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Fixed points of (ψ,ϕ,θ)-contractive mappings in partially ordered b-metric spaces and application to quadratic integral equations

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Abstract

We prove some coupled coincidence and coupled common fixed point theorems for mappings satisfying (ψ,ϕ,θ)-contractive conditions in partially ordered complete b-metric spaces. The obtained results extend and improve many existing results from the literature. As an application, we prove the existence of a unique solution to a class of nonlinear quadratic integral equations.

MSC: Primary 47H10; secondary 54H25

Keywords: coupled common fixed point; coupled fixed point; coupled coincidence point; mixed g-monotone property; b-metric space; partially ordered set

1 Introduction and preliminaries

In [1, 2], Czerwik introduced the notion of a b-metric space, which is a generalization of the usual metric space, and generalized the Banach contraction principle in the context of complete b-metric spaces. After that, many authors have carried out further studies on b-metric spaces and their topological properties (see, *e.g.*, [1–14]). In this paper, some coupled coincidence and coupled common fixed point theorems for mappings satisfying (ψ,ϕ,θ)-contractive conditions in partially ordered complete b-metric spaces are proved. Also, we apply our results to study the existence of a unique solution to a large class of nonlinear quadratic integral equations. There are many papers in the literature concerning coupled fixed points introduced by Bhaskar and Lakshmikantham [15] and their applications in the existence and uniqueness of solutions for boundary value problems. A number of articles on this topic have been dedicated to the improvement and generalization; see [16–20] and references therein. Also, to see some results on common fixed points for generalized contraction mappings, we refer the reader to [21–23]. For the sake of convenience, some definitions and notations are recalled from [1, 3, 24] and [25].

Definition 1.1 [1] Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \longrightarrow \mathbb{R}^+$ is said to be a b-metric space iff for all $x, y, z \in X$, the following conditions are satisfied:

(i) $d(x, y) = 0$ iff $x = y$,

(ii) $d(x,y) = d(y,x)$,
(iii) $d(x,y) \leq s[d(x,z) + d(z,y)]$.

The pair (X,d) is called a b-metric space with the parameter s.

It should be noted that the class of b-metric spaces is effectively larger than that of metric spaces since a b-metric is a metric when $s = 1$.

The following example shows that, in general, a b-metric need not necessarily be a metric (see also [14]).

Example 1.2 [3] Let (X,d) be a metric space and $\rho(x,y) = (d(x,y))^p$, where $p > 1$ is a real number. Then ρ is a b-metric with $s = 2^{p-1}$. However, if (X,d) is a metric space, then (X,ρ) is not necessarily a metric space. For example, if $X = \mathbb{R}$ is the set of real numbers and $d(x,y) = |x-y|$ is the usual Euclidean metric, then $\rho(x,y) = (x-y)^s$ is a b-metric on $\mathbb{R}$ with $s = 2$, but is not a metric on $\mathbb{R}$.

Also, the following example of a b-metric space is given in [26].

Example 1.3 [26] Let X be the set of Lebesgue measurable functions on $[0,1]$ such that $\int_0^1 |f(x)|^2\,dx < \infty$. Define $D : X \times X \longrightarrow [0,\infty)$ by $D(f,g) = \int_0^1 |f(x) - g(x)|^2\,dx$. As $(\int_0^1 |f(x) - g(x)|^2\,dx)^{\frac{1}{2}}$ is a metric on X, then, from the previous example, D is a b-metric on X, with $s = 2$.

Khamsi [27] also showed that each cone metric space over a normal cone has a b-metric structure.

Since, in general, a b-metric is not continuous, we need the following simple lemma about the b-convergent sequences in the proof of our main result.

Lemma 1.4 [3] *Let (X,d) be a b-metric space with $s \geq 1$, and suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x, y, respectively. Then we have*

$$\frac{1}{s^2}d(x,y) \leq \liminf d(x_n,y_n) \leq \limsup d(x_n,y_n) \leq s^2 d(x,y).$$

In particular, if $x = y$, then we have $\lim d(x_n,y_n) = 0$. Moreover, for each $z \in X$, we have

$$\frac{1}{s}d(x,z) \leq \liminf d(x_n,z) \leq \limsup d(x_n,z) \leq sd(x,z).$$

In [25], Lakshmikantham and Ćirić introduced the concept of mixed g-monotone property as follows.

Definition 1.5 [25] Let $(X,\leq)$ be a partially ordered set and $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$. We say F has the mixed g-monotone property if F is non-decreasing g-monotone in its first argument and is non-increasing g-monotone in its second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad gx_1 \leq gx_2 \quad \Longrightarrow \quad F(x_1,y) \leq F(x_2,y)$$

and

$$y_1, y_2 \in X, \quad gy_1 \leq gy_2 \quad \Longrightarrow \quad F(x, y_1) \geq F(x, y_2).$$

Note that if g is an identity mapping, then F is said to have the mixed monotone property (see also [15]).

Definition 1.6 [25] An element $(x,y) \in X \times X$ is called a coupled coincidence point of a mapping $F : X \times X \longrightarrow X$ and a mapping $g : X \longrightarrow X$ if

$$F(x,y) = gx, \qquad F(y,x) = gy.$$

Similarly, note that if g is an identity mapping, then (x,y) is called a coupled fixed point of the mapping F (see also [15]).

Definition 1.7 [24] An element $x \in X$ is called a common fixed point of a mapping $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ if

$$F(x,x) = gx = x. \tag{1.1}$$

Definition 1.8 [25] Let X be a nonempty set and $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$. One says that F and g are commutative if for all $x, y \in X$,

$$F(gx, gy) = g\big(F(x,y)\big).$$

Definition 1.9 [28] The mappings F and g, where $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$, are said to be compatible if

$$\lim_{n \longrightarrow \infty} d\big(g\big(F(x_n, y_n)\big), F(gx_n, gy_n)\big) = 0$$

and

$$\lim_{n \longrightarrow \infty} d\big(g\big(F(y_n, x_n)\big), F(gy_n, gx_n)\big) = 0,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n \longrightarrow \infty} F(x_n, y_n) = \lim_{n \longrightarrow \infty} gx_n = x$ and $\lim_{n \longrightarrow \infty} F(y_n, x_n) = \lim_{n \longrightarrow \infty} gy_n = y$ for all $x, y \in X$.

2 Main results

Throughout the paper, let Ψ be a family of all functions $\psi : [0,\infty) \longrightarrow [0,\infty)$ satisfying the following conditions:

(a) ψ is continuous,
(b) ψ non-decreasing,
(c) $\psi(t) = 0$ if and only if $t = 0$.

We denote by Φ the set of all functions $\phi : [0,\infty) \longrightarrow [0,\infty)$ satisfying the following conditions:

(a) ϕ is lower semi-continuous,
(b) $\phi(t) = 0$ if and only if $t = 0$,

and Θ the set of all continuous functions $\theta : [0,\infty) \longrightarrow [0,\infty)$ with $\theta(t) = 0$ if and only if $t = 0$.

Let $(X,d,\leq)$ be a partially ordered b-metric space, and let $T : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ be two mappings. Set

$$M_{s,T,g}(x,y,u,v) = \max\Big\{d(gx,gu), d(gy,gv), d\big(gx, T(x,y)\big), \frac{1}{2s}d\big(gu, T(u,v)\big), d\big(gy, T(y,x)\big), \frac{1}{2s}d\big(gv, T(v,u)\big), \frac{d(gx,T(u,v)) + d(gu,T(x,y))}{2s}, \frac{d(gy,T(v,u)) + d(gv,T(y,x))}{2s}\Big\}$$

and

$$N_{T,g}(x,y,u,v) = \min\big\{d\big(gx,T(x,y)\big), d\big(gu,T(u,v)\big), d\big(gu,T(x,y)\big), d\big(gx,T(u,v)\big)\big\}.$$

Now, we introduce the following definition.

Definition 2.1 Let $(X,d,\leq)$ be a partially ordered b-metric space and $\psi \in \Psi$, $\phi \in \Phi$ and $\theta \in \Theta$. We say that $T : X \times X \longrightarrow X$ is an almost generalized (ψ,ϕ,θ)-contractive mapping with respect to $g : X \longrightarrow X$ if there exists $L \geq 0$ such that

$$\psi\big(s^3 d\big(T(x,y), T(u,v)\big)\big) \leq \psi\big(M_{s,T,g}(x,y,u,v)\big) - \phi\big(M_{s,T,g}(x,y,u,v)\big) + L\theta\big(N_{T,g}(x,y,u,v)\big) \tag{2.1}$$

for all $x,y,u,v \in X$ with $gx \leq gu$ and $gy \geq gv$.

Now, we establish some results for the existence of a coupled coincidence point and a coupled common fixed point of mappings satisfying almost generalized (ψ,ϕ,θ)-contractive condition in the setup of partially ordered b-metric spaces. The first result in this paper is the following coupled coincidence theorem.

Theorem 2.2 *Suppose that $(X,d,\leq)$ is a partially ordered complete b-metric space. Let $T : X \times X \longrightarrow X$ be an almost generalized (ψ,ϕ,θ)-contractive mapping with respect to $g : X \longrightarrow X$, and T and g are continuous such that T has the mixed g-monotone property and commutes with g. Also, suppose $T(X \times X) \subseteq g(X)$. If there exists $(x_0,y_0) \in X \times X$ such that $gx_0 \leq T(x_0,y_0)$ and $gy_0 \geq T(y_0,x_0)$, then T and g have coupled coincidence point in X.*

Proof By the given assumptions, there exists $(x_0,y_0) \in X \times X$ such that $gx_0 \leq T(x_0,y_0)$ and $gy_0 \geq T(y_0,x_0)$. Since $T(X \times X) \subseteq g(X)$, we can define $(x_1,y_1) \in X \times X$ such that $gx_1 = T(x_0,y_0)$ and $gy_1 = T(y_0,x_0)$, then $gx_0 \leq T(x_0,y_0) = gx_1$ and $gy_0 \geq T(y_0,x_0) = gy_1$. Also, there exists $(x_2,y_2) \in X \times X$ such that $gx_2 = T(x_1,y_1)$ and $gy_2 = T(y_1,x_1)$. Since T has the mixed g-monotone property, we have

$$gx_1 = T(x_0,y_0) \leq T(x_0,y_1) \leq T(x_1,y_1) = gx_2$$

and

$$gy_2 = T(y_1,x_1) \le T(y_0,x_1) \le T(y_0,x_0) = gy_1.$$

Continuing in this way, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$gx_{n+1} = T(x_n,y_n) \quad \text{and} \quad gy_{n+1} = T(y_n,x_n) \quad \text{for all } n = 0,1,2,\ldots \tag{2.2}$$

for which

$$\begin{aligned} &gx_0 \le gx_1 \le gx_2 \le \cdots \le gx_n \le gx_{n+1} \le \cdots, \\ &gy_0 \ge gy_1 \ge gy_2 \ge \cdots \ge gy_n \ge gy_{n+1} \ge \cdots. \end{aligned} \tag{2.3}$$

From (2.2) and (2.3) and inequality (2.1) with $(x,y) = (x_n,y_n)$ and $(u,v) = (x_{n+1},y_{n+1})$, we obtain

$$\begin{aligned} \psi\bigl(d(gx_{n+1},gx_{n+2})\bigr) &\le \psi\bigl(s^3 d(gx_{n+1},gx_{n+2})\bigr) = \psi\bigl(s^3 d\bigl(T(x_n,y_n),T(x_{n+1},y_{n+1})\bigr)\bigr) \\ &\le \psi\bigl(M_{s,T,g}(x_n,y_n,x_{n+1},y_{n+1})\bigr) - \phi\bigl(M_{s,T,g}(x_n,y_n,x_{n+1},y_{n+1})\bigr) \\ &\quad + L\theta\bigl(N_{T,g}(x_n,y_n,x_{n+1},y_{n+1})\bigr), \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} M_{s,T,g}(x_n,y_n,x_{n+1},y_{n+1}) = \max\biggl\{ & d(gx_n,gx_{n+1}), d(gy_n,gy_{n+1}), d\bigl(gx_n,T(x_n,y_n)\bigr), \\ & \frac{1}{2s} d\bigl(gx_{n+1},T(x_{n+1},y_{n+1})\bigr), d\bigl(gy_n,T(y_n,x_n)\bigr), \\ & \frac{1}{2s} d\bigl(gy_{n+1},T(y_{n+1},x_{n+1})\bigr), \\ & \frac{d(gx_n,T(x_{n+1},y_{n+1})) + d(gx_{n+1},T(x_n,y_n))}{2s}, \\ & \frac{d(gy_n,T(y_{n+1},x_{n+1})) + d(gy_{n+1},T(y_n,x_n))}{2s} \biggr\} \\ = \max\biggl\{ & d(gx_n,gx_{n+1}), d(gy_n,gy_{n+1}), \\ & \frac{1}{2s} d(gx_{n+1},gx_{n+2}), \frac{1}{2s} d(gy_{n+1},gy_{n+2}), \\ & \frac{d(gx_n,gx_{n+2})}{2s}, \frac{d(gy_n,gy_{n+2})}{2s} \biggr\} \end{aligned}$$

and

$$\begin{aligned} N_{T,g}(x_n,y_n,x_{n+1},y_{n+1}) = \min\bigl\{ & d\bigl(gx_n,T(x_n,y_n)\bigr), d\bigl(gx_{n+1},T(x_{n+1},y_{n+1})\bigr), \\ & d\bigl(gx_{n+1},T(x_n,y_n)\bigr), d\bigl(gx_{n+1},T(x_{n+1},y_{n+1})\bigr)\bigr\} = 0. \end{aligned}$$

Since

$$\frac{d(gx_n,gx_{n+2})}{2s} \le \frac{d(gx_n,gx_{n+1}) + d(gx_{n+1},gx_{n+2})}{2} \le \max\bigl\{d(gx_n,gx_{n+1}), d(gx_{n+1},gx_{n+2})\bigr\}$$

and

$$\frac{d(gy_n,gy_{n+2})}{2s} \leq \frac{d(gy_n,gy_{n+1}) + d(gy_{n+1},gy_{n+2})}{2} \leq \max\bigl\{d(gy_n,gy_{n+1}), d(gy_{n+1},gy_{n+2})\bigr\},$$

then we get

$$\begin{aligned} M_{s,T,g}(x_n,y_n,x_{n+1},y_{n+1}) \leq \max\bigl\{&d(gx_n,gx_{n+1}), d(gx_{n+1},gx_{n+2}), \\ &d(gy_n,gy_{n+1}), d(gy_{n+1},gy_{n+2})\bigr\}, \end{aligned} \tag{2.5}$$
$$N_{T,g}(x_n,y_n,x_{n+1},y_{n+1}) = 0.$$

By (2.4) and (2.5), we have

$$\begin{aligned} &\psi\bigl(d(gx_{n+1},gx_{n+2})\bigr) \\ &\quad\leq \psi\bigl(\max\bigl\{d(gx_n,gx_{n+1}), d(gx_{n+1},gx_{n+2}), d(gy_n,gy_{n+1}), d(gy_{n+1},gy_{n+2})\bigr\}\bigr) \\ &\qquad - \phi\bigl(\max\bigl\{d(gx_n,gx_{n+1}), d(gx_{n+1},gx_{n+2}), d(gy_n,gy_{n+1}), d(gy_{n+1},gy_{n+2})\bigr\}\bigr). \end{aligned} \tag{2.6}$$

Similarly, we can show that

$$\begin{aligned} &\psi\bigl(d(gy_{n+1},gy_{n+2})\bigr) \\ &\quad\leq \psi\bigl(\max\bigl\{d(gx_n,gx_{n+1}), d(gx_{n+1},gx_{n+2}), d(gy_n,gy_{n+1}), d(gy_{n+1},gy_{n+2})\bigr\}\bigr) \\ &\qquad - \phi\bigl(\max\bigl\{d(gx_n,gx_{n+1}), d(gx_{n+1},gx_{n+2}), d(gy_n,gy_{n+1}), d(gy_{n+1},gy_{n+2})\bigr\}\bigr). \end{aligned} \tag{2.7}$$

Now, denote

$$\delta_n = \max\bigl\{d(gx_n,gx_{n+1}), d(gy_n,gy_{n+1})\bigr\}. \tag{2.8}$$

Combining (2.6), (2.7) and the fact that $\max\{\psi(a),\psi(b)\} = \psi(\max\{a,b\})$ for $a,b \in [0,+\infty)$, we have

$$\psi(\delta_{n+1}) = \max\bigl\{\psi\bigl(d(gx_{n+1},gx_{n+2})\bigr), \psi\bigl(d(gy_{n+1},gy_{n+2})\bigr)\bigr\}. \tag{2.9}$$

So, using (2.6), (2.7), (2.8) together with (2.9), we obtain

$$\begin{aligned} &\psi(\delta_{n+1}) \\ &\quad\leq \psi\bigl(\max\bigl\{d(gx_n,gx_{n+1}), d(gx_{n+1},gx_{n+2}), d(gy_n,gy_{n+1}), d(gy_{n+1},gy_{n+2})\bigr\}\bigr) \\ &\qquad - \phi\bigl(\max\bigl\{d(gx_n,gx_{n+1}), d(gx_{n+1},gx_{n+2}), d(gy_n,gy_{n+1}), d(gy_{n+1},gy_{n+2})\bigr\}\bigr). \end{aligned} \tag{2.10}$$

Now we prove that for all $n \in \mathbb{N}$,

$$\begin{aligned} &\max\bigl\{d(gx_n,gx_{n+1}), d(gx_{n+1},gx_{n+2}), d(gy_n,gy_{n+1}), d(gy_{n+1},gy_{n+2})\bigr\} \\ &\quad = \delta_n \quad \text{and} \quad \delta_{n+1} \leq \delta_n. \end{aligned} \tag{2.11}$$

For this purpose, consider the following three cases.

Case 1. If $\max\{d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2}), d(gy_n, gy_{n+1}), d(gy_{n+1}, gy_{n+2})\} = \delta_n$, then by (2.10) we have

$$\psi(\delta_{n+1}) \leq \psi(\delta_n) - \phi(\delta_n) < \psi(\delta_n), \tag{2.12}$$

so (2.11) obviously holds.

Case 2. If $\max\{d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2}), d(gy_n, gy_{n+1}), d(gy_{n+1}, gy_{n+2})\} = d(gx_{n+1}, gx_{n+2}) > 0$, then by (2.6) we have

$$\psi\big(d(gx_{n+1}, gx_{n+2})\big) \leq \psi\big(d(gx_{n+1}, gx_{n+2})\big) - \phi\big(d(gx_{n+1}, gx_{n+2})\big) < \psi\big(d(gx_{n+1}, gx_{n+2})\big),$$

which is a contradiction.

Case 3. If $\max\{d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2}), d(gy_n, gy_{n+1}), d(gy_{n+1}, gy_{n+2})\} = d(gy_{n+1}, gy_{n+2}) > 0$, then from (2.7) we have

$$\psi\big(d(gy_{n+1}, gy_{n+2})\big) \leq \psi\big(d(gy_{n+1}, gy_{n+2})\big) - \phi\big(d(gy_{n+1}, gy_{n+2})\big) < \psi\big(d(gy_{n+1}, gy_{n+2})\big),$$

which is again a contradiction.

Thus, in all the cases, (2.11) holds for each $n \in \mathbb{N}$. It follows that the sequence $\{\delta_n\}$ is a monotone decreasing sequence of nonnegative real numbers and, consequently, there exists $\delta \geq 0$ such that

$$\lim_{n \longrightarrow \infty} \delta_n = \delta. \tag{2.13}$$

We show that $\delta = 0$. Suppose, on the contrary, that $\delta > 0$. Taking the limit as $n \longrightarrow \infty$ in (2.12) and using the properties of the function ϕ, we get

$$\psi(\delta) \leq \psi(\delta) - \phi(\delta) < \psi(\delta),$$

which is a contradiction. Therefore $\delta = 0$, that is,

$$\lim_{n \longrightarrow \infty} \delta_n = \lim_{n \longrightarrow \infty} \max\big\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\big\} = 0,$$

which implies that

$$\lim_{n \longrightarrow \infty} d(gx_n, gx_{n+1}) = 0 \quad \text{and} \quad \lim_{n \longrightarrow \infty} d(gy_n, gy_{n+1}) = 0. \tag{2.14}$$

Now, we claim that

$$\lim_{n,m \longrightarrow \infty} \max\big\{d(gx_n, gx_m), d(gy_n, gy_m)\big\} = 0. \tag{2.15}$$

Assume, on the contrary, that there exist $\epsilon > 0$ and subsequences $\{gx_{m(k)}\}$, $\{gx_{n(k)}\}$ of $\{gx_n\}$ and $\{gy_{m(k)}\}$, $\{gy_{n(k)}\}$ of $\{gy_n\}$ with $m(k) > n(k) \geq k$ such that

$$\max\big\{d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)})\big\} \geq \epsilon. \tag{2.16}$$

Additionally, corresponding to $n(k)$, we may choose $m(k)$ such that it is the smallest integer satisfying (2.16) and $m(k) > n(k) \geq k$. Thus,

$$\max\bigl\{d(gx_{n(k)}, gx_{m(k)-1}), d(gy_{n(k)}, gy_{m(k)-1})\bigr\} < \epsilon. \tag{2.17}$$

Using the triangle inequality in a b-metric space and (2.16) and (2.17), we obtain that

$$\begin{aligned}\epsilon &\leq d(gx_{m(k)}, gx_{n(k)}) \leq sd(gx_{m(k)}, gx_{m(k)-1}) + sd(gx_{m(k)-1}, gx_{n(k)}) \\ &< sd(gx_{m(k)}, gx_{m(k)-1}) + s\epsilon.\end{aligned}$$

Taking the upper limit as $k \longrightarrow \infty$ and using (2.14), we obtain

$$\epsilon \leq \limsup_{k\longrightarrow\infty} d(gx_{n(k)}, gx_{m(k)}) \leq s\epsilon. \tag{2.18}$$

Similarly, we obtain

$$\epsilon \leq \limsup_{k\longrightarrow\infty} d(gy_{n(k)}, gy_{m(k)}) \leq s\epsilon. \tag{2.19}$$

Also,

$$\begin{aligned}\epsilon &\leq d(gx_{n(k)}, gx_{m(k)}) \leq sd(gx_{n(k)}, gx_{m(k)+1}) + sd(gx_{m(k)+1}, gx_{m(k)}) \\ &\leq s^2 d(gx_{n(k)}, gx_{m(k)}) + s^2 d(gx_{m(k)}, gx_{m(k)+1}) + sd(gx_{m(k)+1}, gx_{m(k)}) \\ &\leq s^2 d(gx_{n(k)}, gx_{m(k)}) + \bigl(s^2 + s\bigr) d(gx_{m(k)}, gx_{m(k)+1}).\end{aligned}$$

So, from (2.14) and (2.18), we have

$$\frac{\epsilon}{s} \leq \limsup_{k\longrightarrow\infty} d(gx_{n(k)}, gx_{m(k)+1}) \leq s^2\epsilon. \tag{2.20}$$

Similarly, we obtain

$$\frac{\epsilon}{s} \leq \limsup_{k\longrightarrow\infty} d(gy_{n(k)}, gy_{m(k)+1}) \leq s^2\epsilon. \tag{2.21}$$

Also,

$$\begin{aligned}\epsilon &\leq d(gx_{m(k)}, gx_{n(k)}) \leq sd(gx_{m(k)}, gx_{n(k)+1}) + sd(gx_{n(k)+1}, gx_{n(k)}) \\ &\leq s^2 d(gx_{m(k)}, gx_{n(k)}) + s^2 d(gx_{n(k)}, gx_{n(k)+1}) + sd(gx_{n(k)+1}, gx_{n(k)}) \\ &\leq s^2 d(gx_{m(k)}, gx_{n(k)}) + \bigl(s^2 + s\bigr) d(gx_{n(k)}, gx_{n(k)+1}).\end{aligned}$$

So, from (2.14) and (2.18), we have

$$\frac{\epsilon}{s} \leq \limsup_{k\longrightarrow\infty} d(gx_{m(k)}, gx_{n(k)+1}) \leq s^2\epsilon. \tag{2.22}$$

In a similar way, we obtain

$$\frac{\epsilon}{s} \leq \limsup_{k\longrightarrow\infty} d(gy_{m(k)}, gy_{n(k)+1}) \leq s^2\epsilon. \tag{2.23}$$

Also,

$$d(gx_{n(k)+1}, gx_{m(k)}) \le sd(gx_{n(k)+1}, gx_{m(k)+1}) + sd(gx_{m(k)+1}, gx_{m(k)}).$$

So, from (2.14) and (2.22), we have

$$\frac{\epsilon}{s^2} \le \limsup_{k\longrightarrow\infty} d(gx_{n(k)+1}, gx_{m(k)+1}). \tag{2.24}$$

Similarly, we obtain

$$\frac{\epsilon}{s^2} \le \limsup_{k\longrightarrow\infty} d(gy_{n(k)+1}, gy_{m(k)+1}). \tag{2.25}$$

Linking (2.14), (2.18), (2.19), (2.20), (2.21), (2.22) together with (2.23), we get

$$\begin{aligned}
\frac{\epsilon}{s^2} &= \min\left\{\epsilon, \epsilon, \frac{\frac{\epsilon}{s} + \frac{\epsilon}{s}}{2s}, \frac{\frac{\epsilon}{s} + \frac{\epsilon}{s}}{2s}\right\} \\
&\le \max\left\{\limsup_{k\longrightarrow\infty} d(gx_{n(k)}, gx_{m(k)}), \limsup_{k\longrightarrow\infty} d(gy_{n(k)}, gy_{m(k)}),\right. \\
&\quad \frac{\limsup_{k\longrightarrow\infty} d(gx_{n(k)}, gx_{m(k)+1}) + \limsup_{k\longrightarrow\infty} d(gx_{m(k)}, gx_{n(k)+1})}{2s}, \\
&\quad \left.\frac{\limsup_{k\longrightarrow\infty} d(gy_{n(k)}, gy_{m(k)+1}) + \limsup_{k\longrightarrow\infty} d(gy_{m(k)}, gy_{n(k)+1})}{2s}\right\} \\
&\le \max\left\{s\epsilon, s\epsilon, \frac{s^2\epsilon + s^2\epsilon}{2s}, \frac{s^2\epsilon + s^2\epsilon}{2s}\right\} = s\epsilon.
\end{aligned}$$

So,

$$\frac{\epsilon}{s^2} \le \limsup_{k\longrightarrow\infty} M_{s,T,g}(x_{n(k)}, y_{n(k)}, x_{m(k)}, y_{m(k)}) \le \epsilon s. \tag{2.26}$$

Similarly, we have

$$\frac{\epsilon}{s^2} \le \liminf_{k\longrightarrow\infty} M_{s,T,g}(x_{n(k)}, y_{n(k)}, x_{m(k)}, y_{m(k)}) \le \epsilon s \tag{2.27}$$

and

$$\lim_{k\longrightarrow\infty} N_{T,g}(x_{n(k)}, y_{n(k)}, x_{m(k)}, y_{m(k)}) = 0. \tag{2.28}$$

Since $m(k) > n(k)$, from (2.2) we have

$$gx_{n(k)} \le gx_{m(k)}, \qquad gy_{n(k)} \ge gy_{m(k)}.$$

Thus,

$$\begin{aligned}
\psi\left(s^3 d(gx_{n(k)+1}, gx_{m(k)+1})\right) &= \psi\left(s^3 d\left(T(x_{n(k)}, y_{n(k)}), T(x_{m(k)}, y_{m(k)})\right)\right) \\
&\le \psi\left(M_{s,T,g}(x_{n(k)}, y_{n(k)}, x_{m(k)}, y_{m(k)})\right)
\end{aligned}$$

$$\begin{aligned}
&\quad - \phi\big(M_{s,T,g}(x_{n(k)}, y_{n(k)}, x_{m(k)}, y_{m(k)})\big) \\
&\quad + L\theta\big(N_{T,g}(x_{n(k)}, y_{n(k)}, x_{m(k)}, y_{m(k)})\big), \\
\psi\big(s^3 d(gy_{n(k)+1}, gy_{m(k)+1})\big) &= \psi\big(s^3 d\big(T(y_{n(k)}, x_{n(k)}), T(y_{m(k)}, x_{m(k)})\big)\big) \\
&\leq \psi\big(M_{s,T,g}(x_{n(k)}, y_{n(k)}, x_{m(k)}, y_{m(k)})\big) \\
&\quad - \phi\big(M_{s,T,g}(x_{n(k)}, y_{n(k)}, x_{m(k)}, y_{m(k)})\big) \\
&\quad + L\theta\big(N_{T,g}(x_{n(k)}, y_{n(k)}, x_{m(k)}, y_{m(k)})\big).
\end{aligned}$$

Since ψ is a non-decreasing function, we have

$$\begin{aligned}
&\max\big\{\psi\big(s^3 d(gx_{n(k)+1}, gx_{m(k)+1})\big), \psi\big(s^3 d(gy_{n(k)+1}, gy_{m(k)+1})\big)\big\} \\
&\quad = \psi\big(s^3 \max\big\{d(gx_{n(k)+1}, gx_{m(k)+1}), d(gy_{n(k)+1}, gy_{m(k)+1})\big\}\big).
\end{aligned}$$

Taking the upper limit as $k \longrightarrow \infty$ and using (2.25) and (2.26), we get

$$\begin{aligned}
\psi(s\epsilon) &\leq \psi\Big(s^3 \max\Big\{\limsup_{k\longrightarrow\infty} d(gx_{n(k)+1}, gx_{m(k)+1}), \limsup_{k\longrightarrow\infty} d(gy_{n(k)+1}, gy_{m(k)+1})\Big\}\Big) \\
&\leq \psi\Big(\limsup_{k\longrightarrow\infty} M_{s,T,g}(x_{n(k)}, y_{n(k)}, x_{m(k)}, y_{m(k)})\Big) \\
&\quad - \phi\Big(\liminf_{k\longrightarrow\infty} M_{s,T,g}(x_{n(k)}, y_{n(k)}, x_{m(k)}, y_{m(k)})\Big) \\
&\quad + L\theta\Big(\limsup_{k\longrightarrow\infty} N_{T,g}(x_{n(k)}, y_{n(k)}, x_{m(k)}, y_{m(k)})\Big) \\
&\leq \psi(s\epsilon) - \phi\Big(\liminf_{k\longrightarrow\infty} M_{s,T,g}(x_{n(k)}, y_{n(k)}, x_{m(k)}, y_{m(k)})\Big),
\end{aligned}$$

which implies that

$$\phi\Big(\liminf_{k\longrightarrow\infty} M_{s,T,g}(x_{n(k)}, y_{n(k)}, x_{m(k)}, y_{m(k)})\Big) = 0,$$

so

$$\liminf_{k\longrightarrow\infty} M_{s,T,g}(x_{n(k)}, y_{n(k)}, x_{m(k)}, y_{m(k)}) = 0,$$

a contradiction to (2.27). Therefore, (2.15) holds and we have

$$\lim_{n,m\longrightarrow\infty} d(gx_n, gx_m) = 0 \quad \text{and} \quad \lim_{n,m\longrightarrow\infty} d(gy_n, gy_m) = 0.$$

Since X is a complete b-metric space, there exist $x, y \in X$ such that

$$\lim_{n\longrightarrow\infty} gx_{n+1} = x \quad \text{and} \quad \lim_{n\longrightarrow\infty} gy_{n+1} = y. \tag{2.29}$$

From the commutativity of T and g, we have

$$g(gx_{n+1}) = g\big(T(x_n, y_n)\big) = T(gx_n, gy_n), \qquad g(gy_{n+1}) = g\big(T(y_n, x_n)\big) = T(gy_n, gx_n). \tag{2.30}$$

Now, we shall show that

$$gx = T(x,y) \quad \text{and} \quad gy = T(y,x).$$

Letting $n \longrightarrow \infty$ in (2.30), from the continuity of T and g, we get

$$\begin{aligned} gx &= \lim_{n\longrightarrow\infty} g(gx_{n+1}) = \lim_{n\longrightarrow\infty} T(gx_n, gy_n) = T\Big(\lim_{n\longrightarrow\infty} gx_n, \lim_{n\longrightarrow\infty} gy_n\Big) = T(x,y), \\ gy &= \lim_{n\longrightarrow\infty} g(gy_{n+1}) = \lim_{n\longrightarrow\infty} T(gy_n, gx_n) = T\Big(\lim_{n\longrightarrow\infty} gy_n, \lim_{n\longrightarrow\infty} gx_n\Big) = T(y,x). \end{aligned}$$

This implies that (x,y) is a coupled coincidence point of T and g. This completes the proof. □

Corollary 2.3 *Let $(X,d,\leq)$ be a partially ordered complete b-metric space, and let $T: X \times X \longrightarrow X$ be a continuous mapping such that T has the mixed monotone property. Suppose that there exist $\psi \in \Psi$, $\phi \in \Phi$, $\theta \in \Theta$ and $L \geq 0$ such that*

$$\psi\left(s^3 d\big(T(x,y), T(u,v)\big)\right) \leq \psi\big(M_s(x,y,u,v)\big) - \phi\big(M_s(x,y,u,v)\big) + L\theta\big(N(x,y,u,v)\big),$$

where

$$\begin{aligned} M_s(x,y,u,v) = \max\Big\{ & d(x,u), d(y,v), d\big(x, T(x,y)\big), \\ & \frac{1}{2s} d\big(u, T(u,v)\big), d\big(y, T(y,x)\big), \frac{1}{2s} d\big(v, T(v,u)\big), \\ & \frac{d(x, T(u,v)) + d(u, T(x,y))}{2s}, \frac{d(y, T(v,u)) + d(v, T(y,x))}{2s} \Big\} \end{aligned}$$

and

$$N(x,y,u,v) = \min\big\{ d\big(x, T(x,y)\big), d\big(u, T(u,v)\big), d\big(u, T(x,y)\big), d\big(x, T(u,v)\big) \big\}$$

for all $x,y,u,v \in X$ with $x \leq u$ and $y \geq v$. If there exists $(x_0,y_0) \in X \times X$ such that $x_0 \leq T(x_0,y_0)$ and $y_0 \geq T(y_0,x_0)$, then T has a coupled fixed point in X.

Proof Take $g = I_X$ and apply Theorem 2.2. □

The following result is the immediate consequence of Corollary 2.3.

Corollary 2.4 *Let $(X,d,\leq)$ be a partially ordered complete b-metric space. Let $T: X \times X \longrightarrow X$ be a continuous mapping such that T has the mixed monotone property. Suppose that there exists $\phi \in \Phi$ such that*

$$d\big(T(x,y), T(u,v)\big) \leq \frac{1}{s^3} M_s(x,y,u,v) - \frac{1}{s^3}\phi\big(M_s(x,y,u,v)\big), \tag{2.31}$$

where

$$M_s(x,y,u,v) = \max\bigg\{d(x,u), d(y,v), d\big(x, T(x,y)\big), \frac{1}{2s}d\big(u, T(u,v)\big), d\big(y, T(y,x)\big), \frac{1}{2s}d\big(v, T(v,u)\big), \frac{d(x,T(u,v)) + d(u,T(x,y))}{2s}, \frac{d(y,T(v,u)) + d(v,T(y,x))}{2s}\bigg\}$$

for all $x, y, u, v \in X$ with $x \leq u$ and $y \geq v$. If there exists $(x_0, y_0) \in X \times X$ such that $x_0 \leq T(x_0, y_0)$ and $y_0 \geq T(y_0, x_0)$, then T has a coupled fixed point in X.

3 Uniqueness of a common fixed point

In this section we shall provide some sufficient conditions under which T and g have a unique common fixed point. Note that if $(X, \leq)$ is a partially ordered set, then we endow the product $X \times X$ with the following partial order relation, for all $(x,y), (z,t) \in X \times X$,

$$(x,y) \leq (z,t) \quad \Longleftrightarrow \quad x \leq z, \quad y \geq t.$$

From Theorem 2.2, it follows that the set $C(T,g)$ of coupled coincidences is nonempty.

Theorem 3.1 *By adding to the hypotheses of Theorem* 2.2, *the condition*: *for every (x,y) and (z,t) in $X \times X$, there exists $(u,v) \in X \times X$ such that $(T(u,v), T(v,u))$ is comparable to $(T(x,y), T(y,x))$ and to $(T(z,t), T(t,z))$, then T and g have a unique coupled common fixed point*; *that is, there exists a unique $(x,y) \in X \times X$ such that*

$$x = gx = T(x,y), \qquad y = gy = T(y,x).$$

Proof We know, from Theorem 2.2, that there exists at least a coupled coincidence point. Suppose that (x,y) and (z,t) are coupled coincidence points of T and g, that is, $T(x,y) = gx$, $T(y,x) = gy$, $T(z,t) = gz$ and $T(t,z) = gt$. We shall show that $gx = gz$ and $gy = gt$. By the assumptions, there exists $(u,v) \in X \times X$ such that $(T(u,v), T(v,u))$ is comparable to $(T(x,y), T(y,x))$ and to $(T(z,t), T(t,z))$. Without any restriction of the generality, we can assume that

$$\big(T(x,y), T(y,x)\big) \leq \big(T(u,v), T(v,u)\big) \quad \text{and} \quad \big(T(z,t), T(t,z)\big) \leq \big(T(u,v), T(v,u)\big).$$

Put $u_0 = u$, $v_0 = v$ and choose $(u_1, v_1) \in X \times X$ such that

$$gu_1 = T(u_0, v_0), \qquad gv_1 = T(v_0, u_0).$$

For $n \geq 1$, continuing this process, we can construct sequences $\{gu_n\}$ and $\{gv_n\}$ such that

$$gu_{n+1} = T(u_n, v_n), \qquad gv_{n+1} = T(v_n, u_n) \quad \text{for all } n.$$

Further, set $x_0 = x$, $y_0 = y$ and $z_0 = z$, $t_0 = t$ and in the same way define sequences $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$, $\{gt_n\}$. Then it is easy to see that

$$gx_n \longrightarrow T(x,y), \qquad gy_n \longrightarrow T(y,x) \quad \text{and} \quad gz_n \longrightarrow T(z,t), \qquad gt_n \longrightarrow T(t,z) \tag{3.1}$$

for all $n \geq 1$. Since $(T(x,y), T(y,x)) = (gx, gy) = (gx_1, gy_1)$ is comparable to $(T(u,v), T(v,u)) = (gu, gv) = (gu_1, gv_1)$, then it is easy to show $(gx, gy) \leq (gu, gv)$. Recursively, we get that

$$(gx_n, gy_n) \leq (gu_n, gv_n) \quad \text{for all } n. \tag{3.2}$$

Thus from (2.1) we have

$$\begin{aligned}
\psi\big(d(gx, gu_{n+1})\big) &\leq \psi\big(s^3 d(gx, gu_{n+1})\big) = \psi\big(s^3 d\big(T(x,y), T(u_n, v_n)\big)\big) \\
&\leq \psi\big(M_{s,T,g}(x,y,u_n,v_n)\big) - \phi\big(M_{s,T,g}(x,y,u_n,v_n)\big) \\
&\quad + L\theta\big(N_{T,g}(x,y,u_n,v_n)\big),
\end{aligned}$$

where

$$\begin{aligned}
M_{s,T,g}(x,y,u_n,v_n) = \max\bigg\{ & d(gx, gu_n), d(gy, gv_n), d\big(gx, T(x,y)\big), \\
& \frac{1}{2s} d\big(gu_n, T(u_n, v_n)\big), d\big(gy, T(y,x)\big), \\
& \frac{1}{2s} d\big(gv_n, T(v_n, u_n)\big), \frac{d(gx, T(u_n, v_n)) + d(gu_n, T(x,y))}{2s}, \\
& \frac{d(gy, T(v_n, u_n)) + d(gv_n, T(y,x))}{2s} \bigg\} \\
\leq \max\big\{ & d(gx, gu_n), d(gy, gv_n), d(gy, gv_{n+1}), d(gx, gu_{n+1})\big\}.
\end{aligned}$$

It is easy to show that

$$M_{s,T,g}(x,y,u_n,v_n) \leq \max\big\{d(gx, gu_n), d(gy, gv_n)\big\}$$

and

$$N_{T,g}(x,y,u_n,v_n) = 0.$$

Hence,

$$\begin{aligned}
\psi\big(d(gx, gu_{n+1})\big) \leq{}& \psi\big(\max\big\{d(gx, gu_n), d(gy, gv_n)\big\}\big) \\
&- \phi\big(\max\big\{d(gx, gu_n), d(gy, gv_n)\big\}\big).
\end{aligned} \tag{3.3}$$

Similarly, one can prove that

$$\begin{aligned}
\psi\big(d(gy, gv_{n+1})\big) \leq{}& \psi\big(\max\big\{d(gx, gu_n), d(gy, gv_n)\big\}\big) \\
&- \phi\big(\max\big\{d(gx, gu_n), d(gy, gv_n)\big\}\big).
\end{aligned} \tag{3.4}$$

Combining (3.3), (3.4) and the fact that $\max\{\psi(a), \psi(b)\} = \psi(\max\{a,b\})$ for $a, b \in [0, +\infty)$, we have

$$\begin{aligned}
&\psi\big(\max\big\{d(gx, gu_{n+1}), d(gy, gv_{n+1})\big\}\big) \\
&\quad = \max\big\{\psi\big(d(gx, gu_{n+1})\big), \psi\big(d(gy, gv_{n+1})\big)\big\}
\end{aligned}$$

$$\begin{aligned} &\leq \psi\bigl(\max\bigl\{d(gx,gu_n),d(gy,gv_n)\bigr\}\bigr)\phi\bigl(\max\bigl\{d(gx,gu_n),d(gy,gv_n)\bigr\}\bigr) \\ &\leq \psi\bigl(\max\bigl\{d(gx,gu_n),d(gy,gv_n)\bigr\}\bigr). \end{aligned} \tag{3.5}$$

Using the non-decreasing property of ψ, we get that

$$\max\bigl\{d(gx,gu_{n+1}),d(gy,gv_{n+1})\bigr\} \leq \max\bigl\{d(gx,gu_n),d(gy,gv_n)\bigr\}$$

implies that $\max\{d(gx,gu_n),d(gy,gv_n)\}$ is a non-increasing sequence. Hence, there exists $r \geq 0$ such that

$$\lim_{n\longrightarrow\infty} \max\bigl\{d(gx,gu_n),d(gy,gv_n)\bigr\} = r.$$

Passing the upper limit in (3.5) as $n \longrightarrow \infty$, we obtain

$$\psi(r) \leq \psi(r) - \phi(r),$$

which implies that $\phi(r) = 0$, and then $r = 0$. We deduce that

$$\lim_{n\longrightarrow\infty} \max\bigl\{d(gx,gu_n),d(gy,gv_n)\bigr\} = 0,$$

which concludes

$$\lim_{n\longrightarrow\infty} d(gx,gu_n) = \lim_{n\longrightarrow\infty} d(gy,gv_n) = 0. \tag{3.6}$$

Similarly, one can prove that

$$\lim_{n\longrightarrow\infty} d(gz,gu_n) = \lim_{n\longrightarrow\infty} d(gt,gv_n) = 0. \tag{3.7}$$

From (3.6) and (3.7), we have $gx = gz$ and $gy = gt$. Since $gx = T(x,y)$ and $gy = T(y,x)$, by the commutativity of T and g, we have

$$g(gx) = g\bigl(T(x,y)\bigr) = T(gx,gy), \qquad g(gy) = g\bigl(T(y,x)\bigr) = T(gy,gx). \tag{3.8}$$

Denote $gx = a$ and $gy = b$. Then from (3.8) we have

$$g(a) = T(a,b), \qquad g(b) = T(b,a). \tag{3.9}$$

Thus, (a,b) is a coupled coincidence point. It follows that $ga = gz$ and $gb = gy$, that is,

$$g(a) = a, \qquad g(b) = b. \tag{3.10}$$

From (3.9) and (3.10), we obtain

$$a = g(a) = T(a,b), \qquad b = g(b) = T(b,a). \tag{3.11}$$

Therefore, (a, b) is a coupled common fixed point of T and g. To prove the uniqueness of the point (a, b), assume that (c, d) is another coupled common fixed point of T and g. Then we have

$$c = gc = T(c, d), \qquad d = gd = T(d, c).$$

Since (c, d) is a coupled coincidence point of T and g, we have $gc = gx = a$ and $gd = gy = b$. Thus $c = gc = ga = a$ and $d = gd = gb = b$, which is the desired result. □

Theorem 3.2 *In addition to the hypotheses of Theorem* 3.1, *if gx_0 and gy_0 are comparable, then T and g have a unique common fixed point, that is, there exists $x \in X$ such that $x = gx = T(x, x)$.*

Proof Following the proof of Theorem 3.1, T and g have a unique coupled common fixed point (x, y). We only have to show that $x = y$. Since gx_0 and gy_0 are comparable, we may assume that $gx_0 \preceq gy_0$. By using the mathematical induction, one can show that

$$gx_n \preceq gy_n \quad \text{for all } n \geq 0, \tag{3.12}$$

where $\{gx_n\}$ and $\{gy_n\}$ are defined by (2.2). From (2.29) and Lemma 1.4, we have

$$\begin{aligned}
\psi\big(sd(x, y)\big) = \psi\left(s^3 \frac{1}{s^2} d(x, y)\right) &\leq \limsup_{n \longrightarrow \infty} \psi\big(s^3 d(gx_{n+1}, gy_{n+1})\big) \\
&= \limsup_{n \longrightarrow \infty} \psi\big(s^3 d\big(T(x_n, y_n), T(y_n, x_n)\big)\big) \\
&\leq \limsup_{n \longrightarrow \infty} \psi\big(M_{s,T,g}(x_n, y_n, y_n, x_n)\big) - \liminf_{n \longrightarrow \infty} \phi\big(M_{s,T,g}(x_n, y_n, y_n, x_n)\big) \\
&\quad + \limsup_{n \longrightarrow \infty} L\theta\big(N_{T,g}(x_n, y_n, y_n, x_n)\big) \\
&\leq \psi\big(d(x, y)\big) - \liminf_{n \longrightarrow \infty} \phi\big(M_s(x_n, y_n, y_n, x_n)\big) \\
&< \psi\big(d(x, y)\big),
\end{aligned}$$

a contradiction. Therefore, $x = y$, that is, T and g have a common fixed point. □

Remark 3.3 Since a b-metric is a metric when $s = 1$, from the results of Jachymski [29], the condition

$$\psi\big(d\big(F(x, y), F(u, v)\big)\big) \leq \psi\big(\max\big\{d(gx, gu), d(gy, gv)\big\}\big) - \phi\big(\max\big\{d(gx, gu), d(gy, gv)\big\}\big)$$

is equivalent to

$$d\big(F(x, y), F(u, v)\big) \leq \varphi\big(\max\big\{d(gx, gu), d(gy, gv)\big\}\big),$$

where $\psi \in \Psi$, $\phi \in \Phi$ and $\varphi : [0, \infty) \longrightarrow [0, \infty)$ is continuous, $\varphi(t) < t$ for all $t > 0$ and $\varphi(t) = 0$ if and only if $t = 0$. So, our results can be viewed as a generalization and extension of the corresponding results in [15, 25, 30–32] and several other comparable results.

4 Application to integral equations

Here, in this section, we wish to study the existence of a unique solution to a nonlinear quadratic integral equation, as an application to our coupled fixed point theorem. Consider the nonlinear quadratic integral equation

$$x(t) = h(t) + \lambda \int_0^1 k_1(t,s)f_1\big(s,x(s)\big)\,ds \int_0^1 k_2(t,s)f_2\big(s,x(s)\big)\,ds, \quad t \in I = [0,1], \lambda \geq 0. \quad (4.1)$$

Let Γ denote the class of those functions $\gamma : [0,+\infty) \longrightarrow [0,+\infty)$ which satisfy the following conditions:

(i) γ is non-decreasing and $(\gamma(t))^p \leq \gamma(t^p)$ for all $p \geq 1$.

(ii) There exists $\phi \in \Phi$ such that $\gamma(t) = t - \phi(t)$ for all $t \in [0,+\infty)$.

For example, $\gamma_1(t) = kt$, where $0 \leq k < 1$ and $\gamma_2(t) = \frac{t}{t+1}$ are in Γ.

We will analyze Eq. (4.1) under the following assumptions:

(a_1) $f_i : I \times \mathbb{R} \longrightarrow \mathbb{R}$ $(i = 1,2)$ are continuous functions, $f_i(t,x) \geq 0$ and there exist two functions $m_i \in L^1(I)$ such that $f_i(t,x) \leq m_i(t)$ $(i = 1,2)$.

(a_2) $f_1(t,x)$ is monotone non-decreasing in x and $f_2(t,y)$ is monotone non-increasing in y for all $x,y \in \mathbb{R}$ and $t \in I$.

(a_3) $h : I \longrightarrow \mathbb{R}$ is a continuous function.

(a_4) $k_i : I \times I \longrightarrow \mathbb{R}$ $(i = 1,2)$ are continuous in $t \in I$ for every $s \in I$ and measurable in $s \in I$ for all $t \in I$ such that

$$\int_0^1 k_i(t,s)m_i(s)\,ds \leq K, \quad i = 1,2,$$

and $k_i(t,x) \geq 0$.

(a_5) There exist constants $0 \leq L_i < 1$ $(i = 1,2)$ and $\gamma \in \Gamma$ such that for all $x,y \in \mathbb{R}$ and $x \geq y$,

$$\big|f_i(t,x) - f_i(t,y)\big| \leq L_i\gamma(x-y) \quad (i = 1,2).$$

(a_6) There exist $\alpha, \beta \in C(I)$ such that

$$\begin{aligned}\alpha(t) &\leq h(t) + \lambda \int_0^1 k_1(t,s)f_1\big(s,\alpha(s)\big)\,ds \int_0^1 k_2(t,s)f_2\big(s,\beta(s)\big)\,ds \\ &\leq h(t) + \lambda \int_0^1 k_1(t,s)f_1\big(s,\beta(s)\big)\,ds \int_0^1 k_2(t,s)f_2\big(s,\alpha(s)\big)\,ds \leq \beta(t).\end{aligned}$$

(a_7) $\max\{L_1^p, L_2^p\}\lambda^p K^{2p} \leq \frac{1}{2^{4p-3}}$.

Consider the space $X = C(I)$ of continuous functions defined on $I = [0,1]$ with the standard metric given by

$$\rho(x,y) = \sup_{t\in I}\big|x(t) - y(t)\big| \quad \text{for } x,y \in C(I).$$

This space can also be equipped with a partial order given by

$$x,y \in C(I), \quad x \leq y \quad \Longleftrightarrow \quad x(t) \leq y(t) \quad \text{for any } t \in I.$$

Now, for $p \geq 1$, we define

$$d(x,y) = \bigl(\rho(x,y)\bigr)^p = \Bigl(\sup_{t\in I}\bigl|x(t)-y(t)\bigr|\Bigr)^p = \sup_{t\in I}\bigl|x(t)-y(t)\bigr|^p \quad \text{for } x,y \in C(I).$$

It is easy to see that (X,d) is a complete b-metric space with $s = 2^{p-1}$ [3].

Also, $X \times X = C(I) \times C(I)$ is a partially ordered set if we define the following order relation:

$$(x,y),(u,v) \in X \times X, \quad (x,y) \leq (u,v) \quad \Longleftrightarrow \quad x \leq u \quad \text{and} \quad y \geq v.$$

For any $x,y \in X$ and each $t \in I$, $\max\{x(t),y(t)\}$ and $\min\{x(t),y(t)\}$ belong to X and are upper and lower bounds of x, y, respectively. Therefore, for every $(x,y),(u,v) \in X \times X$, one can take $(\max\{x,u\},\min\{y,v\}) \in X \times X$ which is comparable to (x,y) and (u,v). Now, we formulate the main result of this section.

Theorem 4.1 *Under assumptions* (a_1)-(a_7), *Eq.* (4.1) *has a unique solution in* $C(I)$.

Proof We consider the operator $T : X \times X \longrightarrow X$ defined by

$$T(x,y)(t) = h(t) + \lambda \int_0^1 k_1(t,s)f_1\bigl(s,x(s)\bigr)\,ds \int_0^1 k_2(t,s)f_2\bigl(s,y(s)\bigr)\,ds \quad \text{for } t \in I.$$

By virtue of our assumptions, T is well defined (this means that if $x,y \in X$, then $T(x,y) \in X$). Firstly, we prove that T has the mixed monotone property. In fact, for $x_1 \leq x_2$ and $t \in I$, we have

$$\begin{aligned}
T(x_1,y)(t) - T(x_2,y)(t) &= h(t) + \lambda \int_0^1 k_1(t,s)f_1\bigl(s,x_1(s)\bigr)\,ds \int_0^1 k_2(t,s)f_2\bigl(s,y(s)\bigr)\,ds \\
&\quad - h(t) - \lambda \int_0^1 k_1(t,s)f_1\bigl(s,x_2(s)\bigr)\,ds \int_0^1 k_2(t,s)f_2\bigl(s,y(s)\bigr)\,ds \\
&= \lambda \int_0^1 k_1(t,s)\bigl[f_1\bigl(s,x_1(s)\bigr) - f_1\bigl(s,x_2(s)\bigr)\bigr]\,ds \int_0^1 k_2(t,s)f_2\bigl(s,y(s)\bigr)\,ds \\
&\leq 0.
\end{aligned}$$

Similarly, if $y_1 \geq y_2$ and $t \in I$, then $T(x,y_1)(t) \leq T(x,y_2)(t)$. Therefore, T has the mixed monotone property. Also, for $(x,y) \leq (u,v)$, that is, $x \leq u$ and $y \geq v$, we have

$$\begin{aligned}
&\bigl|T(x,y)(t) - T(u,v)(t)\bigr| \\
&\quad \leq \biggl|\lambda \int_0^1 k_1(t,s)f_1\bigl(s,x(s)\bigr)\,ds \int_0^1 k_2(t,s)\bigl[f_2\bigl(s,y(s)\bigr) - f_2\bigl(s,v(s)\bigr)\bigr]\,ds \\
&\qquad + \lambda \int_0^1 k_2(t,s)f_2\bigl(s,v(s)\bigr)\,ds \int_0^1 k_1(t,s)\bigl[f_1\bigl(s,x(s)\bigr) - f_1\bigl(s,u(s)\bigr)\bigr]\,ds\biggr| \\
&\quad \leq \lambda \int_0^1 k_1(t,s)f_1\bigl(s,x(s)\bigr)\,ds \int_0^1 k_2(t,s)\bigl|f_2\bigl(s,y(s)\bigr) - f_2\bigl(s,v(s)\bigr)\bigr|\,ds \\
&\qquad + \lambda \int_0^1 k_2(t,s)f_2\bigl(s,v(s)\bigr)\,ds \int_0^1 k_1(t,s)\bigl|f_1\bigl(s,x(s)\bigr) - f_1\bigl(s,u(s)\bigr)\bigr|\,ds
\end{aligned}$$

$$\begin{aligned}
&\leq \lambda \int_0^1 k_1(t,s)m_1(s)\,ds \int_0^1 k_2(t,s)L_2\gamma\big(y(s)-v(s)\big)\,ds \\
&\quad + \lambda \int_0^1 k_2(t,s)m_2(s)\,ds \int_0^1 k_1(t,s)L_1\gamma\big(u(s)-x(s)\big)\,ds.
\end{aligned}$$

Since the function γ is non-decreasing and $x \leq u$ and $y \geq v$, we have

$$\gamma\big(u(s)-x(s)\big) \leq \gamma\Big(\sup_{t\in I}\big|x(s)-u(s)\big|\Big) = \gamma\big(\rho(x,u)\big)$$

and

$$\gamma\big(y(s)-v(s)\big) \leq \gamma\Big(\sup_{t\in I}\big|y(s)-v(s)\big|\Big) = \gamma\big(\rho(y,v)\big),$$

hence

$$\begin{aligned}
\big|T(x,y)(t)-T(u,v)(t)\big| &\leq \lambda K \int_0^1 k_2(t,s)L_2\gamma\big(\rho(y,v)\big)\,ds + \lambda K \int_0^1 k_1(t,s)L_1\gamma\big(\rho(u,x)\big)\,ds \\
&\leq \lambda K^2 \max\{L_1,L_2\}\big[\gamma\big(\rho(u,x)\big)+\gamma\big(\rho(y,v)\big)\big].
\end{aligned}$$

Then we can obtain

$$\begin{aligned}
d\big(T(x,y),T(u,v)\big) &= \sup_{t\in I}\big|T(x,y)(t)-T(u,v)(t)\big|^p \\
&\leq \big\{\lambda K^2 \max\{L_1,L_2\}\big[\gamma\big(\rho(u,x)\big)+\gamma\big(\rho(y,v)\big)\big]\big\}^p \\
&= \lambda^p K^{2p} \max\big\{L_1^p,L_2^p\big\}\big[\gamma\big(\rho(u,x)\big)+\gamma\big(\rho(y,v)\big)\big]^p,
\end{aligned}$$

and using the fact that $(a+b)^p \leq 2^{p-1}(a^p+b^p)$ for $a,b\in(0,+\infty)$ and $p>1$, we have

$$\begin{aligned}
d\big(T(x,y),T(u,v)\big) &\leq 2^{p-1}\lambda^p K^{2p} \max\big\{L_1^p,L_2^p\big\}\big[\big(\gamma\big(\rho(u,x)\big)\big)^p+\big(\gamma\big(\rho(y,v)\big)\big)^p\big] \\
&\leq 2^{p-1}\lambda^p K^{2p} \max\big\{L_1^p,L_2^p\big\}\big[\gamma\big(d(u,x)\big)+\gamma\big(d(y,v)\big)\big] \\
&\leq 2^{p}\lambda^p K^{2p} \max\big\{L_1^p,L_2^p\big\}\big[\gamma\big(M_s(x,y,u,v)\big)\big] \\
&\leq 2^{p}\lambda^p K^{2p} \max\big\{L_1^p,L_2^p\big\}\big[M_s(x,y,u,v)-\phi\big(M_s(x,y,u,v)\big)\big] \\
&\leq \frac{1}{2^{3p-3}}M_s(x,y,u,v) - \frac{1}{2^{3p-3}}\phi\big(M_s(x,y,u,v)\big).
\end{aligned}$$

This proves that the operator T satisfies the contractive condition (2.31) appearing in Corollary 2.4.

Finally, let α, β be the functions appearing in assumption (a_6); then, by (a_6), we get

$$\alpha \leq T(\alpha,\beta) \leq T(\beta,\alpha) \leq \beta.$$

Theorem 3.1 gives us that T has a unique coupled fixed point $(x^*,y^*)\in X\times X$. Since $\alpha\leq\beta$, Theorem 3.2 says that $x^* = y^*$ and this implies $x^* = T(x^*,x^*)$. So, $x^*\in C(I)$ is the unique solution of Eq. (4.1) and the proof is complete. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Each author contributed equally in the development of this manuscript. Both authors read and approved the final version of this manuscript.

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