phi, C)$ when the bifunction f is relaxed ξ -monotone by using the well-known KKM technique. In Section 4, we prove that our proposed hybrid iterative scheme converges strongly to a common element of the set of $MEP(f, \phi, C)$ and the set of common fixed points of a countable family of quasi- ϕ -asymptotically nonexpansive multivalued mappings. In Section 5, we give an application for a system of mixed equilibrium problems. The last section is the conclusions.

2 Preliminaries

Throughout this paper, let E be a real Banach space and E^* be its dual space. We write $x_n \rightharpoonup x$ (respectively $x_n \rightharpoonup^* x$) to indicate that the sequence $\{x_n\}$ weakly (respectively weak*) converges to x ; as usual $x_n \rightarrow x$ will symbolize strong convergence. Let $S(E) = \{x \in E : \|x\| = 1\}$ denote the unit sphere of a Banach space E .

A Banach space E is said to have a Gâteaux differentiable norm (we also say that E is smooth) if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in S(E)$. A Banach space E is said to have a uniformly Gâteaux differentiable norm if for each y in $S(E)$, the limit (2.1) is uniformly attained for $x \in S(E)$; a Fréchet differentiable norm if for each $x \in S(E)$, the limit (2.1) is attained uniformly for $y \in S(E)$; a uniformly Fréchet differentiable norm (we also say that E is uniformly smooth) if the limit (2.1) is attained uniformly for $(x, y) \in S(E) \times S(E)$.

A Banach space E is said to strictly convex if $\frac{\|x+y\|}{2} < 1$ for $x, y \in S(E)$, $x \neq y$; uniformly convex if, for any $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in S(E)$, $\|x - y\| \geq \varepsilon$ implies $\|\frac{x+y}{2}\| \leq 1 - \delta$.

Remark 2.1 The following basic properties for the Banach space E and for the normalized duality mapping J can be found in Cioranescu [23].

- (i) E (E^* , resp.) is uniformly convex if and only if E (E^* , resp.) is uniformly smooth.
- (ii) If E is smooth, then J is single-valued and norm-to-weak continuous.
- (iii) If E is reflexive, then J is onto.
- (iv) If E is strictly convex, then $Jx \cap Jy \neq \emptyset$ for all $x, y \in E$.
- (v) Each uniformly convex Banach space E has the Kadec-Klee property, that is, for any sequence $\{x_n\} \subset E$, if $x_n \rightarrow x \in E$ and $\|x_n\| \rightarrow \|x\|$, then $x_n \rightarrow x \in E$.
- (vi) If E is a strictly convex reflexive Banach space, then J^{-1} is hemicontinuous, that is, J^{-1} is norm-to-weak*-continuous.

Recall the Lyapunov function ϕ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E.$$

It follows from the definition of the function ϕ that

$$\phi(x, J^{-1}(\lambda Jy + (1 - \lambda)Jz)) \leq \lambda\phi(x, y) + (1 - \lambda)\phi(x, z), \quad \forall \lambda \in [0, 1] \text{ and } x, y, z \in E. \quad (2.2)$$

Following Alber [24], the generalized projection Π_C from E onto C is defined by

$$\Pi_C x = \arg \min_{y \in C} \phi(y, x), \quad \forall x \in E.$$

If E is a Hilbert space, then $\phi(y, x) = \|y - x\|^2$ and Π_C is the metric projection of H onto C . We know the following lemma for generalized projection.

Lemma 2.2 [24] *Let E be a smooth, strictly convex and reflexive Banach space, and let C be a nonempty, closed and convex subset of E . Then the following conclusions hold:*

- (i) $\phi(x, y) = 0$ if and only if $x = y$ for all $x, y \in E$;
- (ii) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y)$ for all $x \in C$, for all $y \in E$;
- (iii) if $x \in E$ and $z \in C$, then $z = \Pi_C x \Leftrightarrow \langle z - y, Jx - Jz \rangle \geq 0$ for all $y \in C$.

Now, let us recall the following useful concepts and results.

Definition 2.3 Let B be a subset of a topological vector space X . A mapping $G : B \rightarrow 2^X$ is called a KKM mapping if $\text{co}\{x_1, x_2, \dots, x_m\} \subset \bigcup_{i=1}^m G(x_i)$ for $x_i \in B$ and $i = 1, 2, \dots, m$, where $\text{co}A$ denotes the convex hull of the set A .

Lemma 2.4 [25] *Let B be a nonempty subset of a Hausdorff topological vector space X , and let $G : B \rightarrow 2^X$ be a KKM mapping. If $G(x)$ is closed for all $x \in B$ and is compact for at least one $x \in B$, then $\bigcap_{x \in B} G(x) \neq \emptyset$.*

Definition 2.5 A multivalued mapping $S : C \rightarrow CB(C)$ is said to be closed if for any sequence $\{x_n\} \subset C$ with $x_n \rightarrow x \in C$ and $d(y, S(x_n)) \rightarrow 0$, $d(y, S(x)) = 0$.

The following lemmas can be found in [21].

Lemma 2.6 [21, Lemma 2.8] *Let E be a real uniformly smooth and strictly convex Banach space with the Kadec-Klee property, and let C be a nonempty, closed and convex subset of E . Let $\{x_n\}$ and $\{y_n\}$ be two sequences in C such that $x_n \rightarrow p$ and $\phi(x_n, y_n) \rightarrow 0$, then $y_n \rightarrow p$.*

Lemma 2.7 [21, Lemma 2.9] *Let E be a real uniformly smooth and strictly convex Banach space with the Kadec-Klee property, and let C be a nonempty, closed and convex subset of E . Let $S : C \rightarrow CB(C)$ be a closed and quasi- ϕ -asymptotically nonexpansive multivalued mapping with nonnegative real sequences $\{l_n\} \subset [1, \infty)$, if $l_n \rightarrow 1$, then the fixed point set $F(S)$ of S is a closed and convex subset of C .*

3 Existence results of mixed equilibrium problem

In this chapter, we prove the existence theorem for $MEP(f, \varphi, C)$ by using the KKM technique.

Before solving mixed equilibrium problem (1.2), let us assume the following conditions for a bifunction $f : C \times C \rightarrow \mathbb{R}$:

- (A1) $f(x, x) = 0$ for all $x \in C$;
- (A2) f is relaxed ξ -monotone, i.e.,

$$f(x, y) + f(y, x) \leq \xi(y - x);$$

- (A3) for all $y \in C$, $f(\cdot, y)$ is upper hemicontinuous, i.e., for all $x, z \in C$,

$$\limsup_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y);$$

- (A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower-semicontinuous.

Lemma 3.1 *Let C be a nonempty, closed and convex subset of a real smooth, strictly convex and reflexive Banach space E , let f be a bifunction from $C \times C$ to \mathbb{R} satisfying (A1)-(A4), and let φ be a lower semicontinuous and convex function from C to \mathbb{R} . Let $z \in C$. Then the following problems (3.1) and (3.2) are equivalent:*

$$\begin{aligned} \text{Find } \tilde{x} \in C \text{ such that } f(\tilde{x}, y) + \varphi(y) + \frac{1}{2} \|y\|^2 - \frac{1}{2} \|\tilde{x}\|^2 - \langle y - \tilde{x}, Jz \rangle \geq \varphi(\tilde{x}), \\ \forall y \in C. \end{aligned} \tag{3.1}$$

$$\begin{aligned} \text{Find } \tilde{x} \in C \text{ such that } \varphi(y) + \frac{1}{2} \|y\|^2 - \frac{1}{2} \|\tilde{x}\|^2 - \langle y - \tilde{x}, Jz \rangle + \xi(y - x) \geq f(y, \tilde{x}) + \varphi(\tilde{x}), \\ \forall y \in C. \end{aligned} \tag{3.2}$$

Proof Let $\tilde{x} \in C$ be a solution of problem (3.1). It follows from (A2) that

$$\begin{aligned} 0 &\leq f(\tilde{x}, y) + \varphi(y) - \varphi(\tilde{x}) + \frac{1}{2} \|y\|^2 - \frac{1}{2} \|\tilde{x}\|^2 - \langle y - \tilde{x}, Jz \rangle \\ &= f(\tilde{x}, y) + f(y, \tilde{x}) - f(y, \tilde{x}) + \varphi(y) - \varphi(\tilde{x}) + \frac{1}{2} \|y\|^2 - \frac{1}{2} \|\tilde{x}\|^2 - \langle y - \tilde{x}, Jz \rangle \\ &\leq \xi(y - \tilde{x}) - f(y, \tilde{x}) + \varphi(y) - \varphi(\tilde{x}) + \frac{1}{2} \|y\|^2 - \frac{1}{2} \|\tilde{x}\|^2 - \langle y - \tilde{x}, Jz \rangle, \quad \forall y \in C. \end{aligned} \tag{3.3}$$

Thus $\tilde{x} \in C$ is a solution of problem (3.2).

Conversely, let $\tilde{x} \in C$ be a solution of problem (3.2). For any $t \in (0, 1]$, we put

$$y_t = (1 - t)\tilde{x} + ty. \tag{3.4}$$

Then $y_t \in C$, because of the convexity of C . Since $\tilde{x} \in C$ is a solution of problem (3.2), it follows that

$$\begin{aligned} f(y_t, \tilde{x}) &\leq \varphi(y_t) - \varphi(\tilde{x}) + \frac{1}{2} \|y_t\|^2 - \frac{1}{2} \|\tilde{x}\|^2 - \langle y_t - \tilde{x}, Jz \rangle + \xi(y_t - \tilde{x}) \\ &= \varphi(y_t) - \varphi(\tilde{x}) + \frac{1}{2} \|y_t\|^2 - \frac{1}{2} \|\tilde{x}\|^2 - \langle y_t - \tilde{x}, Jz \rangle + t^p \xi(y - \tilde{x}). \end{aligned} \tag{3.5}$$

Using (A1) and (A4), we have

$$0 = f(y_t, y_t) \leq (1 - t)f(y_t, \tilde{x}) + tf(y_t, y),$$

and so

$$t(f(y_t, \tilde{x}) - f(y_t, y)) \leq f(y_t, \tilde{x}). \tag{3.6}$$

The convexity of the function φ implies that

$$\varphi(y_t) = \varphi((1 - t)\tilde{x} + ty) \leq (1 - t)\varphi(\tilde{x}) + t\varphi(y). \tag{3.7}$$

It follows from (3.5)-(3.7) and the convexity of $\frac{1}{2} \|\cdot\|^2$ that

$$\begin{aligned} 0 &= f(y_t, y_t) \\ &\leq (1 - t)f(y_t, \tilde{x}) + tf(y_t, y) \\ &\leq (1 - t) \left[\varphi(y_t) - \varphi(\tilde{x}) + \frac{1}{2} \|y_t\|^2 - \frac{1}{2} \|\tilde{x}\|^2 - \langle y_t - \tilde{x}, Jz \rangle + t^p \xi(y - \tilde{x}) \right] + tf(y_t, y) \\ &\leq tf(y_t, y) + (1 - t) \left[(1 - t)\varphi(\tilde{x}) + t\varphi(y) - \varphi(\tilde{x}) + \frac{(1 - t)}{2} \|\tilde{x}\|^2 + \frac{t}{2} \|y\|^2 \right. \\ &\quad \left. - \frac{1}{2} \|\tilde{x}\|^2 - t \langle y - \tilde{x}, Jz \rangle + t^p \xi(y - \tilde{x}) \right] \\ &\leq tf(y_t, y) + (1 - t)t \left[\varphi(y) - \varphi(\tilde{x}) + \frac{1}{2} \|y\|^2 - \frac{1}{2} \|\tilde{x}\|^2 - \langle y - \tilde{x}, Jz \rangle + t^{p-1} \xi(y - \tilde{x}) \right]. \end{aligned} \tag{3.8}$$

This implies that

$$0 \leq f(y_t, y) + (1 - t) \left[\varphi(y) - \varphi(\tilde{x}) + \frac{1}{2} \|y\|^2 - \frac{1}{2} \|\tilde{x}\|^2 - \langle y - \tilde{x}, Jz \rangle + t^{p-1} \xi(y - \tilde{x}) \right]. \quad (3.9)$$

Taking the upper limit $t \rightarrow 0$ in (3.9), by (A3) and $p > 1$, we get that

$$\varphi(\tilde{x}) \leq f(\tilde{x}, y) + \varphi(y) + \frac{1}{2} \|y\|^2 - \frac{1}{2} \|\tilde{x}\|^2 - \langle y - \tilde{x}, Jz \rangle, \quad \forall y \in C.$$

Therefore, \tilde{x} is also a solution of problem (3.1). This completes the proof. □

Lemma 3.2 *Let C be a nonempty, bounded, closed and convex subset of a real smooth, strictly convex and reflexive Banach space E , let f be a bifunction from $C \times C$ to \mathbb{R} satisfying (A1) and (A4), and let φ be a lower semicontinuous and convex function from C to \mathbb{R} . Let $z \in C$. Assume that*

- (i) $\xi : E \rightarrow \mathbb{R}$ is weakly upper semicontinuous; that is, for any net $\{x_\beta\}$, x_β converges to x in $\sigma(E, E^*)$, which implies that $\xi(x) \leq \liminf \xi(x_\beta)$.

Then the solution set of problem (3.1) is nonempty; that is, there exists $\tilde{x} \in C$ such that

$$f(\tilde{x}, y) + \varphi(y) + \frac{1}{2} \|y\|^2 - \frac{1}{2} \|\tilde{x}\|^2 - \langle y - \tilde{x}, Jz \rangle \geq \varphi(\tilde{x}), \quad \forall y \in C. \quad (3.10)$$

Proof Let $z \in C$. Define two set-valued mappings $F_z, G_z : C \rightarrow 2^C$ as follows:

$$F_z(y) = \left\{ x \in C : f(x, y) + \varphi(y) + \frac{1}{2} \|y\|^2 - \frac{1}{2} \|x\|^2 - \langle y - x, Jz \rangle \geq \varphi(x) \right\}$$

and

$$G_z(y) = \left\{ x \in C : \varphi(y) + \frac{1}{2} \|y\|^2 - \frac{1}{2} \|x\|^2 - \langle y - x, Jz \rangle + \xi(y - x) \geq f(y, x) + \varphi(x) \right\}$$

for every $y \in C$. It is easily seen that $y \in F_z(y)$ and $y \in G_z(y)$, and hence $F_z(y)$ and $G_z(y)$ are nonempty.

(a) We claim that F_z is a KKM mapping. If F_z is not a KKM mapping, then there exist $\{y_1, \dots, y_n\} \subset C$ and $\mu_i > 0, i = 1, \dots, n$, such that

$$\sum_{i=1}^n \mu_i = 1, \quad y_0 = \sum_{i=1}^n \mu_i y_i \notin \bigcup_{i=1}^n F_z(y_i). \quad (3.11)$$

By the definition of F_z , we have

$$f(y_0, y_i) + \varphi(y_i) - \varphi(y_0) + \frac{1}{2} \|y_i\|^2 - \frac{1}{2} \|y_0\|^2 - \langle y_i - y_0, Jz \rangle < 0 \quad (3.12)$$

for all $i = 1, \dots, n$. It follows from (A1), (A4), the convexity of φ and $\frac{1}{2} \|\cdot\|^2$ that

$$\begin{aligned} 0 &= f(y_0, y_0) + \varphi(y_0) - \varphi(y_0) + \frac{1}{2} \|y_0\|^2 - \frac{1}{2} \|y_0\|^2 - \langle y_0 - y_0, Jz \rangle \\ &\leq \sum_{i=1}^n \mu_i \left(f(y_0, y_i) + \varphi(y_i) - \varphi(y_0) + \frac{1}{2} \|y_i\|^2 - \frac{1}{2} \|y_0\|^2 - \langle y_i - y_0, Jz \rangle \right) < 0, \end{aligned} \quad (3.13)$$

which is a contradiction. This implies that F_z is a KKM mapping.

(b) We claim that G_z is a KKM mapping. It is sufficient to show that

$$F_z(y) \subset G_z(y), \quad \forall y \in C.$$

For any given $y \in C$, taking $x \in F_z(y)$, then

$$f(x, y) + \varphi(y) - \varphi(x) + \frac{1}{2} \|y\|^2 - \frac{1}{2} \|x\|^2 - \langle y - x, Jz \rangle \geq 0. \tag{3.14}$$

It follows from the relaxed ξ -monotonicity of f that

$$\varphi(y) + \frac{1}{2} \|y\|^2 - \frac{1}{2} \|x\|^2 - \langle y - x, Jz \rangle + \xi(y - x) \geq f(y, x) + \varphi(x).$$

It follows that $x \in G_z(y)$ and so

$$F_z(y) \subset G_z(y), \quad \forall y \in C.$$

This implies that G_z is also a KKM mapping.

(c) We show that $G_z(y)$ is weakly closed for all $y \in C$. Let $\{u_n\}$ be a sequence in $G_z(y)$ such that $u_n \rightharpoonup u$ as $n \rightarrow \infty$. It then follows from $u_n \in G_z(y)$ that

$$\varphi(y) + \frac{1}{2} \|y\|^2 - \frac{1}{2} \|u_n\|^2 - \langle y - u_n, Jz \rangle + \xi(y - u_n) \geq f(y, u_n) + \varphi(u_n), \quad \forall y \in C. \tag{3.15}$$

By (A4), the weak lower semicontinuity of φ and $\|\cdot\|^2$, and the weak upper semicontinuity of ξ , we obtain from (3.15) that

$$\begin{aligned} f(y, u) + \varphi(u) &\leq \liminf_{n \rightarrow \infty} f(y, u_n) + \liminf_{n \rightarrow \infty} \varphi(u_n) \\ &\leq \liminf_{n \rightarrow \infty} \{f(y, u_n) + \varphi(u_n)\} \\ &\leq \limsup_{n \rightarrow \infty} \left\{ \varphi(y) + \frac{1}{2} \|y\|^2 - \frac{1}{2} \|u_n\|^2 - \langle y - u_n, Jz \rangle + \xi(y - u_n) \right\} \\ &\leq \varphi(y) + \frac{1}{2} \|y\|^2 - \liminf_{n \rightarrow \infty} \frac{1}{2} \|u_n\|^2 - \liminf_{n \rightarrow \infty} \langle y - u_n, Jz \rangle \\ &\quad + \limsup_{n \rightarrow \infty} \xi(y - u_n) \\ &= \varphi(y) + \frac{1}{2} \|y\|^2 - \frac{1}{2} \|u\|^2 - \langle y - u, Jz \rangle + \xi(y - u). \end{aligned}$$

This shows that $u \in G_z(y)$ and hence $G_z(y)$ is weakly closed for all $y \in C$.

(d) We prove that $G_z(y)$ is weakly compact. Since C is a closed, bounded and convex subset of a reflexive Banach space E , it is weakly compact. Again, since $G_z(y)$ is a weakly closed subset of C , we also have $G_z(y)$ is weakly compact.

By using (a)-(d) and Lemma 2.4 and Lemma 3.1 that

$$\bigcap_{y \in C} F_z(y) = \bigcap_{y \in C} G_z(y) \neq \emptyset.$$

Hence, there exists $x_0 \in C$ satisfying inequality (3.10). This completes the proof. □

Proposition 3.3 *Let C be a nonempty, closed and convex subset of a real smooth, strictly convex and reflexive Banach space E , $\psi : C \rightarrow \mathbb{R}$ be a convex function. The following two inequalities are equivalent:*

$$\text{there exists } \tilde{x} \in C \text{ such that } \psi(\tilde{x}) = 0 \text{ and } \psi(y) + \langle y - \tilde{x}, J\tilde{x} \rangle \geq 0, \quad \forall y \in C \quad (3.16)$$

and

$$\text{there exists } \tilde{x} \in C \text{ such that } \psi(\tilde{x}) = 0 \text{ and } \psi(y) + \frac{1}{2}\|y\|^2 - \frac{1}{2}\|\tilde{x}\|^2 \geq 0, \quad \forall y \in C. \quad (3.17)$$

Proof Let \tilde{x} satisfy (3.16). It is well known that

$$\langle y - x, Jx \rangle \leq \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x\|^2, \quad \forall x, y \in E.$$

Then

$$\begin{aligned} 0 &\leq \psi(y) + \langle y - \tilde{x}, J\tilde{x} \rangle \\ &= \psi(y) + \langle y - \tilde{x}, J\tilde{x} \rangle \\ &\leq \psi(y) + \frac{1}{2}\|y\|^2 - \frac{1}{2}\|\tilde{x}\|^2. \end{aligned}$$

Let \tilde{x} satisfy (3.17). For any $t \in (0, 1]$, let $y_t = (1 - t)\tilde{x} + ty$. Then $y_t \in C$ because of the convexity of C , and so

$$\psi(y_t) + \frac{1}{2}\|y_t\|^2 - \frac{1}{2}\|\tilde{x}\|^2 - \langle y_t - \tilde{x}, Jz \rangle \geq 0.$$

Notice that in a real smooth, strictly convex reflexive Banach space E , the duality mapping J is single-valued, 1-1, and onto. Since $\frac{1}{2}\|\cdot\|^2$ is continuous and Gâteaux differentiable, from the mean value theorem, there exists $\alpha_t \in (0, 1)$ such that

$$\frac{1}{2}\|y_t\|^2 - \frac{1}{2}\|\tilde{x}\|^2 = \langle y_t - \tilde{x}, Jw_{\alpha_t} \rangle,$$

where $w_{\alpha_t} = \alpha_t\tilde{x} + (1 - \alpha_t)y_t$. Hence

$$\begin{aligned} 0 &\leq \psi(y) + \frac{1}{2}\|y\|^2 - \frac{1}{2}\|\tilde{x}\|^2 \\ &= \psi(y_t) + \langle y_t - \tilde{x}, Jw_{\alpha_t} \rangle \\ &\leq (1 - t)\psi(\tilde{x}) + t\psi(y) + t\langle y - \tilde{x}, Jw_{\alpha_t} \rangle \\ &= t\psi(y) + t\langle y - \tilde{x}, Jw_{\alpha_t} \rangle. \end{aligned}$$

Dividing t in the above inequality, we get

$$0 \leq \psi(y) + \langle y - \tilde{x}, Jw_{\alpha_t} \rangle.$$

By the existence of w_{α_t} , $w_{\alpha_t} \rightarrow \tilde{x}$ as $t \rightarrow 0$. Since J is norm-to-weak* continuous, we have that

$$0 \leq \psi(y) + \langle y - \tilde{x}, J\tilde{x} \rangle.$$

This completes the proof. □

Remark 3.4 If $\tilde{x} \in C$ is a solution of (3.1), then

$$f(\tilde{x}, y) + \varphi(y) + \frac{1}{2}\|y\|^2 - \frac{1}{2}\|\tilde{x}\|^2 - \langle y - \tilde{x}, Jz \rangle \geq \varphi(\tilde{x}), \quad \forall y \in C.$$

For any fixed $z \in E$, we put $\psi_{\tilde{x}}(y) = f(\tilde{x}, y) + \varphi(y) - \varphi(\tilde{x}) - \langle y - \tilde{x}, Jz \rangle$ for all $y \in C$. Obviously, $\psi_{\tilde{x}}(\tilde{x}) = 0$ and $\psi_{\tilde{x}}$ is a convex function, since the linearity of a duality mapping and the convexity of $f(\tilde{x}, \cdot)$ and φ . Hence

$$\psi_{\tilde{x}}(y) + \frac{1}{2}\|y\|^2 - \frac{1}{2}\|\tilde{x}\|^2 \geq 0, \quad \forall y \in C.$$

It then follows from Proposition 3.3 that \tilde{x} is a solution of the following problem:

$$f(\tilde{x}, y) + \varphi(y) + \langle y - \tilde{x}, J\tilde{x} - Jz \rangle \geq \varphi(\tilde{x}), \quad \forall y \in C.$$

The existence result relaxes the results of Ceng and Yao [1] and Cholamjiak and Suantai [2], because of the ξ -monotonicity of f .

Lemma 3.5 *Let C be a closed, bounded and convex subset of a uniformly smooth, strictly convex Banach space E , let f be a bifunction from $C \times C$ to \mathbb{R} satisfying (A1)-(A4), and let φ be a lower semicontinuous and convex function from C to \mathbb{R} . Suppose further that*

- (i) $\xi : E \rightarrow \mathbb{R}$ is weakly upper semicontinuous;
- (ii) $\xi(y - x) + \xi(x - y) \leq 0$.

Define a mapping $T_r(z) : E \rightarrow C$ as follows:

$$T_r(z) = \left\{ x \in C : f(x, y) + \varphi(y) + \frac{1}{r}\langle y - x, Jx - Jz \rangle \geq \varphi(x), \forall y \in C \right\}. \quad (3.18)$$

Then

- (1) T_r is single-valued;
- (2) T_r is a firmly nonexpansive mapping, i.e., for all $x, y \in E$,

$$\langle T_r(x) - T_r(y), JT_r(x) - JT_r(y) \rangle \leq \langle T_r(x) - T_r(y), Jx - Jy \rangle;$$

- (3) $F(T_r) = \text{MEP}(f, \varphi, C)$;
- (4) $\text{MEP}(f, \varphi, C)$ is closed and convex;
- (5) $\phi(p, T_r x) + \phi(T_r x, x) \leq \phi(p, x)$.

Proof If $f(x, y) : C \times C \rightarrow \mathbb{R}$ satisfies (A1)-(A4) and $\varphi : C \rightarrow \mathbb{R}$ is convex lower semicontinuous, then for any $r > 0$, we know that $rf(x, y)$ and $r\varphi$ are also. We have from Lemma 3.2 and Remark 3.4 that $T_r z$ is nonempty for all $z \in E$.

(1) For each $z \in E$, let $x_1, x_2 \in T_r(z)$. Then

$$f(x_1, x_2) + \varphi(x_2) + \frac{1}{r} \langle x_2 - x_1, Jx_1 - Jz \rangle \geq \varphi(x_1) \tag{3.19}$$

and

$$f(x_2, x_1) + \varphi(x_1) + \frac{1}{r} \langle x_1 - x_2, Jx_2 - Jz \rangle \geq \varphi(x_2). \tag{3.20}$$

Adding (3.19) and (3.20), we have

$$f(x_2, x_1) + f(x_1, x_2) + \frac{1}{r} \langle x_1 - x_2, Jx_2 - Jz \rangle + \frac{1}{r} \langle x_2 - x_1, Jx_1 - Jz \rangle \geq 0,$$

and so

$$f(x_2, x_1) + f(x_1, x_2) + \frac{1}{r} \langle x_2 - x_1, Jx_1 - Jx_2 \rangle \geq 0. \tag{3.21}$$

By the relaxed ξ -monotonicity of f , we know that

$$\langle x_2 - x_1, Jx_1 - Jx_2 \rangle \geq -r\xi(y - x). \tag{3.22}$$

In (3.21) exchanging the position of x_1 and x_2 , we get

$$f(x_1, x_2) + f(x_2, x_1) + \frac{1}{r} \langle x_1 - x_2, Jx_2 - Jx_1 \rangle \geq 0, \tag{3.23}$$

and so

$$\langle x_1 - x_2, Jx_2 - Jx_1 \rangle \geq -r\xi(y - x). \tag{3.24}$$

Now, adding inequalities (3.22) and (3.24), by using (ii), we have

$$\langle x_2 - x_1, Jx_1 - Jx_2 \rangle \geq -\frac{r}{2} (\xi(y - x) + \xi(y - x)) \geq 0.$$

Since J is monotone and E is strictly convex, we have $x_1 = x_2$, and so $T_r(z)$ is single-valued.

(2) For $x, y \in C$, we have that

$$f(T_r x, T_r y) + \varphi(T_r y) - \varphi(T_r x) + \frac{1}{r} \langle T_r y - T_r x, J T_r x - Jx \rangle \geq 0$$

and

$$f(T_r y, T_r x) + \varphi(T_r x) - \varphi(T_r y) + \frac{1}{r} \langle T_r x - T_r y, J T_r y - Jy \rangle \geq 0.$$

Adding the two inequalities above, we get that

$$f(T_r x, T_r y) + f(T_r y, T_r x) + \frac{1}{r} \langle T_r y - T_r x, J T_r x - J T_r y - Jx + Jy \rangle \geq 0.$$

From (A2), we obtain that

$$\xi(T_r y - T_r x) + \frac{1}{r} \langle T_r y - T_r x, JT_r x - JT_r y - Jx + Jy \rangle \geq 0. \tag{3.25}$$

In (3.25), interchanging the position of $T_r x$ and $T_r y$, we get

$$\xi(T_r x - T_r y) + \frac{1}{r} \langle T_r x - T_r y, JT_r y - JT_r x - Jy + Jx \rangle \geq 0. \tag{3.26}$$

Again adding (3.24) and (3.25), we get

$$\xi(T_r y - T_r x) + \xi(T_r x - T_r y) + \frac{2}{r} \langle T_r y - T_r x, JT_r x - JT_r y - Jx + Jy \rangle \geq 0.$$

It follows from (ii) and $r > 0$ that

$$\langle T_r x - T_r y, JT_r x - JT_r y \rangle \leq \langle T_r y - T_r x, Jx - Jy \rangle.$$

(3) Indeed, we have the following:

$$\begin{aligned} p \in F(T_r) &\Leftrightarrow p = T_r p \\ &\Leftrightarrow f(p, y) + \varphi(y) - \varphi(p) + \frac{1}{r} \langle y - p, Jp - Jp \rangle \geq 0, \quad \forall y \in C \\ &\Leftrightarrow f(p, y) + \varphi(y) - \varphi(p) \geq 0, \quad \forall y \in C \\ &\Leftrightarrow p \in MEP(f, \varphi, C). \end{aligned}$$

(4) We claim that $MEP(f, \varphi, C)$ is closed and convex. Indeed, from (3) we have $MEP(f, \varphi, C) = F(T_r)$. From (2) we have, for all $x, y \in C$,

$$\langle T_r(x) - T_r(y), JT_r(x) - JT_r(y) \rangle \leq \langle T_r(x) - T_r(y), Jx - Jy \rangle.$$

Moreover, we have

$$\begin{aligned} \phi(T_r x, T_r y) + \phi(T_r y, T_r x) &= 2\|T_r x\|^2 - \langle T_r x, JT_r y \rangle - \langle T_r y, JT_r x \rangle + 2\|T_r y\|^2 \\ &= 2\langle T_r x, JT_r x - JT_r y \rangle + 2\langle T_r y, T_r y - JT_r x \rangle \\ &= 2\langle T_r x - T_r y, JT_r x - JT_r y \rangle \end{aligned}$$

and

$$\begin{aligned} &\phi(T_r x, y) + \phi(T_r y, x) - \phi(T_r x, x) - \phi(T_r y, y) \\ &= \|T_r x\|^2 - 2\langle T_r x, Jy \rangle + \|y\|^2 + \|T_r y\|^2 - \langle T_r y, Jx \rangle + \|x\|^2 \\ &= -\|T_r x\|^2 + 2\langle T_r x, Jx \rangle - \|x\|^2 - \|T_r y\|^2 + \langle T_r y, Jy \rangle - \|y\|^2 \\ &= 2\langle T_r x, Jx - Jy \rangle - 2\langle T_r y, Jx - Jy \rangle \\ &= 2\langle T_r x - T_r y, Jx - Jy \rangle. \end{aligned}$$

Hence, we have

$$\phi(T_r x, T_r y) + \phi(T_r y, T_r x) \leq \phi(T_r x, y) + \phi(T_r y, x) - \phi(T_r x, x) - \phi(T_r y, y). \tag{3.27}$$

So, we have, for any $x, y \in C$,

$$\phi(T_r x, T_r y) + \phi(T_r y, T_r x) \leq \phi(T_r x, y) + \phi(T_r y, x).$$

Next, we show that $\hat{F}(T_r) = MEP(f, \varphi, C)$. Let $p \in \hat{F}(T_r)$. Then there exists $\{u_n\} \subset E$ such that $z_n \rightharpoonup p$ and $\lim_{n \rightarrow \infty} (u_n - T_r u_n) = 0$. Moreover, we get that $T_r u_n \rightharpoonup p$. Since the duality mapping J is uniformly continuous on a bounded set, we get

$$\lim_{n \rightarrow \infty} \frac{1}{r} \|Ju_n - JT_r u_n\| = 0.$$

From the definition of T_r , we have

$$f(T_r u_n, y) + \frac{1}{r} \langle y - T_r u_n, JT_r u_n - Ju_n \rangle \geq \varphi(T_r u_n).$$

Since

$$\begin{aligned} & \frac{1}{r} \langle y - T_r u_n, JT_r u_n - Ju_n \rangle \\ & \geq \varphi(T_r u_n) - \varphi(y) - f(T_r u_n, y) \\ & \geq \varphi(T_r u_n) - \varphi(y) + f(y, T_r u_n) - \xi(y - T_r u_n). \end{aligned}$$

By (A4), the convexity and lower semicontinuity of $\varphi(x)$ and the weak upper semicontinuity of ξ , we can obtain that

$$\begin{aligned} 0 & \geq \liminf_{n \rightarrow \infty} \{ \varphi(T_r u_n) - \varphi(y) + f(y, T_r u_n) - \xi(y - T_r u_n) \} \\ & = \liminf_{n \rightarrow \infty} \varphi(T_r u_n) - \varphi(y) + \liminf_{n \rightarrow \infty} f(y, T_r u_n) - \limsup_{n \rightarrow \infty} \xi(y - T_r u_n) \\ & \geq \varphi(p) - \varphi(y) + f(y, p) - \xi(y - p). \end{aligned}$$

This implies that

$$\varphi(p) - \varphi(y) + f(y, p) - \xi(y - p) \leq 0, \quad \forall y \in C. \tag{3.28}$$

Let $y \in K$ and set $p_t = ty + (1 - t)p$ for $t \in (0, 1]$. It follows from (A1) and (A4) that

$$\begin{aligned} 0 & = f(p_t, p_t) \\ & \leq (1 - t)f(p_t, p) + tf(p_t, y). \end{aligned} \tag{3.29}$$

The convexity of the function φ implies that

$$\varphi(p_t) = \varphi((1 - t)p + ty) \leq (1 - t)\varphi(p) + t\varphi(y). \tag{3.30}$$

It follows from (3.28)-(3.30) that

$$0 = f(p_t, p_t) \tag{3.31}$$

$$\begin{aligned} &\leq (1-t)f(p_t, p) + tf(p_t, y) \\ &\leq tf(p_t, y) + (1-t)[\varphi(p_t) - \varphi(p) + t^p \xi(y-p)] \\ &\leq tf(p_t, y) + (1-t)[(1-t)\varphi(p) + t\varphi(y) - \varphi(p) + t^p \xi(y-p)] \\ &= tf(p_t, y) + (1-t)t[\varphi(y) - \varphi(p) + t^{p-1} \xi(y-p)]. \end{aligned} \tag{3.32}$$

Dividing by t , we have

$$0 \leq f(p_t, y) + (1-t)[\varphi(y) - \varphi(p) + t^{p-1} \xi(y-p)]. \tag{3.33}$$

By (A3) and $p > 1$, taking the upper limit $t \rightarrow 0$ in (3.33), we get

$$f(p, y) + \varphi(y) \geq \varphi(p), \quad \forall y \in C.$$

Hence, $p \in MEP(f, \varphi, C)$, and so $F(T_r) = MEP(f, \varphi, C) = \hat{F}(T_r)$. Therefore, we have T_r is a relatively nonexpansive mapping. From Lemma 1.2, $MEP(f, \varphi, C) = F(T_r)$ is closed and convex.

(5) From (3.27) we have, for all $x, y \in E$,

$$\phi(T_r x, T_r y) + \phi(T_r y, T_r x) \leq \phi(T_r x, y) + \phi(T_r y, x) - \phi(T_r x, x) - \phi(T_r y, y).$$

Letting $y = p \in F(T_r)$, we have

$$\phi(p, T_r x) + \phi(T_r x, x) \leq \phi(p, x).$$

This completes the proof. □

4 Strong convergence theorems

Before proving the convergence theorem, we recall some definitions of a countable family of multivalued mappings $\{S_i : C \rightarrow CB(C)\}_{i=1}^\infty$.

Definition 4.1 $\{S_i\}_{i=1}^\infty$ is said to be a family of uniformly quasi- ϕ -asymptotically nonexpansive mappings [15] if $\mathcal{F} := \bigcap_{i=1}^\infty F(S_i) \neq \emptyset$ and there exists a sequence $\{l_n\} \subset [1, \infty)$ with $l_n \rightarrow 1$ such that for each $i \geq 1$,

$$\phi(p, z_{n,i}) \leq l_n \phi(p, x), \quad \forall p \in \mathcal{F}, z_{n,i} \in S_i^n x, \forall n \geq 1.$$

Definition 4.2 A mapping $S : C \rightarrow CB(C)$ is said to be uniformly L -Lipschitz continuous [15] if there exists a constant $L > 0$ such that $\|x_n - y_n\| \leq L\|x - y\|$ for all $x, y \in C, x_n \in T^n x, y_n \in T^n y$.

Theorem 4.3 Let E be a real uniformly smooth and strictly convex Banach space with the Kadec-Klee property, and let C be a nonempty, bounded, closed and convex subset of E . Let

f be a bifunction from $C \times C$ to \mathbb{R} satisfying (A1)-(A4), and let φ be a lower semicontinuous and convex function from C to \mathbb{R} . Let $\{S_i : C \rightarrow CB(C)\}_{i=1}^\infty$ be a family of closed and uniformly quasi- ϕ -asymptotically nonexpansive multivalued mappings with a sequence $\{l_n\} \subset [1, \infty)$, $l_n \rightarrow 1$. Suppose that for each $i \geq 1$, S_i is uniformly L_i -Lipschitz continuous and $\mathcal{F} \cap MEP(f, \varphi, C) \neq \emptyset$.

Let $\{x_n\}$ be a sequence in C generated by

$$\begin{cases} x_0 \in C, & C_0 = C, \\ w_n = J^{-1}(\alpha_{n,0}Jx_0 + \sum_{i=1}^\infty \alpha_{n,i}Jz_{n,i}), & z_{n,i} \in S_i^n x_n, \\ u_n \in K & \text{such that} \\ f(u_n, y) + \varphi(y) + \frac{1}{r_n} \langle y - u_n, Ju_n - Jw_n \rangle \geq \varphi(u_n), & \forall y \in C, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \alpha_{n,0}\phi(v, x_0) + (1 - \alpha_{n,0})\phi(v, x_n) + \zeta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, & n \geq 0, \end{cases} \tag{4.1}$$

where for each $\{r_n\}_{n=0}^\infty \subset [a, \infty)$ for some $a > 0$, $\{\alpha_{n,i}\}_{n=0,i=0}^\infty \subset [0, 1]$, and $\zeta_n = \sup_{u \in \mathcal{F}} (l_n - 1)\phi(u, x_n)$. If $\sum_{i=0}^\infty \alpha_{n,i} = 1$, $\forall n \geq 0$, $\lim_{n \rightarrow \infty} \alpha_{n,0} = 0$, then $\{x_n\}$ converges strongly as $n \rightarrow \infty$ to $\Pi_\Omega x_0$, where $\Omega := \mathcal{F} \cap MEP(f, \varphi, C)$.

Proof We divide the proof into five steps. Firstly, we rewrite algorithm (4.1) as follows:

$$\begin{cases} x_0 \in C, & C_0 = C, \\ w_n = J^{-1}(\alpha_{n,0}Jx_0 + \sum_{i=1}^\infty \alpha_{n,i}Jz_{n,i}), & z_{n,i} \in S_i^n x_n, \\ C_{n+1} = \{v \in C_n : \phi(v, T_{r_n} w_n) \leq \alpha_{n,0}\phi(v, x_0) + (1 - \alpha_{n,0})\phi(v, x_n) + \zeta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, & n \geq 0, \end{cases} \tag{4.2}$$

where T_r is the mapping defined by (3.18) for all $r > 0$.

Step 1. We first show that the sequence $\{x_n\}$ is well defined. It suffices to prove that C_n is closed and convex and that $\Omega \subset C_n$ for all $n \geq 0$. Suppose that C_n is closed and convex for some $n \geq 0$. By the definition of ϕ , we have

$$\begin{aligned} C_{n+1} &= \{v \in C_n : \phi(v, u_n) \leq \alpha_{n,0}\phi(v, x_0) + (1 - \alpha_{n,0})\phi(v, x_n) + \zeta_n\} \\ &= \{v \in C : \phi(v, u_n) \leq \alpha_{n,0}\phi(v, x_0) + (1 - \alpha_{n,0})\phi(v, x_n) + \zeta_n\} \cap C_n \\ &= \{v \in C : 2\alpha_{n,0}\langle v, Jx_0 \rangle + 2(1 - \alpha_{n,0})\langle v, Jx_n \rangle - 2\langle v, Ju_n \rangle \\ &\quad \leq \alpha_{n,0}\|x_0\|^2 + (1 - \alpha_{n,0})\|x_n\|^2 - \|y_n\|^2 + \zeta_n\} \cap C_n. \end{aligned}$$

This shows that C_{n+1} is closed and convex. The conclusions are proved.

Step 2. Next, we prove that $\Omega \subset C_n$ for all $n \geq 0$. In fact, it is obvious that $\Omega \subset C_0 := C$. Suppose $\Omega \subset C_n$ for some $n \geq 0$. Hence, for any $p \in \Omega$, it follows from Lemma 3.5(5) that

$$\begin{aligned} \phi(p, u_n) &= \phi(p, T_{r_n} w_n) \leq \phi(p, w_n) \\ &= \phi\left(p, J^{-1}\left(\alpha_{n,0}Jx_0 + \sum_{i=1}^\infty \alpha_{n,i}Jz_{n,i}\right)\right) \\ &\leq \alpha_{n,0}\phi(p, x_0) + \sum_{i=1}^\infty \alpha_{n,i}\phi(p, z_{n,i}) \end{aligned}$$

$$\begin{aligned}
 &\leq \alpha_{n,0}\phi(p, x_0) + (1 - \alpha_{n,0})l_n\phi(p, x_n) \\
 &\leq \alpha_{n,0}\phi(p, x_0) + (1 - \alpha_{n,0})\{\phi(p, x_n) + (l_n - 1)\phi(p, x_n)\} \\
 &\leq \alpha_{n,0}\phi(p, x_0) + (1 - \alpha_{n,0})\left\{\phi(p, x_n) + (l_n - 1)\sup_{p \in \mathcal{F}}\phi(p, x_n)\right\} \\
 &= \alpha_{n,0}\phi(p, x_0) + (1 - \alpha_{n,0})\phi(p, x_n) + \zeta_n.
 \end{aligned} \tag{4.3}$$

This shows that $p \in \Omega \subset C_{n+1}$. Hence $\Omega \subset C_n$ for all $n \geq 0$.

Step 3. We show that $\{x_n\}$ converges strongly to some point \tilde{x} . Since $x_n = \Pi_{C_n}x_0$, from Lemma 2.2(iii), we have that

$$\langle x_n - y, Jx_0 - Jx_n \rangle \geq 0, \quad \forall y \in C_n.$$

Also, since $\mathcal{F} \subset C_n$, we have

$$\langle x_n - p, Jx_0 - Jx_n \rangle \geq 0, \quad \forall p \in \mathcal{F}.$$

It follows from Lemma 2.2(ii) that for each $p \in \mathcal{F}$ and for each $n \geq 1$,

$$\phi(x_n, x_0) = \phi(\Pi_{C_n}x_0, x_0) \leq \phi(p, x_0) - \phi(p, x_n) \leq \phi(p, x_0).$$

Therefore, $\{\phi(x_n, x_0)\}$ is bounded, and so is $\{x_n\}$. Since $x_n = \Pi_{C_n}x_0$ and $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we have

$$\phi(x_n, p) \leq \phi(x_{n+1}, p), \quad \forall n \geq 0.$$

That is, $\{\phi(x_n, p)\}$ is a nondecreasing sequence, and so $\lim_{n \rightarrow \infty} \phi(x_n, p)$ exists. Since E is reflexive, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup \tilde{x} \in C$. Since C_n is closed, convex and $C_{n+1} \subset C_n$, we get that C_n is weakly closed and $\tilde{x} \in C_n$ for all $n \geq 0$. Since $x_{n_i} = \Pi_{C_{n_i}}x_0$, we have

$$\phi(x_{n_i}, x_0) \leq \phi(\tilde{x}, x_0), \quad \forall n_i \geq 0.$$

By the weak lower semicontinuity of $\|\cdot\|$, we have

$$\begin{aligned}
 \liminf_{n_i \rightarrow \infty} \phi(x_{n_i}, x_0) &= \liminf_{n_i \rightarrow \infty} (\|x_{n_i}\|^2 - 2\langle x_{n_i}, Jx_0 \rangle + \|x_0\|^2) \\
 &\geq \|\tilde{x}\|^2 - 2\langle \tilde{x}, Jx_0 \rangle + \|x_0\|^2 \\
 &= \phi(\tilde{x}, x_0),
 \end{aligned}$$

and so

$$\phi(\tilde{x}, x_0) \leq \liminf_{i \rightarrow \infty} \phi(x_{n_i}, x_0) \leq \limsup_{i \rightarrow \infty} \phi(x_{n_i}, x_0) \leq \phi(\tilde{x}, x_0).$$

Therefore $\lim_{i \rightarrow \infty} \phi(x_{n_i}, x_0) = \phi(\tilde{x}, x_0)$, and so $\|x_{n_i}\| \rightarrow \|\tilde{x}\|$. Since E has the Kadec-Klee property and $x_{n_i} \rightharpoonup \tilde{x}$, we obtain that $x_{n_i} \rightarrow \tilde{x}$. Since the limit of $\{\phi(x_n, x_0)\}$ exists, this

together with $\lim_{n \rightarrow \infty} \phi(x_{n_i}, x_0) = \phi(\tilde{x}, x_0)$ implies that $\lim_{n \rightarrow \infty} \phi(x_n, x_0) = \phi(\tilde{x}, x_0)$. If there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} \rightarrow \hat{x}$. Then by Lemma 2.2(ii) we have that

$$\begin{aligned} \phi(\tilde{x}, \hat{x}) &= \lim_{i,j \rightarrow \infty} \phi(x_{n_i}, x_{n_j}) = \lim_{i,j \rightarrow \infty} \phi(x_{n_i}, \Pi_{C_{n_j}} x_0) \\ &\leq \lim_{i,j \rightarrow \infty} \{ \phi(x_{n_i}, x_0) - \phi(\Pi_{C_{n_j}} x_0, x_0) \} = \lim_{i,j \rightarrow \infty} \{ \phi(x_{n_i}, x_0) - \phi(x_{n_j}, x_0) \} \\ &= \phi(\tilde{x}, x_0) - \phi(\tilde{x}, x_0). \end{aligned}$$

Hence $\tilde{x} = \hat{x}$, and so

$$\lim_{n \rightarrow \infty} x_n = \tilde{x}. \tag{4.4}$$

This implies that

$$\zeta_n = (I_n - 1) \sup_{p \in \mathcal{F}} \phi(p, x_n) \rightarrow 0. \tag{4.5}$$

Step 4. We prove that $\tilde{x} \in \mathcal{F}$. Since $x_n = \Pi_{C_n} x_0$, by Lemma 2.2(ii) we get that

$$\phi(x, x_n) \leq \phi(x, x_0) - \phi(x_n, x_0), \quad \forall x \in C_n.$$

Since $x_{n+1} \in C_{n+1} \subset C_n$, we get that

$$\phi(x_{n+1}, x_n) \leq \phi(x_{n+1}, x_0) - \phi(x_n, x_0).$$

Since $\lim_{n \rightarrow \infty} \phi(x_n, x_0) = \phi(\tilde{x}, x_0)$, we get that

$$\lim_{n \rightarrow \infty} \phi(x_{n+1}, x_n) = 0. \tag{4.6}$$

Since $x_{n+1} \in C_{n+1}$, we have from (4.1) that

$$\phi(x_{n+1}, u_n) \leq \alpha_{n,0} \phi(v, x_0) + (1 - \alpha_{n,0}) \phi(x_{n+1}, x_n) + \zeta_n. \tag{4.7}$$

By $\lim_{n \rightarrow \infty} \alpha_{n,0} = 0$, (4.5) and (4.6), it implies that

$$\lim_{n \rightarrow \infty} \phi(x_{n+1}, u_n) = 0, \tag{4.8}$$

which together with (4.4) and Lemma 2.6 give that

$$\lim_{n \rightarrow \infty} u_n = \tilde{x}. \tag{4.9}$$

Recall from (4.3) that

$$\phi(p, u_n) \leq \phi(p, w_n) \leq \alpha_{n,0} \phi(p, x_0) + (1 - \alpha_{n,0}) \phi(p, x_n) + \zeta_n.$$

It follows from $\lim_{n \rightarrow \infty} \alpha_{n,0} = 0$, (4.4), (4.5) and (4.9) that

$$\lim_{n \rightarrow \infty} \phi(p, w_n) = \phi(p, \tilde{x}).$$

By Lemma 3.5, we get that

$$\phi(u_n, w_n) \leq \phi(p, w_n) - \phi(p, u_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{4.10}$$

It follows from Lemma 2.6 and $\lim_{n \rightarrow \infty} u_n = \tilde{x}$ that

$$\lim_{n \rightarrow \infty} w_n = \tilde{x}. \tag{4.11}$$

From (4.1), we get that

$$\|Jw_n - Jz_{n,i}\| = \alpha_{n,0} \|Jx_0 - Jz_{n,i}\|, \quad \forall i \geq 1.$$

Since $\lim_{n \rightarrow \infty} \alpha_{n,0} = 0$ and the boundedness of $\{z_{n,i}\}$,

$$\lim_{n \rightarrow \infty} \|Jw_n - Jz_{n,i}\| = 0, \quad \forall i \geq 1. \tag{4.12}$$

Since $Jw_n \rightarrow J\tilde{x}$, we have from the uniform continuity of J that $Jz_{n,i} \rightarrow J\tilde{x}$ for all $i \geq 1$. Remark 2.1(vi) gives that

$$z_{n,i} \rightarrow \tilde{x}, \quad \forall i \geq 1. \tag{4.13}$$

Again, since

$$\| \|z_{n,i}\| - \|\tilde{x}\| \| = \| \|Jz_{n,i}\| - \|J\tilde{x}\| \| \leq \|Jz_{n,i} - J\tilde{x}\| \rightarrow 0 \quad \text{as } n \rightarrow \infty, \forall i \geq 1, \tag{4.14}$$

it follows from (4.13), (4.14) and the Kadec-Klee property of E that

$$\lim_{n \rightarrow \infty} z_{n,i} = \tilde{x}, \quad \forall i \geq 1. \tag{4.15}$$

For all $i \geq 1$, we consider

$$\begin{aligned} d(S_i z_{n,i}, z_{n,i}) &\leq d(S_i z_{n,i}, z_{n+1,i}) + \|z_{n+1,i} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - z_{n,i}\| \\ &\leq d(S_i^{n+1} x_n, S_i^{n+1} x_{n+1}) + \|z_{n+1,i} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - z_{n,i}\| \\ &\leq L_i \|x_n - x_{n+1}\| + \|z_{n+1,i} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - z_{n,i}\| \\ &= (L_i + 1) \|x_n - x_{n+1}\| + \|z_{n+1,i} - x_{n+1}\| + \|x_n - z_{n,i}\|. \end{aligned}$$

From (4.4) and (4.15), we get that $d(S_i z_{n,i}, z_{n,i}) \rightarrow 0$ as $n \rightarrow \infty$ for all $i \geq 1$. By (4.15) and the closedness of S_i , we have $\tilde{x} \in S_i \tilde{x}$ for all $i \geq 1$, and so $\tilde{x} \in \mathcal{F}$.

Step 5. We show that $\tilde{x} \in MEP(f, \varphi, C)$. Since $u_n = T_{r_n} w_n$, we derive

$$f(u_n, y) + \varphi(y) + \frac{1}{r_n} (y - u_n, Ju_n - Jw_n) \geq \varphi(u_n), \quad \forall y \in C.$$

From relaxed ξ -monotonicity of f , we have

$$\frac{1}{r_n} (y - u_n, Ju_n - Jw_n) + \varphi(y) - \varphi(u_n) \geq -f(u_n, y) \geq f(y, u_n) - \xi(y - u_n), \quad \forall y \in C.$$

Since $\frac{Ju_n - Jw_n}{r_n} \rightarrow 0$ and $u_n \rightarrow \tilde{x}$. By (A4), the lower semicontinuity of φ and the upper semicontinuity of ξ , we have

$$\begin{aligned} f(y, \tilde{x}) - \xi(y - \tilde{x}) &\leq \liminf_{n \rightarrow \infty} f(y, u_n) - \limsup_{n \rightarrow \infty} \xi(y - u_n) \\ &\leq \liminf_{n \rightarrow \infty} \{f(y, u_n) - \xi(y - u_n)\} \\ &\leq \liminf_{n \rightarrow \infty} \left\{ \frac{1}{r_n} \langle y - u_n, Ju_n - Jw_n \rangle + \varphi(y) - \varphi(u_n) \right\} \\ &= \varphi(y) - \limsup_{n \rightarrow \infty} \varphi(u_n) \\ &= \varphi(y) - \varphi(\tilde{x}), \quad \forall y \in C. \end{aligned}$$

For all $y \in C$ and $t \in (0, 1]$, let $y_t = ty + (1 - t)\tilde{x}$. Since C is convex, we have $y_t \in C$, and then

$$f(y_t, \tilde{x}) - \varphi(y_t) + \varphi(\tilde{x}) \leq \xi(y_t - \tilde{x}).$$

Using (A1) and (A4), we have

$$0 = f(y_t, y_t) \leq (1 - t)f(y_t, \tilde{x}) + tf(y_t, y). \tag{4.16}$$

The convexity of the function φ implies that

$$\varphi(y_t) = \varphi((1 - t)\tilde{x} + ty) \leq (1 - t)\varphi(\tilde{x}) + t\varphi(y). \tag{4.17}$$

This implies that

$$\begin{aligned} 0 &= f(y_t, y_t) \\ &\leq (1 - t)f(y_t, \tilde{x}) + tf(y_t, y) \\ &\leq tf(y_t, y) + (1 - t)[\varphi(y_t) - \varphi(\tilde{x}) + \xi(y_t - \tilde{x})] \\ &\leq tf(y_t, y) + (1 - t)t[\varphi(y) - \varphi(\tilde{x}) + t^{p-1}\xi(y - \tilde{x})]. \end{aligned}$$

This implies that

$$0 \leq f(y_t, y) + (1 - t)[\varphi(y) - \varphi(\tilde{x}) + t^{p-1}\xi(y - \tilde{x})]. \tag{4.18}$$

By (A3) and $p > 1$, taking the upper limit $t \rightarrow 0$ in (4.18), we get

$$f(\tilde{x}, y) + \varphi(y) - \varphi(\tilde{x}) \geq 0, \quad \forall y \in C.$$

Therefore, $\tilde{x} \in MEP(f, \varphi, C)$.

Step 6. Finally, we prove that $\tilde{x} = \Pi_{\Omega}x_0$. Let $w = \Pi_{\Omega}x_0$. Then $\phi(w, x_0) \leq \phi(\tilde{x}, x_0)$.

Since $w \in \Omega \subset C_n$, for all $n \geq 0$ and $x_n \in \Pi_{C_n}x_0$, we have

$$\phi(x_n, x_0) \leq \phi(w, x_0), \quad \forall n \geq 0.$$

Since $x_n \rightarrow \tilde{x}$, we get that

$$\begin{aligned} \phi(\tilde{x}, x_0) &= \|\tilde{x}\|^2 - 2\langle \tilde{x}, Jx_0 \rangle + \|x_0\|^2 \\ &= \lim_{n \rightarrow \infty} \{ \|x_n\|^2 - 2\langle x_n, Jx_0 \rangle + \|x_0\|^2 \} \\ &= \lim_{n \rightarrow \infty} \phi(x_n, x_0) \leq \phi(w, x_0). \end{aligned}$$

Hence $x_n \rightarrow \tilde{x} = \Pi_{\Omega}x_0$. □

In the case of $f \equiv 0$ and $\varphi \equiv 0$ in Theorem 4.3, we have the following corollary.

Corollary 4.4 *Let E be a real uniformly smooth and strictly convex Banach space with the Kadec-Klee property, and let C be a nonempty, bounded, closed and convex subset of E . Let $\{S_i : C \rightarrow CB(C)\}_{i=1}^{\infty}$ be a family of closed and uniformly quasi- ϕ -asymptotically non-expansive multivalued mappings with a sequence $\{l_n\} \subset [1, \infty)$, $l_n \rightarrow 1$. Suppose that for each $i \geq 1$, S_i is uniformly L_i -Lipschitz continuous and $F \neq \emptyset$. Let $\{x_n\}$ be a sequence in C generated by*

$$\begin{cases} x_0 \in C, & C_0 = C, \\ w_n = J^{-1}(\alpha_{n,0}Jx_0 + \sum_{i=1}^{\infty} \alpha_{n,i}Jz_{n,i}), & z_{n,i} \in S_i^n x_n, \\ C_{n+1} = \{v \in C_n : \phi(v, w_n) \leq \alpha_{n,0}\phi(v, x_0) + (1 - \alpha_{n,0})\phi(v, x_n) + \zeta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, & n \geq 0, \end{cases} \tag{4.19}$$

where $\{\alpha_n\} \subset [0, 1]$, and $\zeta_n = \sup_{u \in F(S)} (l_n - 1)\phi(u, x_n)$. If $\lim_{n \rightarrow \infty} \alpha_n = 0$, then $\{x_n\}$ converges strongly as $n \rightarrow \infty$ to $\Pi_{\Omega}x_0$, where $\Omega := F(S) \cap MEP(f, \varphi, K)$.

If $S_i = S$, for all $i \geq 1$, then the following corollary follows from Theorem 4.3.

Corollary 4.5 *Let E be a real uniformly smooth and strictly convex Banach space with the Kadec-Klee property, and let C be a nonempty, bounded, closed and convex subset of E . Let f be a bifunction from $C \times C$ to \mathbb{R} satisfying (A1)-(A4), and let φ be a lower semicontinuous and convex function from C to \mathbb{R} . Let $S : C \rightarrow CB(C)$ be a closed and uniformly L -Lipschitz continuous quasi- ϕ -asymptotically nonexpansive multivalued mapping with a sequence $\{l_n\} \subset [1, \infty)$, $l_n \rightarrow 1$ with $F(S) \cap MEP(f, \varphi, C) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C generated by*

$$\begin{cases} x_0 \in C, & C_0 = C, \\ w_n = J^{-1}[\alpha_n Jx_0 + (1 - \alpha_n)Jz_n], & z_n \in S^n x_n, \\ u_n \in C & \text{such that} \\ f(u_n, y) + \varphi(y) + \frac{1}{r_n}(y - u_n, Ju_n - Jw_n) \geq \varphi(u_n), & \forall y \in C, \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \alpha_{n,0}\phi(v, x_0) + (1 - \alpha_{n,0})\phi(v, x_n) + \zeta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, & n \geq 0, \end{cases} \tag{4.20}$$

where for each $\{r_n\}_{n=0}^{\infty} \subset [a, \infty)$ for some $a > 0$, $\{\alpha_n\} \subset [0, 1]$, and $\zeta_n = \sup_{u \in F(S)} (l_n - 1)\phi(u, x_n)$. If $\lim_{n \rightarrow \infty} \alpha_n = 0$, then $\{x_n\}$ converges strongly as $n \rightarrow \infty$ to $\Pi_{\Omega}x_0$, where $\Omega := F(S) \cap MEP(f, \varphi, K)$.

In the case of $f \equiv 0$ and $\varphi \equiv 0$ in Corollary 4.4, we can omit the boundedness of C (necessary for the existence of a mapping T_r), so we have the following corollary.

Corollary 4.6 [21, Theorem 3.1] *Let E be a real uniformly smooth and strictly convex Banach space with the Kadec-Klee property, let C be a nonempty, closed and convex subset of E , and let $S : C \rightarrow CB(C)$ be a closed and uniformly L -Lipschitz continuous quasi- ϕ -asymptotically nonexpansive multivalued mapping with nonnegative real sequences $\{l_n\} \subset [1, \infty)$ and $l_n \rightarrow 1$ with $F(S)$ being a nonempty bounded subset. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_0 \in C, & C_0 = C, \\ y_n = J^{-1}[\alpha_n Jx_0 + (1 - \alpha_n)Jz_n], & z_n \in S^n x_n, \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \alpha_n \phi(v, x_0) + (1 - \alpha_n)\phi(v, x_n) + \zeta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, & n \geq 0, \end{cases} \tag{4.21}$$

where $\zeta_n = (l_n - 1) \sup_{p \in F(S)} \phi(p, x_n)$. If $\alpha_n \rightarrow 0$, then $\{x_n\}$ converges strongly to $\Pi_{F(S)} x_0$.

5 Application

We utilize Theorem 4.3 to study a modified Halpern iterative algorithm for a system of mixed equilibrium problems.

Theorem 5.1 *Let E be a real uniformly smooth and strictly convex Banach space with the Kadec-Klee property, and let C be a nonempty, bounded, closed and convex subset of E . Let $f_i : C \times C \rightarrow \mathbb{R}, i = 1, 2, \dots$, be a countable family of bifunctions satisfying conditions (A1)-(A4), and let $\varphi_i : C \rightarrow \mathbb{R}, i = 1, 2, \dots$, be a countable family of lower semicontinuous and convex functions satisfying conditions (i)-(ii) as in Lemma 3.5 with $\Gamma = \bigcap_{i=1}^{\infty} MEP(f_i, \varphi_i, C) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C generated by*

$$\begin{cases} x_0 \in C, & C_0 = C, \\ u_{n,i} \in C & \text{such that} \\ f_i(u_{n,i}, y) + \varphi_i(y) + \frac{1}{r}(y - u_{n,i}, Ju_{n,i} - Jx_{n,i}) \geq \varphi(u_{n,i}), & \forall y \in C, r > 0, i \geq 1, \\ y_n = J^{-1}(\alpha_{n,0} Jx_0 + \sum_{i=1}^{\infty} \alpha_{n,i} J u_{n,i}), \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \alpha_{n,0} \phi(v, x_0) + (1 - \alpha_{n,0})\phi(v, x_n) + \zeta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, & n \geq 0, \end{cases} \tag{5.1}$$

where $\{\alpha_n\} \subset [0, 1]$, and $\zeta_n = \sup_{u \in F(S)} (l_n - 1)\phi(u, x_n)$. If $\lim_{n \rightarrow \infty} \alpha_n = 0$, then $\{x_n\}$ converges strongly as $n \rightarrow \infty$ to $\Pi_{\Gamma} x_0$, which is a common solution of the system of mixed equilibrium problems.

Proof For any $i = 1, 2, \dots$, we define a mapping $T_{r,i}(z) : E \rightarrow C$ as follows:

$$T_{r,i}(z) = \left\{ x \in C : f_i(x, y) + \varphi_i(y) + \frac{1}{r}(y - x, Jx - Jz) \geq \varphi_i(x), \forall y \in C \right\}.$$

From Lemma 3.5, we get that $u_{n,i} = T_{r,i} x_n$ and $F(T_{r,i}) = MEP(f_i, \varphi_i, C)$ for all $i = 1, 2, \dots$, and $T_{r,i}$ is a countable family of closed quasi- ϕ -nonexpansive mappings. Thus, (5.1) can

be written as follows:

$$\begin{cases} x_0 \in C, & C_0 = C, \\ y_n = J^{-1}(\alpha_{n,0}Jx_0 + \sum_{i=1}^{\infty} \alpha_{n,i}JT_{r,i}x_n), \\ C_{n+1} = \{v \in C_n : \phi(v, y_n) \leq \alpha_{n,0}\phi(v, x_0) + (1 - \alpha_{n,0})\phi(v, x_n) + \zeta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}x_0, & n \geq 0. \end{cases}$$

Therefore, the result can be obtained from Corollary 4.4. \square

6 Conclusion

In this paper, we establish the existence of a solution of the mixed equilibrium problem $MEP(f, \varphi, C)$ by using the KKM mapping in a Banach space setting, when f is relaxed ξ -monotone. Then, by virtue of this result, we introduce a hybrid iterative scheme and prove that our proposed iterative scheme converges strongly to a common element of the set of solutions of $MEP(f, \varphi, C)$ and the set of common fixed points of a countable family of quasi- ϕ -asymptotically nonexpansive multivalued mappings.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors read and approved the final manuscript.

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