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Some results on fixed points of α - ψ -Ciric generalized multifunctions

B Mohammadi¹, S Rezapour² and N Shahzad^{3*}

*Correspondence:

nshahzad@kau.edu.sa

³Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah, 21859, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

In 2012, Samet, Vetro and Vetro introduced α - ψ -contractive mappings and gave some results on a fixed point of the mappings (Samet et al. in *Nonlinear Anal.* 75:2154-2165, 2012). In fact, their technique generalized some ordered fixed point results (see (Alikhani et al. in *Filomat*, 2012, to appear) and (Samet et al. in *Nonlinear Anal.* 75:2154-2165, 2012)). By using the main idea of (Samet et al. in *Nonlinear Anal.* 75:2154-2165, 2012), we give some new results for α - ψ -Ciric generalized multifunctions and some related self-maps. Also, we give an affirmative answer to a recent open problem which was raised by Haghi, Rezapour and Shahzad in 2012.

Keywords: α - ψ -Ciric generalized multifunction; fixed point; quasi-contractive multifunction

1 Introduction

In 2012, Samet, Vetro and Vetro introduced α - ψ -contractive mappings and gave fixed point results for such mappings [1]. Their results generalized some ordered fixed point results (see [1]). Immediately, using their idea, some authors presented fixed point results in the field (see, for example, [2]). Denote by Ψ the family of nondecreasing functions $\psi : [0, +\infty) \rightarrow [0, +\infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each $t > 0$. It is well known that $\psi(t) < t$ for all $t > 0$. Let (X, d) be a metric space, $\beta : 2^X \times 2^X \rightarrow [0, +\infty)$ be a mapping and $\psi \in \Psi$. A multivalued operator $T : X \rightarrow 2^X$ is said to be β - ψ -contractive whenever $\beta(Tx, Ty)H(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$, where H is the Hausdorff distance (see [2]). Alikhani, Rezapour and Shahzad proved fixed point results for β - ψ -contractive multifunctions [2]. Let (X, d) be a metric space, $\alpha : X \times X \rightarrow [0, +\infty)$ be a mapping and $\psi \in \Psi$. We say that $T : X \rightarrow 2^X$ is an α - ψ -Ciric generalized multifunction if

$$\alpha(x, y)H(Tx, Ty) \leq \psi \left(\max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\} \right)$$

for all $x, y \in X$. One can find an idea of this notion in [3]. Also, we say that the self-map F on X is α -admissible whenever $\alpha(x, y) \geq 1$ implies $\alpha(Fx, Fy) \geq 1$ [1]. In this paper, we give fixed point results for α - ψ -Ciric generalized multifunctions.

In 2012, Haghi, Rezapour and Shahzad proved that some fixed point generalizations are not real ones [4]. Here, by presenting a result and an example, we are going to show that obtained results in this new field are real generalizations in respect to old similar results in the literature (compare the next result and the example with main results of [5] and [6]). The following result has been proved in [7].

Lemma 1.1 Let (X, d) be a complete metric space, $\alpha : X \times X \rightarrow [0, \infty)$ be a function, $\psi \in \Psi$ and T be a self-map on X such that

$$\alpha(x, y)d(Tx, Ty) \leq \psi \left(\max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2} [d(x, Ty) + d(y, Tx)] \right\} \right)$$

for all $x, y \in X$. Suppose that T is α -admissible and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Assume that if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \rightarrow x$, then $\alpha(x_n, x) \geq 1$ for all n . Then T has a fixed point.

Here, we give the following example which shows that Lemma 1.1 holds while we cannot use some similar old results, for example, Theorem 1 of [5].

Example 1.1 Let $M_1 = \{\frac{m}{n} : m = 0, 1, 3, 9, \dots \text{ and } n = 3k + 1(k \geq 0)\}$, $d(x, y) = |x - y|$, $M_2 = \{\frac{m}{n} : m = 1, 3, 9, 27, \dots \text{ and } n = 3k + 2(k \geq 0)\}$ and $M_3 = \{2k : k \geq 1\}$. Now, put $M = M_1 \cup M_2 \cup M_3$ and define the self-map $T : M \rightarrow M$ by $Tx = \frac{3x}{11}$ whenever $x \in M_1$, $Tx = \frac{x}{8}$ whenever $x \in M_2$ and $Tx = 2x$ whenever $x \in M_3$. Put $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\}$ for all $x, y \in M$. If $x \in M_1$ and $y \in M_2$, then $d(Tx, Ty) = |\frac{3x}{11} - \frac{y}{8}| = \frac{3}{11}|x - \frac{11y}{24}|$. Now, we consider two cases. If $x > \frac{11y}{24}$, then we have

$$\frac{3}{11} \left| x - \frac{11y}{24} \right| = \frac{3}{11} \left(x - \frac{11y}{24} \right) \leq \frac{3}{11} \left(x - \frac{y}{8} \right) = \frac{3}{11} d(x, Ty).$$

Hence, $d(Tx, Ty) \leq \frac{6}{11} \frac{d(x, Ty)}{2} \leq \frac{6}{11} M(x, y)$. If $x < \frac{11y}{24}$, then

$$\frac{3}{11} \left| x - \frac{11y}{24} \right| = \frac{3}{11} \left(\frac{11y}{24} - x \right) \leq \frac{3(y - x)}{11} = \frac{3}{11} d(x, y) \leq \frac{6}{11} M(x, y).$$

Hence, $d(Tx, Ty) \leq \frac{6}{11} M(x, y)$. If $x, y \in M_1$, then

$$d(Tx, Ty) = \left| \frac{3x}{11} - \frac{3y}{11} \right| = \frac{3}{11} d(x, y) \leq \frac{6}{11} M(x, y).$$

If $x, y \in M_2$, then $d(Tx, Ty) = |\frac{x}{8} - \frac{y}{8}| = \frac{1}{8} d(x, y) \leq \frac{6}{11} M(x, y)$. Define $\alpha(x, y) = 1$ whenever $x, y \in M_1 \cup M_2$ and $\alpha(x, y) = 1$ otherwise. Also, put $\psi(t) = \frac{6t}{11}$ for all $t \geq 0$. Then it is easy to see that $\alpha(x, y)d(Tx, Ty) \leq \psi(M(x, y))$ for all $x, y \in M$. Also, $\alpha(1, T1) = \alpha(1, 3/11) \geq 1$. If $\alpha(x, y) \geq 1$, then $x, y \in M_1 \cup M_2$ and so $\alpha(Tx, Ty) \geq 1$. Thus, T is α -admissible. It is easy to show that if $\{x_n\}$ is a sequence in M such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \rightarrow x$, then $\alpha(x_n, x) \geq 1$ for all n . Now, by using Theorem 1.1, T has a fixed point. Now, we show that Theorem 1 in [5] does not apply here. Put $x = 1$ and $y = 2$. Then $d(T1, T2) = |\frac{3}{11} - 4| = \frac{41}{11}$, $d(1, 2) = 1$, $d(1, T1) = d(1, \frac{3}{11}) = \frac{8}{11}$, $d(2, T2) = d(2, 4) = 2$, $d(1, T2) = d(1, 4) = 3$, $d(2, T1) = d(2, \frac{3}{11}) = \frac{19}{11}$. Thus, $N(1, 2) = 3$ and so $d(T1, T2) = \frac{41}{11} > 3 = N(1, 2)$, where $N(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$.

2 An affirmative answer for an open problem

After providing some results on fixed points of quasi-contractions on normal cone metric spaces by Ilic and Rakocevic in 2009 [8], Kadelburg, Radenovic and Rakocevic generalized the results by considering an additional assumption and deleting the assumption on

normality [9]. In 2011, Haghi, Rezapour and Shahzad proved same results without the additional assumption and for $\lambda \in (0, 1)$ [10]. Then Amini-Harandi proved a result on the existence of fixed points of set-valued quasi-contraction maps in metric spaces by using the technique of [10] (see [11]). But similar to [9], he could prove it only for $\lambda \in (0, \frac{1}{2})$ [11]. In 2012, Haghi, Rezapour and Shahzad proved the main result of [11] by using a simple method [12]. Also, they introduced quasi-contraction type multifunctions and showed that the main result of [11] holds for quasi-contraction type multifunctions. They raised an open problem about the difference between quasi-contraction and quasi-contraction type multifunctions [12]. In this section, we give a positive answer to the question. Let (X, d) be a metric space. Recall that the multifunction $T : X \rightarrow 2^X$ is called quasi-contraction whenever there exists $\lambda \in (0, 1)$ such that

$$H(Tx, Ty) \leq \lambda \max \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for all $x, y \in X$ [11]. Also, a multifunction $T : X \rightarrow 2^X$ is called quasi-contraction type whenever there exists $\lambda \in (0, 1)$ such that

$$H(Tx, Ty) \leq \lambda \max \{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for all $x, y \in X$ [12]. It is clear that each quasi-contraction type multifunction is a quasi-contractive multifunction. In 2012, Haghi, Rezapour and Shahzad raised this question that (see [12]): Is there any quasi-contractive multifunction which is not quasi-contraction type? By providing the following example, we give a positive answer to the problem.

Example 2.1 Let $X = [0, 2]$ and $d(x, y) = |x - y|$. Define $T : X \rightarrow CB(X)$ by $Tx = [1, \frac{5}{4}]$ whenever $x \in [0, 3/4]$, $Tx = [\frac{7}{8}, \frac{9}{8}]$ whenever $x \in (\frac{3}{4}, \frac{5}{4})$ and $Tx = [\frac{3}{4}, 1]$ whenever $x \in [\frac{5}{4}, 2]$. We show that T is a quasi-contraction while it is not a quasi-contractive type multifunction. Put $N(x, y) = \max \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$ for all $x, y \in X$. First, we show that T is a quasi-contraction. It is easy to check that $H(Tx, Ty) \leq \frac{1}{2}N(x, y)$ whenever $x, y \in [0, \frac{3}{4}]$ or $x, y \in (\frac{3}{4}, \frac{5}{4})$ or $x, y \in [\frac{5}{4}, 2]$. If $x \in [0, \frac{3}{4}]$ and $y \in [\frac{5}{4}, 2]$, then $H(Tx, Ty) = \frac{1}{4}$ and $d(x, y) \geq \frac{5}{4} - \frac{3}{4} = \frac{1}{2}$. Hence, $H(Tx, Ty) = \frac{1}{4} \leq \frac{1}{2}d(x, y) \leq \frac{1}{2}N(x, y)$. If $x \in [0, \frac{3}{4}]$ and $y \in (\frac{3}{4}, \frac{5}{4})$, then $H(Tx, Ty) = \frac{1}{8}$ and $d(x, Tx) \geq \frac{1}{4}$. Hence,

$$H(Tx, Ty) = \frac{1}{8} \leq \frac{1}{2}d(x, Tx) \leq \frac{1}{2}N(x, y).$$

If $x \in [\frac{5}{4}, 2]$ and $y \in (\frac{3}{4}, \frac{5}{4})$, then $H(Tx, Ty) = H([\frac{3}{4}, 1], [\frac{7}{8}, \frac{9}{8}]) = \frac{1}{8}$ and $d(x, Tx) \geq \frac{1}{4}$. Hence, $H(Tx, Ty) = \frac{1}{8} \leq \frac{1}{2}d(x, Tx) \leq \frac{1}{2}N(x, y)$. Therefore, T is a quasi-contraction with $\lambda = \frac{1}{2}$. Now, put $x = \frac{3}{4}$ and $y = \frac{5}{4}$. Then we have $d(\frac{3}{4}, T\frac{3}{4}) = d(\frac{3}{4}, [1, \frac{5}{4}]) = \frac{1}{4}$, $d(\frac{5}{4}, T\frac{5}{4}) = d(\frac{5}{4}, [\frac{3}{4}, 1]) = \frac{1}{4}$, $d(\frac{3}{4}, T\frac{5}{4}) = d(\frac{3}{4}, [\frac{3}{4}, 1]) = 0$ and $d(\frac{5}{4}, T\frac{3}{4}) = d(\frac{5}{4}, [1, \frac{5}{4}]) = 0$. Hence, $N_1(\frac{3}{4}, \frac{5}{4}) = \max \{d(\frac{3}{4}, T\frac{3}{4}), d(\frac{5}{4}, T\frac{5}{4}), d(\frac{3}{4}, T\frac{5}{4}), d(\frac{5}{4}, T\frac{3}{4})\} = \max \{\frac{1}{4}, 0\} = \frac{1}{4}$. Also, we have $H(T\frac{3}{4}, T\frac{5}{4}) = H([1, \frac{5}{4}], [\frac{3}{4}, 1]) = \frac{1}{4}$. Thus, $H(T\frac{3}{4}, T\frac{5}{4}) > \lambda N_1(\frac{3}{4}, \frac{5}{4})$ for all $\lambda \in (0, 1)$. This shows that T is not a quasi-contractive type multifunction.

3 Main results

Now, we are ready to state and prove our main results. Let (X, d) be a metric space, $\alpha : X \times X \rightarrow [0, \infty)$ be a mapping and $T : X \rightarrow CB(X)$ be a multifunction. We say that X satisfies the condition (C_α) whenever for each sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all n

and $x_n \rightarrow x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all k (see [13] for the idea of this notion). Recall that T is continuous whenever $H(Tx_n, Tx) \rightarrow 0$ for all sequence $\{x_n\}$ in X with $x_n \rightarrow x$. Also, we say that T is α -admissible whenever for each $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq 1$, we have $\alpha(y, z) \geq 1$ for all $z \in Ty$. Note that this notion is different from the notion of α_* -admissible multifunctions which has been provided in [7]. But, by providing a similar proof to that of Theorem 2.1 in [7], we can prove the following result.

Theorem 3.1 *Let (X, d) be a complete metric space, $\alpha : X \times X \rightarrow [0, \infty)$ be a function, $\psi \in \Psi$ be a strictly increasing map and $T : X \rightarrow CB(X)$ be an α -admissible multifunction such that $\alpha(x, y)H(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$ and there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) \geq 1$. If T is continuous or X satisfies the condition (C_α) , then T has a fixed point.*

Corollary 3.2 *Let (X, d) be a complete metric space, $\psi \in \Psi$ be a strictly increasing map, $x^* \in X$ and $T : X \rightarrow CB(X)$ be a multifunction such that $H(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$ with $x^* \in Tx \cap Ty$. Suppose that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $x^* \in Tx_0 \cap Tx_1$. Assume that for each $x \in X$ and $y \in Tx$ with $x^* \in Tx \cap Ty$, we have $x^* \in Ty \cap Tz$ for all $z \in Ty$. If T is continuous or X satisfies the condition (C_α) , then T has a fixed point.*

Proof Define $\alpha : X \times X \rightarrow [0, +\infty)$ by $\alpha(x, y) = 1$ whenever $x^* \in Tx \cap Ty$ and $\alpha(x, y) = 0$ otherwise. Then, by using Theorem 3.1, T has a fixed point. \square

Let (X, \leq) be an ordered set and $A, B \subseteq X$. We say that $A \leq B$ whenever for each $a \in A$ there exists $b \in B$ such that $a \leq b$.

Corollary 3.3 *Let (X, \leq, d) be a complete ordered metric space, $\psi \in \Psi$ be a strictly increasing map and $T : X \rightarrow CB(X)$ be a multifunction such that $H(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$ with $Tx \leq Ty$ or $Ty \leq Tx$. Suppose that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $Tx_0 \leq Tx_1$ or $Tx_1 \leq Tx_0$. Assume that for each $x \in X$ and $y \in Tx$ with $Tx \leq Ty$ or $Ty \leq Tx$, we have $Ty \leq Tz$ or $Tz \leq Ty$ for all $z \in Ty$. If T is continuous or X satisfies the condition (C_α) , then T has a fixed point.*

Now, we give the following result.

Theorem 3.4 *Let (X, d) be a complete metric space, $\alpha : X \times X \rightarrow [0, \infty)$ be a function, $\psi \in \Psi$ be a strictly increasing map and $T : X \rightarrow CB(X)$ be an α -admissible α - ψ -Ciric generalized multifunction, and let there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) \geq 1$. If T is continuous, then T has a fixed point.*

Proof If $x_1 = x_0$, then we have nothing to prove. Let $x_1 \neq x_0$. Then we have

$$\begin{aligned} d(x_1, Tx_1) &\leq \alpha(x_0, x_1)H(Tx_0, Tx_1) \\ &\leq \psi \left(\max \left\{ d(x_0, x_1), d(x_0, Tx_0), d(x_1, Tx_1), \frac{d(x_0, Tx_1) + d(x_1, Tx_0)}{2} \right\} \right) \\ &= \psi \left(\max \left\{ d(x_0, x_1), d(x_1, Tx_1), \frac{d(x_0, Tx_1)}{2} \right\} \right) \\ &\leq \psi \left(\max \left\{ d(x_0, x_1), d(x_1, Tx_1), \frac{d(x_0, x_1) + d(x_1, Tx_1)}{2} \right\} \right) \\ &= \psi \left(\max \{ d(x_0, x_1), d(x_1, Tx_1) \} \right). \end{aligned}$$

If $\max\{d(x_0, x_1), d(x_1, Tx_1)\} = d(x_1, Tx_1)$, then $d(x_1, Tx_1) \leq \psi(d(x_1, Tx_1))$ and so we get $d(x_1, Tx_1) = 0$. Thus, $d(x_0, x_1) = 0$, which is a contradiction. Hence, we obtain $\max\{d(x_0, x_1), d(x_1, Tx_1)\} = d(x_0, x_1)$ and so $d(x_1, Tx_1) \leq \psi(d(x_0, x_1))$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T . Let $x_1 \notin Tx_1$ and $q > 1$. Then

$$0 < d(x_1, Tx_1) < q\psi(d(x_0, x_1)).$$

Put $t_0 = d(x_0, x_1)$. Then $t_0 > 0$ and $d(x_1, Tx_1) < q\psi(t_0)$. Hence, there exists $x_2 \in Tx_1$ such that $d(x_1, x_2) < q\psi(t_0)$ and so $\psi(d(x_1, x_2)) < \psi(q\psi(t_0))$. It is clear that $x_2 \neq x_1$. Put $q_1 = \frac{\psi(q\psi(t_0))}{\psi(d(x_1, x_2))}$. Then $q_1 > 1$ and we have

$$\begin{aligned} d(x_2, Tx_2) &\leq \alpha(x_1, x_2)H(Tx_1, Tx_2) \\ &\leq \psi\left(\max\left\{d(x_1, x_2), d(x_1, Tx_1), d(x_2, Tx_2), \frac{d(x_1, Tx_2) + d(x_2, Tx_1)}{2}\right\}\right) \\ &= \psi\left(\max\left\{d(x_1, x_2), d(x_2, Tx_2), \frac{d(x_1, Tx_2)}{2}\right\}\right) \\ &\leq \psi(\max\{d(x_1, x_2), d(x_2, Tx_2)\}). \end{aligned}$$

Similarly, we should have $\max\{d(x_1, x_2), d(x_2, Tx_2)\} = d(x_2, Tx_2)$ and so we get

$$d(x_2, Tx_2) \leq \psi(d(x_1, x_2)).$$

If $x_2 \in Tx_2$, then x_2 is a fixed point of T . Let $x_2 \notin Tx_2$. Then $0 < d(x_2, Tx_2) < q_1\psi(d(x_1, x_2))$. Hence, there exists $x_3 \in Tx_2$ such that $d(x_2, x_3) < q_1\psi(d(x_1, x_2)) = \psi(q\psi(t_0))$. It is clear that $x_3 \neq x_2$ and $\psi(d(x_2, x_3)) < \psi^2(q\psi(t_0))$. Put $q_2 = \frac{\psi^2(q\psi(t_0))}{\psi(d(x_2, x_3))}$. Then $q_2 > 1$. Also, we have

$$\begin{aligned} d(x_3, Tx_3) &\leq \alpha(x_2, x_3)H(Tx_2, Tx_3) \\ &\leq \psi\left(\max\left\{d(x_2, x_3), d(x_2, Tx_2), d(x_3, Tx_3), \frac{d(x_2, Tx_3) + d(x_3, Tx_2)}{2}\right\}\right) \\ &= \psi\left(\max\left\{d(x_2, x_3), d(x_3, Tx_3), \frac{d(x_2, Tx_3)}{2}\right\}\right) \\ &\leq \psi(\max\{d(x_2, x_3), d(x_3, Tx_3)\}). \end{aligned}$$

By continuing this process, we obtain a sequence $\{x_n\}$ in X such that $x_n \in Tx_{n-1}$, $x_n \neq x_{n-1}$ and $d(x_n, x_{n+1}) \leq \psi^{n-1}(q\psi(t_0))$ for all n . Let $m > n$. Then

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \psi^{i-1}(q\psi(t_0))$$

and so $\{x_n\}$ is a Cauchy sequence in X . Hence, there exists $x^* \in X$ such that $x_n \rightarrow x^*$. If T is continuous, then

$$d(x^*, Tx^*) = \lim_{n \rightarrow \infty} d(x_{n+1}, Tx^*) \leq \lim_{n \rightarrow \infty} H(Tx_n, Tx^*) = 0$$

and so $x^* \in Tx^*$. □

Corollary 3.5 Let (X, d) be a complete metric space, $\psi \in \Psi$ be a strictly increasing map, $x^* \in X$ and $T : X \rightarrow CB(X)$ be a multifunction such that

$$H(Tx, Ty) \leq \psi \left(\max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\} \right)$$

for all $x, y \in X$ with $x^* \in Tx \cap Ty$. Suppose that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $x^* \in Tx_0 \cap Tx_1$. Assume that for each $x \in X$ and $y \in Tx$ with $x_0 \in Tx \cap Ty$, we have $x_0 \in Ty \cap Tz$ for all $z \in Ty$. If T is continuous, then T has a fixed point.

Corollary 3.6 Let (X, \preceq, d) be a complete ordered metric space, $\psi \in \Psi$ be a strictly increasing map and $T : X \rightarrow CB(X)$ be a multifunction such that

$$H(Tx, Ty) \leq \psi \left(\max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\} \right)$$

for all $x, y \in X$ with $Tx \preceq Ty$ or $Ty \preceq Tx$. Suppose that there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $Tx_0 \preceq Tx_1$ or $Tx_1 \preceq Tx_0$. Assume that for each $x \in X$ and $y \in Tx$ with $Tx \preceq Ty$ or $Ty \preceq Tx$, we have $Ty \preceq Tz$ or $Tz \preceq Ty$ for all $z \in Ty$. If T is continuous, then T has a fixed point.

Now, we give the following result about a fixed point of self-maps on complete metric spaces.

Theorem 3.7 Let (X, d) be a complete metric space, $\alpha : X \times X \rightarrow [0, \infty)$ be a mapping, $\phi : [0, \infty) \rightarrow [0, \infty)$ be a continuous and nondecreasing map such that $\phi(t) < t$ for all $t > 0$ and T be a self-map on X such that

$$\alpha(x, y)d(Tx, Ty) \leq \phi \left(\max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\} \right)$$

for all $x, y \in X$. Assume that there exists $x_0 \in X$ such that $\alpha(T^i x_0, T^j x_0) \geq 1$ for all $i, j \geq 0$ with $i \neq j$. Suppose that T is continuous or $\alpha(T^i x_0, x) \geq 1$ for all $i \geq 0$ whenever $T^i x_0 \rightarrow x$. Then T has a fixed point.

Proof It is easy to check that $\lim_{n \rightarrow \infty} \phi^n(t) = 0$ for all $t > 0$. Let $x_n = Tx_{n-1}$ for all $n \geq 1$. If $x_n = x_{n-1}$ for some n , then x_{n-1} is a fixed point of T . Suppose that $x_n \neq x_{n-1}$ for all n . Then we have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \leq \alpha(x_n, x_{n-1})d(Tx_n, Tx_{n-1}) \\ &\leq \phi \left(\max \left\{ d(x_n, x_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1}), \right. \right. \\ &\quad \left. \left. \frac{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})}{2} \right\} \right) \\ &= \phi \left(\max \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2} \right\} \right) \\ &\leq \phi \left(\max \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2} \right\} \right) \\ &= \phi \left(\max \{ d(x_{n-1}, x_n), d(x_n, x_{n+1}) \} \right). \end{aligned}$$

Since $d(x_{n+1}, x_n) \leq \phi(d(x_{n+1}, x_n))$ does not hold, we get $d(x_{n+1}, x_n) \leq \phi(d(x_n, x_{n-1}))$. Since ϕ is nondecreasing, we have

$$d(x_{n+1}, x_n) \leq \phi(d(x_n, x_{n-1})) \leq \phi^2(d(x_{n-1}, x_{n-2})) \leq \dots \leq \phi^n(d(x_1, x_0))$$

for all n . Hence, $d(x_{n+1}, x_n) \rightarrow 0$. If $\{x_n\}$ is not a Cauchy sequence, then there exists $\varepsilon > 0$ and subsequences $\{x_{n_i}\}$ and $\{x_{m_i}\}$ of $\{x_n\}$ with $n_i < m_i$ such that $d(x_{n_i}, x_{m_i}) > \varepsilon$ for all i . For each n_i , put $k_i = \min\{m_i \mid d(x_{n_i}, x_{m_i}) > \varepsilon\}$. Then we have

$$\varepsilon < d(x_{n_i}, x_{k_i}) \leq d(x_{n_i}, x_{k_i-1}) + d(x_{k_i-1}, x_{k_i}) \leq \varepsilon + d(x_{k_i-1}, x_{k_i})$$

and so $d(x_{n_i}, x_{k_i}) \rightarrow \varepsilon$. But we have

$$\begin{aligned} d(x_{n_i}, x_{k_i}) - d(x_{n_i}, x_{n_i+1}) - d(x_{k_i}, x_{k_i+1}) \\ \leq d(x_{n_i+1}, x_{k_i+1}) \leq d(x_{n_i}, x_{k_i}) + d(x_{n_i}, x_{n_i+1}) + d(x_{k_i}, x_{k_i+1}) \end{aligned}$$

and so $d(x_{n_i+1}, x_{k_i+1}) \rightarrow \varepsilon$. Thus, we get

$$\begin{aligned} \max \left\{ d(x_{n_i}, x_{k_i}), d(x_{n_i+1}, x_{n_i}), d(x_{k_i+1}, x_{k_i}), \frac{1}{2} [d(x_{n_i+1}, x_{k_i}) + d(x_{k_i+1}, x_{n_i})] \right\} \\ \leq \max \left\{ d(x_{n_i}, x_{k_i}), d(x_{n_i+1}, x_{n_i}), d(x_{k_i+1}, x_{k_i}), \right. \\ \left. \frac{1}{2} [d(x_{n_i+1}, x_{k_i+1}) + d(x_{k_i+1}, x_{k_i}) + d(x_{k_i+1}, x_{k_i}) + d(x_{k_i}, x_{n_i})] \right\} \\ \leq \max \left\{ d(x_{n_i}, x_{k_i}), d(x_{n_i+1}, x_{n_i}), d(x_{k_i+1}, x_{k_i}), \right. \\ \left. \frac{1}{2} [2d(x_{n_i}, x_{k_i}) + 3d(x_{k_i+1}, x_{k_i}) + d(x_{n_i+1}, x_{n_i})] \right\} \\ \leq d(x_{n_i}, x_{k_i}) + \frac{3}{2} d(x_{k_i+1}, x_{k_i}) + d(x_{n_i+1}, x_{n_i}) \end{aligned}$$

and so $M(x_{n_i}, x_{k_i}) \rightarrow \varepsilon$. On the other hand, we have

$$d(x_{n_i+1}, x_{k_i+1}) = d(Tx_{n_i}, Tx_{k_i}) \leq \alpha(x_{n_i}, x_{k_i})d(Tx_{n_i}, Tx_{k_i}) \leq \phi(M(x_{n_i}, x_{k_i}))$$

and so $\varepsilon \leq \phi(\varepsilon)$. This contradiction shows that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $x \in X$ such that $x_n \rightarrow x$. If T is continuous, then we get

$$d(x, Tx) = \lim_{n \rightarrow \infty} d(x_{n+1}, Tx) = \lim_{n \rightarrow \infty} d(Tx_n, Tx) = 0.$$

Now, suppose that $\alpha(T^i x_0, x) \geq 1$ for all $i \geq 0$ whenever $T^i x_0 \rightarrow x$. Then

$$\begin{aligned} d(x, Tx) &= \lim_{n \rightarrow \infty} d(x_{n+1}, Tx) = \lim_{n \rightarrow \infty} d(Tx_n, Tx) \leq \lim_{n \rightarrow \infty} \alpha(x_n, x)d(Tx_n, Tx) \\ &\leq \lim_{n \rightarrow \infty} \phi(\max\{d(x_n, x), d(x_n, x_{n+1}), d(x, Tx), (d(x_n, Tx) + d(x, x_{n+1}))/2\}) \\ &= \phi(d(x, Tx)). \end{aligned}$$

Thus, $d(x, Tx) = 0$. □

Here, we give the following example to show that there are discontinuous mappings satisfying the conditions of Theorem 3.7.

Example 3.1 Let $X = [0, \infty)$ and $d(x, y) = |x - y|$. Define $T : X \rightarrow X$ by $Tx = x + 2$ whenever $x \in [0, 1]$, $Tx = \frac{3}{2}$ whenever $x \in (1, 2)$ and $Tx = x^3 + 2x + 1$ whenever $x \in [2, \infty)$. Also, define the mappings $\phi : [0, \infty) \rightarrow [0, \infty)$ and $\alpha : X \times X \rightarrow [0, \infty)$ by $\phi(t) = \frac{t}{2}$, $\alpha(x, y) = 1$ whenever $x, y \in (1, 2)$ and $\alpha(x, y) = 0$ otherwise. An easy calculation shows us that

$$\alpha(x, y)d(Tx, Ty) \leq \phi \left(\max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\} \right)$$

for all $x, y \in X$. Put $x_0 = \frac{3}{2}$. Since $T^i x_0 = x_0$ for all $i \geq 1$, $\alpha(T^i x_0, T^j x_0) \geq 1$ for all $i, j \geq 0$ with $i \neq j$ and $\alpha(T^i x_0, x) \geq 1$ for all $i \geq 0$ whenever $T^i x_0 \rightarrow x$. Thus, the map T satisfies the conditions of Theorem 3.7. Note that $x_0 = \frac{3}{2}$ is a fixed point of T .

Corollary 3.8 Let (X, \leq, d) be a complete ordered metric space, $\phi : [0, \infty) \rightarrow [0, \infty)$ be a continuous and nondecreasing map such that $\phi(t) < t$ for all $t > 0$ and T be a self-map on X such that

$$d(Tx, Ty) \leq \phi \left(\max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\} \right)$$

for all comparable elements $x, y \in X$. Assume that there exists $x_0 \in X$ such that $T^i x_0$ and $T^j x_0$ are comparable for all $i, j \geq 0$ with $i \neq j$. Suppose that T is continuous or $T^i x_0$ and x are comparable for all $i \geq 0$ whenever $T^i x_0 \rightarrow x$. Then T has a fixed point.

Proof Define the mapping $\alpha : X \times X \rightarrow [0, +\infty)$ by $\alpha(x, y) = 1$ whenever x and y are comparable and $\alpha(x, y) = 0$ otherwise. Then, by using Theorem 3.7, T has a fixed point. \square

Corollary 3.9 Let (X, \leq, d) be a complete ordered metric space, $z \in X$, $\phi : [0, \infty) \rightarrow [0, \infty)$ be a continuous and nondecreasing map such that $\phi(t) < t$ for all $t > 0$ and T be a self-map on X such that

$$d(Tx, Ty) \leq \phi \left(\max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\} \right)$$

for all $x, y \in X$ which are comparable with z . Assume that there exists $x_0 \in X$ such that $T^i x_0$ and $T^j x_0$ are comparable with z for all $i, j \geq 0$ with $i \neq j$. Suppose that T is continuous or $T^i x_0$ and x are comparable with z for all $i \geq 0$ whenever $T^i x_0 \rightarrow x$. Then T has a fixed point.

Proof Define the mapping $\alpha : X \times X \rightarrow [0, +\infty)$ by $\alpha(x, y) = 1$ whenever x and y are comparable with z and $\alpha(x, y) = 0$ otherwise. Then, by using Theorem 3.7, T has a fixed point. \square

Finally, by using [14], we can find also some equivalent conditions for some presented results. We give two following results in this way. This shows us the importance of the main results of [14]. Also, Example 3.1 leads us to the fact that there are discontinuous mappings satisfying the conditions of the following results.

Proposition 3.10 Let (X, d) be a complete metric space, $\alpha : X \times X \rightarrow [0, \infty)$ be a mapping, $\psi : [0, \infty) \rightarrow [0, \infty)$ be a lower semi-continuous function, $\eta : [0, \infty) \rightarrow [0, \infty)$ be a map such that $\eta^{-1}(\{0\}) = \{0\}$ and $\liminf_{t \rightarrow \infty} \eta(t) > 0$ for all $t > 0$ and T be a self-map on X such that

$$\psi(\alpha(x, y)d(Tx, Ty)) \leq \psi(M(x, y)) - \eta(M(x, y))$$

for all $x, y \in X$. Assume that there exists $x_0 \in X$ such that $\alpha(T^i x_0, T^j x_0) \geq 1$ for all $i, j \geq 0$ with $i \neq j$. Suppose that T is continuous or $\alpha(T^i x_0, x) \geq 1$ for all $i \geq 0$ whenever $T^i x_0 \rightarrow x$. Then T has a fixed point.

Proposition 3.11 Let (X, d) be a complete metric space, $\alpha : X \times X \rightarrow [0, \infty)$ be a mapping, $\psi \in \Psi$ be a nondecreasing map which is continuous from right at each point, $\phi : [0, \infty) \rightarrow [0, \infty)$ be a map such that $\phi(t) < t$ for all $t > 0$ and T be a self-map on X such that $\psi(\alpha(x, y)d(Tx, Ty)) \leq \phi(\psi(M(x, y)))$ for all $x, y \in X$. Assume that there exists $x_0 \in X$ such that $\alpha(T^i x_0, T^j x_0) \geq 1$ for all $i, j \geq 0$ with $i \neq j$. Suppose that T is continuous or $\alpha(T^i x_0, x) \geq 1$ for all $i \geq 0$ whenever $T^i x_0 \rightarrow x$. Then T has a fixed point.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Author details

¹Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran. ²Department of Mathematics, Azarbaijan Shahid Madani University, Tabriz, Azarshahr, Iran. ³Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah, 21859, Saudi Arabia.

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