Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Nonlinear ergodic theorems and weak convergence theorems for reversible semigroup of asymptotically nonexpansive mappings in Banach spaces

Lanping Zhu, Qianglian Huang* and Gang Li

*Correspondence:
qlhmath@outlook.com
College of Mathematics, Yangzhou University, Yangzhou, 225002, China

Abstract

In this paper, we provide the nonlinear ergodic theorems and weak convergence theorems for almost orbits of a reversible semigroup of asymptotically nonexpansive mappings in a uniformly convex Banach space without assuming that X has a Fréchet differentiable norm. Since almost orbits in this paper are not almost asymptotically isometric, new methods have to be introduced and used for the proofs. Our main results include many well-known results as special cases and are new even for reversible semigroup of nonexpansive mappings.

MSC: 47H20

Keywords: reversible semigroups; Kadec-Klee property; asymptotically nonexpansive mapping; almost orbit; uniformly convex Banach space

1 Introduction

Baillon [1] proved the first nonlinear ergodic theorem for nonexpansive mappings in the framework of Hilbert space. Baillon's theorem was extended to various semigroups in Hilbert spaces [2–4] or Banach spaces [5–13]. For instance, Takahashi [2] proved the ergodic theorem for right reversible semigroups of nonexpansive mappings in a Hilbert space by using the methods of invariant means. Lau *et al.* [5] studied the existence of nonexpansive retractions for amenable semigroups of nonexpansive mappings and provided the nonlinear ergodic theorems in Banach spaces. Kim and Li [6] proved the ergodic theorem for the almost asymptotically isometric almost orbits of right reversible semigroups of asymptotically nonexpansive mappings in a uniformly convex Banach space with a Fréchet differentiable norm. Many papers about weak convergence of asymptotically nonexpansive semigroups in a uniformly convex Banach space with a Fréchet differentiable norm have appeared [6, 10, 11, 14–16]. In 2001, Falset, *et al.* [14], Kaczor [15] proved the weak convergence theorems of almost orbits of commutative semigroups of asymptotically nonexpansive mappings under the assumptions that the Banach space is uniformly convex, and its dual space has the Kadec-Klee property.

This paper is devoted to the study of the nonlinear ergodic theorem and weak convergence for almost orbits of reversible semigroups of asymptotically nonexpansive mappings. Using the technique of product net, we first obtain the nonlinear ergodic theorems without assuming that the uniformly convex Banach space has a Fréchet differentiable norm, which extend and unify many previously known results in [2, 6, 10, 11, 16]. Next, we establish the convergence theorem in the case of reversible semigroup and the uniformly convex Banach space whose dual space has the Kadec-Klee property, which improves the known ones (see [2, 11, 14–16]) for commutative semigroups of asymptotically nonexpansive mappings in a uniformly convex Banach space. It is safe to say that the many general and key assumptions in the situation of reversible semigroup, such as the almost orbit $u(\cdot)$ is almost asymptotically isometric, and the subspace D has a left invariant mean (see [2, 6, 10]), are not necessary in this paper. Our main results are new even for the reversible semigroup of nonexpansive mappings.

2 Preliminaries

Let C be a nonempty bounded closed convex subset of a Banach space X. Let X^* be the dual of X, then the value of $x^* \in X^*$ at $x \in X$ will be denoted by $\langle x, x^*\rangle$, and we associate the set

$$J(x) = \left\{x^* \in X^* : \langle x, x^*\rangle = \|x\|^2 = \left\|x^*\right\|^2\right\}.$$

It is clear from the Hahn-Banach theorem that $J(x) \neq \emptyset$ for all $x \in X$. Then the multi-valued operator $J : X \mapsto X^*$ is called the normalized duality mapping of X. We say that X has a Fréchet differentiable norm, *i.e.*, for each $x \neq 0$, $\lim_{t\to 0}(\|x + ty\| - \|x\|)/t$ exists uniformly in $y \in B_r = \{z \in X : \|z\| \leq r\}$, $r > 0$. We say that X has the Kadec-Klee property if for every sequence $\{x_n\}_{n\in N}$ in X, whenever ω-$\lim_{n\to\infty} x_n = x$ with $\lim_{n\to\infty} \|x_n\| = \|x\|$, it follows that $\lim_{n\to\infty} x_n = x$. Recall that X has the Kadec property if for every net $\{x_\alpha\}_{\alpha\in I}$ in X, whenever ω-$\lim_{\alpha\in I} x_\alpha = x$ with $\lim_{\alpha\in I} \|x_\alpha\| = \|x\|$, it follows that $\lim_{\alpha\in I} x_\alpha = x$, where I is a directed system. It is well known that within the class of reflexive spaces, the Kadec-Klee property is equivalent to the Kadec property [17]. We also would like to remark that a uniformly convex Banach space with a Fréchet differentiable norm implies that its dual has Kadec-Klee property, while the converse implication fails [14, 15].

Let G be a semitopological semigroup, *i.e.*, G is a semigroup with a Hausdorff topology such that for each $t \in G$, the mappings $s \mapsto st$ and $s \mapsto ts$ from G to G are continuous. G is called right reversible if any two closed left ideals of G have nonvoid intersection. In this case, $(G, \leq)$ is a directed system when the binary relation $\leq$ on G is defined by $s \leq t$ if and only if $\{s\} \cup \overline{Gs} \supseteq \{t\} \cup \overline{Gt}$, $s, t \in G$. Right reversible semitopological semigroups include all commutative semigroups and all semitopological semigroups, which are right amenable as discrete semigroups.

Let $m(G)$ be the Banach space of all bounded real valued functions on G with the supremum norm. Then for each $s \in G$ and $f \in m(G)$, we can define $l_s f$ in $m(G)$ by $(l_s f)(t) = f(st)$ for all $t \in G$. Let D be a subspace of $m(G)$ containing constant functions and invariant under l_s for every $s \in G$. Let D^* be the dual space of D, then the value of $\mu \in D^*$ at $f \in D$ will be denoted by $\mu(f) = \int f(t)\,d\mu(t) = \mu(t)f(t)$. A linear function μ on D is called a mean on D if $\|\mu\| = \mu(1) = 1$. Further, a mean μ on D is left invariant if for all $s \in G$ and $f \in D$, $\mu(l_s f) = \mu(f)$. For each $s \in G$, we define a point evaluation δ_s on D by $\delta_s(f) = f(s)$ for every $f \in D$. A convex combination of point evaluation is called a finite mean on G.

By Day [18], if D has a left invariant mean, then there exists a net $\{\lambda_\alpha : \alpha \in A\}$ of finite means on G such that

$$\lim_{\alpha\in A}\left\|\lambda_\alpha - l_s^*\lambda_\alpha\right\| = 0$$

for every $s \in G$, where A is a directed system, and l_s^* is the conjugate operator of l_s.

Let $\Im = \{T(t) : t \in G\}$ be a semigroup acting on C, *i.e.*, $T(ts)x = T(t)T(s)x$ for all $t, s \in G$ and $x \in C$. Recall that $\Im$ is said to be asymptotically nonexpansive [19–21] if there exists a function $\alpha(\cdot) : G \mapsto [0, +\infty)$ with $\limsup_{t\in G} \alpha(t) = 0$ such that for all $x, y \in C$ and $t \in G$,

$$\|T(t)x - T(t)y\| \leq (1 + \alpha(t))\|x - y\|.$$

If $\alpha(t) \equiv 0$ for every $t \in G$, then $\Im$ is said to be nonexpansive. It should be pointed out that there is a notion of asymptotically nonexpansive mappings defined dependent on right ideals in a semigroup in [22, 23].

A function $u(\cdot) : G \mapsto C$ is said to be an almost orbit of $\Im$ [16] if

$$\limsup_{t\in G}\Big[\sup_{h\in G}\|u(ht) - T(h)u(t)\|\Big] = 0.$$

Suppose that $u(\cdot)$ is an almost orbit of $\Im$ such that for each $x^* \in X^*$, the function $h_{x^*} : t \mapsto \langle u(t), x^*\rangle$ is in D. For each $\mu \in D^*$, since X is reflexive, there exists a unique u_μ in X such that $\langle u_\mu, x^*\rangle = \int \langle u(t), x^*\rangle \, d\mu(t)$ for all $x^* \in X^*$. We denote u_μ by $\mu(t)\langle u(t)\rangle$. If λ is a finite mean on G, say $\lambda = \sum_{i=1}^n a_i\delta_{s_i}$, where $s_i \in G$, $a_i \geq 0$, $i = 1, 2, \ldots, n$, and $\sum_{i=1}^n a_i = 1$, then

$$\lambda(t)\langle u(t)\rangle = \sum_{i=1}^n a_i u(s_i).$$

Throughout this paper, let C be a nonempty bounded closed convex subset of uniformly convex Banach space X, and let $\Im = \{T(t) : t \in G\}$ be a reversible semigroup of asymptotically nonexpansive mappings acting on C. Let $F(\Im)$ denote the set of all fixed points of $\Im$, *i.e.*, $F(\Im) = \{x \in C : T(t)x = x \text{ for all } t \in G\}$. For each $\varepsilon > 0$ and $h \in G$, we set

$$F_\varepsilon(T(h)) = \{x \in C : \|T(h)x - x\| \leq \varepsilon\}.$$

It should be noted that if for any $\varepsilon > 0$, there exists $h_\varepsilon \in G$ such that for all $h \geq h_\varepsilon$, $x \in F_\varepsilon(T(h))$, then $\lim_{h\in G} T(h)x = x$, and thus $x \in F(\Im)$ by the continuity of $\{T(h), h \in G\}$.

We denote by $\mathrm{AO}(\Im)$ the set of all almost orbits of $\Im$ and by $\mathrm{LAO}(\Im)$ the set $\{T(h)u(\cdot) : h \in G, u \in \mathrm{AO}(\Im)\}$. Denote by $\omega_\omega(u)$ the set of all weak limit points of subnets of the net $\{u(t)\}_{t\in G}$.

3 Lemmas

In this section, we prove some lemmas, which play a crucial role in the proof of our main theorems in next section.

Lemma 3.1 [24] *Let X be a Banach space and J be the normalized duality mapping. Then for given $x, y \in X$, the following inequality holds*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle$$

for all $j(x + y) \in J(x + y)$.

Lemma 3.2 [25] *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X. Then there exists a strictly increasing continuous convex function $\gamma : [0,+\infty) \mapsto [0,+\infty)$ with $\gamma(0) = 0$ such that*

$$\gamma\left(\left\| T\left(\sum_{i=1}^{n} a_i x_i\right) - \sum_{i=1}^{n} a_i T x_i \right\|\right) \le \max_{1\le i,j\le n} \left\{ \|x_i - x_j\| - \|Tx_i - Tx_j\| \right\}$$

for all integers $n \ge 1$, $a_1,\dots,a_n \ge 0$ with $\sum_{i=1}^{n} a_i = 1$, $x_1,\dots,x_n \in C$, and all nonexpansive mapping $T : C \mapsto C$.

From Lemma 3.2, we can get for all $a_1,\dots,a_n \ge 0$ with $\sum_{i=1}^{n} a_i = 1$, $x_1,\dots,x_n \in C$,

$$\begin{aligned}
&\left\| T(h)\left(\sum_{i=1}^{n} a_i x_i\right) - \sum_{i=1}^{n} a_i T(h) x_i \right\| \\
&\quad \le \bigl(1+\alpha(h)\bigr)\gamma^{-1}\left(\max_{1\le i,j\le n}\left\{ \|x_i - x_j\| - \frac{1}{1+\alpha(h)}\bigl\| T(h)x_i - T(h)x_j \bigr\| \right\}\right) \\
&\quad \le \bigl(1+\alpha(h)\bigr)\gamma^{-1}\Bigl(\max_{1\le i,j\le n}\left\{ \|x_i - x_j\| - \bigl\| T(h)x_i - T(h)x_j \bigr\| \right\} + d\cdot\alpha(h)\Bigr),
\end{aligned}$$

where $d = 4\sup\{\|x\| : x \in C\} + 1$.

To simplify, in the following, for each $\varepsilon \in (0,1]$, we define

$$a(\varepsilon) = \min\left\{ \frac{\varepsilon^2}{(d+2)^2}, \frac{\varepsilon^3}{(3d+2)^2}\gamma\left(\frac{\varepsilon}{4}\right)\right\}$$

and

$$G_\varepsilon = \bigl\{ h \in G : \alpha(h) \le \varepsilon \bigr\},$$

where $\gamma(\cdot)$ is as in Lemma 3.2. Then G_ε is nonempty for each $\varepsilon > 0$, and if $h \in G_\varepsilon$, then for all $t \ge h$, $t \in G_\varepsilon$. And it should also be noted $G_{a(\varepsilon)} \subset G_\varepsilon$ for all $\varepsilon \in (0,1]$.

Lemma 3.3 *For all $h \in G_{a(\varepsilon)}$,*

$$\overline{\mathrm{co}} F_{a(\varepsilon)}\bigl(T(h)\bigr) \subset F_\varepsilon\bigl(T(h)\bigr).$$

Proof Since $F_\varepsilon(T(h))$ is closed, we only need to prove that for all $h \in G_{a(\varepsilon)}$,

$$\mathrm{co}\, F_{a(\varepsilon)}\bigl(T(h)\bigr) \subset F_\varepsilon\bigl(T(h)\bigr).$$

Let $y = \sum_{i=1}^{n} a_i y_i$, $y_i \in F_{a(\varepsilon)}(T(h))$, $a_i \ge 0$, $i = 1,\dots,n$, and $\sum_{i=1}^{n} a_i = 1$. Then

$$\begin{aligned}
&\bigl\| T(h)y - y \bigr\| \\
&\quad = \left\| T(h)\sum_{i=1}^{n} a_i y_i - \sum_{i=1}^{n} a_i y_i \right\| \\
&\quad \le \left\| T(h)\sum_{i=1}^{n} a_i y_i - \sum_{i=1}^{n} a_i T(h) y_i \right\| + \left\| \sum_{i=1}^{n} a_i T(h) y_i - \sum_{i=1}^{n} a_i y_i \right\|
\end{aligned}$$

$$
\begin{aligned}
&\le 2\gamma^{-1}\Big(\max_{1\le i,j\le n}\big\{\|y_i - y_j\| - \|T(h)y_i - T(h)y_j\|\big\} + d\cdot\alpha(h)\Big) + a(\varepsilon)\\
&\le 2\gamma^{-1}\Big(\max_{1\le i,j\le n}\big\{\|y_i - T(h)y_i\| + \|y_j - T(h)y_j\|\big\} + d\cdot\alpha(h)\Big) + a(\varepsilon)\\
&\le 2\gamma^{-1}\big(2a(\varepsilon) + d\cdot a(\varepsilon)\big) + a(\varepsilon)\\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}
$$

This completes the proof. □

Lemma 3.4 *For all $h \in G_{\frac{\varepsilon}{4}}$,*

$$
F_{\frac{\varepsilon}{4}}\big(T(h)\big) + B\left(0, \frac{\varepsilon}{4}\right) \subset F_{\varepsilon}\big(T(h)\big).
$$

Proof Let $h \in G_{\frac{\varepsilon}{4}}$ and $x = y + z \in F_{\frac{\varepsilon}{4}}(T(h)) + B(0, \frac{\varepsilon}{4})$, where $y \in F_{\frac{\varepsilon}{4}}(T(h))$ and $z \in B(0, \frac{\varepsilon}{4})$, then

$$
\begin{aligned}
&\|T(h)x - x\|\\
&\quad = \|T(h)(y+z) - (y+z)\|\\
&\quad \le \|T(h)(y+z) - T(h)y\| + \|T(h)y - y\| + \|z\|\\
&\quad \le 2\|z\| + \|T(h)y - y\| + \|z\|\\
&\quad \le 3\cdot\frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}
$$

This completes the proof. □

Lemma 3.5 *Let $\varepsilon \in (0,1]$ and $h \in G_{a(a(\frac{\varepsilon}{4}))}$, then there exists an $n_0 \in N$ such that for all $n \ge n_0$ and $x \in C$,*

$$
\frac{1}{n}\sum_{i=1}^{n} T\big(h^i\big)x \in F_{\varepsilon}\big(T(h)\big).
$$

Proof Let $\varepsilon \in (0,1]$ and $m = \frac{2d+1}{a(\frac{\varepsilon}{4})}$, there is an $n_0 \in N$ satisfying

$$
n_0 \ge \max\left\{\frac{12md}{\varepsilon}, 32m^2 d(d+1)\left(\gamma\left(\frac{a(\frac{\varepsilon}{4})}{2}\right)\varepsilon\right)^{-1}\right\}.
$$

For any given $n \ge n_0$ and $h \in G_{a(a(\frac{\varepsilon}{4}))}$, we can take a number

$$
K = m^2 d\big(1 + 2n\alpha(h)\big)\left(\gamma\left(\frac{a(\frac{\varepsilon}{4})}{2}\right)\right)^{-1} \quad \left(K < \frac{n}{2}\right).
$$

For each $i \in N$ and $x \in C$, we set

$$
a_i(x) = \gamma\left(\frac{8}{9}\left\|\frac{1}{m}\sum_{j=1}^{m} T\big(h^{i+j+1}\big)x - T(h)\frac{1}{m}\sum_{j=1}^{m} T\big(h^{i+j}\big)x\right\|\right).
$$

Noting $\alpha(h) \leq \frac{1}{8}$ and

$$\begin{aligned} a_i(x) &\leq \max_{1\leq j,k\leq m} \left\{ \left\| T\left(h^{i+j}\right)x - T\left(h^{i+k}\right)x \right\| - \left\| T\left(h^{i+j+1}\right)x - T\left(h^{i+k+1}\right)x \right\| + d\cdot\alpha(h) \right\} \\ &\leq \sum_{1\leq j<k\leq m} \left(\left\| T\left(h^{i+j}\right)x - T\left(h^{i+k}\right)x \right\| - \left\| T\left(h^{i+j+1}\right)x - T\left(h^{i+k+1}\right)x \right\| + d\cdot\alpha(h) \right), \end{aligned}$$

we get

$$\begin{aligned} &\sum_{i=1}^{n} a_i(x) \\ &\quad\leq \sum_{i=1}^{n} \sum_{1\leq j<k\leq m} \left(\left\| T\left(h^{i+j}\right)x - T\left(h^{i+k}\right)x \right\| - \left\| T\left(h^{i+j+1}\right)x - T\left(h^{i+k+1}\right)x \right\| + d\cdot\alpha(h) \right) \\ &\quad= \sum_{1\leq j<k\leq m} \sum_{i=1}^{n} \left(\left\| T\left(h^{i+j}\right)x - T\left(h^{i+k}\right)x \right\| - \left\| T\left(h^{i+j+1}\right)x - T\left(h^{i+k+1}\right)x \right\| + d\cdot\alpha(h) \right) \\ &\quad\leq \sum_{1\leq j<k\leq m} \left(d + nd\cdot\alpha(h) \right) \leq m^2 d\left(1 + n\alpha(h)\right). \end{aligned}$$

Suppose that there are k elements in $\{a_i(x) : i = 1, 2, \ldots, 2n\}$ such that $a_i(x) \geq \gamma(\frac{a(\frac{\varepsilon}{4})}{2})$, then

$$k\gamma\left(\frac{a(\frac{\varepsilon}{4})}{2}\right) \leq m^2 d\left(1 + 2n\alpha(h)\right).$$

Hence

$$k \leq m^2 d\left(1 + 2n\alpha(h)\right)\left(\gamma\left(\frac{a(\frac{\varepsilon}{4})}{2}\right)\right)^{-1} = K.$$

Thus, there are at most $N = [K]$ terms in $\{a_i(x) : i = 1, 2, \ldots, 2n\}$ with $a_i(x) \geq \gamma(\frac{a(\frac{\varepsilon}{4})}{2})$. Therefore, for each $i \in \{1, 2, \ldots, n\}$, there is at least one term $a_{i+j_0}(x)$ $(0 \leq j_0 \leq N)$ in $\{a_{i+j}(x) : j = 0, 1, \ldots, N\}$ satisfying $a_{i+j_0}(x) < \gamma(\frac{a(\frac{\varepsilon}{4})}{2})$.

Putting

$$l_i = \min\left\{ j : a_{i+j}(x) < \gamma\left(\frac{a(\frac{\varepsilon}{4})}{2}\right), 0 \leq j \leq N \right\},$$

$i = 1, 2, \ldots, n$. It is easy to see that there are at most N elements in $\{i : i = 1, 2, \ldots, n\}$ such that $l_i \neq 0$. Since

$$\begin{aligned} &\left\| T(h)\frac{1}{m}\sum_{j=1}^{m} T\left(h^{i+l_i+j}\right)x - \frac{1}{m}\sum_{j=1}^{m} T\left(h^{i+l_i+j}\right)x \right\| \\ &\quad\leq \left\| T(h)\frac{1}{m}\sum_{j=1}^{m} T\left(h^{i+l_i+j}\right)x - \frac{1}{m}\sum_{j=1}^{m} T\left(h^{i+l_i+j+1}\right)x \right\| \\ &\qquad+ \left\| \frac{1}{m}\sum_{j=1}^{m} T\left(h^{i+l_i+j}\right)x - \frac{1}{m}\sum_{j=1}^{m} T\left(h^{i+l_i+j+1}\right)x \right\| \end{aligned}$$

$$\begin{aligned}
&\leq \frac{9}{8}\gamma^{-1}\bigl(a_{i+l_i}(x)\bigr) + \frac{d}{2m} \\
&\leq \frac{9}{16}a\left(\frac{\varepsilon}{4}\right) + \frac{1}{4}a\left(\frac{\varepsilon}{4}\right) < a\left(\frac{\varepsilon}{4}\right),
\end{aligned}$$

we can conclude that for all $h \in G_{a(a(\frac{\varepsilon}{4}))}$,

$$\frac{1}{m}\sum_{j=1}^{m} T\bigl(h^{i+l_i+j}\bigr)x \in F_{a(\frac{\varepsilon}{4})}\bigl(T(h)\bigr).$$

By Lemma 3.3, we get for all $h \in G_{a(a(\frac{\varepsilon}{4}))} \subset G_{a(\frac{\varepsilon}{4})}$,

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{m}\sum_{j=1}^{m} T\bigl(h^{i+l_i+j}\bigr)x \in \operatorname{co} F_{a(\frac{\varepsilon}{4})}\bigl(T(h)\bigr) \subset F_{\frac{\varepsilon}{4}}\bigl(T(h)\bigr).$$

Using Lemma 3.4 and

$$\begin{aligned}
&\left\| \frac{1}{n}\sum_{i=1}^{n} T\bigl(h^{i}\bigr)x - \frac{1}{n}\sum_{i=1}^{n}\frac{1}{m}\sum_{j=1}^{m} T\bigl(h^{i+l_i+j}\bigr)x \right\| \\
&\quad\leq \frac{1}{mn}\sum_{j=1}^{m}\left\| \sum_{i=1}^{n} T\bigl(h^{i}\bigr)x - \sum_{i=1}^{n} T\bigl(h^{i+l_i+j}\bigr)x \right\| \\
&\quad\leq \frac{1}{mn}\sum_{j=1}^{m}\left\| \sum_{i=1}^{n} T\bigl(h^{i}\bigr)x - \sum_{i=1}^{n} T\bigl(h^{i+j}\bigr)x \right\| + \frac{1}{mn}\sum_{j=1}^{m}\left\| \sum_{i=1}^{n} T\bigl(h^{i+j}\bigr)x - \sum_{i=1}^{n} T\bigl(h^{i+l_i+j}\bigr)x \right\| \\
&\quad\leq \frac{md}{n} + \frac{Nd}{n} \\
&\quad\leq \frac{\varepsilon}{12} + \frac{m^2 d^2 (\gamma(\frac{a(\frac{\varepsilon}{4})}{2}))^{-1}}{n} + 2m^2 d^2 \alpha(h)\left(\gamma\left(\frac{a(\frac{\varepsilon}{4})}{2}\right)\right)^{-1} \\
&\quad< \frac{\varepsilon}{12} + \frac{\varepsilon}{32} + \frac{\varepsilon}{8} < \frac{\varepsilon}{4},
\end{aligned}$$

we obtain

$$\frac{1}{n}\sum_{i=1}^{n} T\bigl(h^{i}\bigr)x \in F_{\frac{\varepsilon}{4}}\bigl(T(h)\bigr) + B\left(0, \frac{\varepsilon}{4}\right) \subset F_{\varepsilon}\bigl(T(h)\bigr).$$

This completes the proof. □

Lemma 3.6 *Let $u(\cdot)$ be an almost orbits of $\Im$. Then*

$$\lim_{t\in G}\bigl\| \lambda u(t) + (1-\lambda)f - g \bigr\|$$

exists for all $\lambda \in (0,1)$ and $f, g \in F(\Im)$.

Proof We only need to show that

$$\inf_{s\in G}\sup_{t\in G}\bigl\| \lambda u(ts) + (1-\lambda)f - g \bigr\| \leq \sup_{s\in G}\inf_{t\in G}\bigl\| \lambda u(ts) + (1-\lambda)f - g \bigr\|.$$

In fact, for any $\varepsilon > 0$, there are t_0 and $s_0 \in G$ such that for any $t \in G$, $\alpha(tt_0) < \frac{\varepsilon}{1+d}$ and $\varphi(ts_0) < \varepsilon$, where $\varphi(t) = \sup_{h\in G} \|u(ht) - T(h)u(t)\|$. Then for all $a \in G$,

$$\begin{aligned}
&\inf_{s\in G}\sup_{t\in G}\left\|u(tss_0) - f\right\| \\
&\quad\le \sup_{t\in G}\left\|u(tt_0as_0) - f\right\| \\
&\quad\le \sup_{t\in G}\left\|u(tt_0as_0) - T(tt_0)u(as_0)\right\| + \sup_{t\in G}\left\|T(tt_0)u(as_0) - f\right\| \\
&\quad\le \varphi(as_0) + \sup_{t\in G}\bigl(1 + \alpha(tt_0)\bigr)\cdot\left\|u(as_0) - f\right\| \\
&\quad\le \left\|u(as_0) - f\right\| + 2\varepsilon.
\end{aligned}$$

Hence $\inf_{s\in G}\sup_{t\in G}\|u(tss_0) - f\| \le \inf_{a\in G}\|u(as_0) - f\| + 2\varepsilon$. Thus, there exists $s_1 \in G$ such that

$$\sup_{t\in G}\left\|u(ts_1s_0) - f\right\| < \inf_{a\in G}\left\|u(as_0) - f\right\| + 3\varepsilon.$$

So, for any $a \in G$, we get

$$\begin{aligned}
&\inf_{s\in G}\sup_{t\in G}\left\|\lambda u(ts) + (1-\lambda)f - g\right\| \\
&\quad\le \sup_{t\in G}\left\|\lambda u(tt_0as_1s_0) + (1-\lambda)f - g\right\| \\
&\quad\le \lambda\sup_{t\in G}\left\|u(tt_0as_1s_0) - T(tt_0)u(as_1s_0)\right\| + \sup_{t\in G}\left\|\lambda T(tt_0)u(as_1s_0) + (1-\lambda)f - g\right\| \\
&\quad\le \varphi(as_1s_0) + \sup_{t\in G}\left\|\lambda T(tt_0)u(as_1s_0) + (1-\lambda)f - T(tt_0)\bigl(\lambda u(as_1s_0) + (1-\lambda)f\bigr)\right\| \\
&\qquad + \sup_{t\in G}\left\|T(tt_0)\bigl(\lambda u(as_1s_0) + (1-\lambda)f\bigr) - g\right\| \\
&\quad\le \varepsilon + \sup_{t\in G}\bigl(1 + \alpha(tt_0)\bigr)\gamma^{-1}\bigl(\left\|u(as_1s_0) - f\right\| - \left\|T(tt_0)u(as_1s_0) - f\right\| + d\cdot\alpha(tt_0)\bigr) \\
&\qquad + \sup_{t\in G}\bigl(1 + \alpha(tt_0)\bigr)\left\|\lambda u(as_1s_0) + (1-\lambda)f - g\right\| \\
&\quad\le \varepsilon + (1+\varepsilon)\sup_{t\in G}\gamma^{-1}\bigl(\left\|u(as_1s_0) - f\right\| - \left\|u(tt_0as_1s_0) - f\right\| + \varphi(as_1s_0) + \varepsilon\bigr) \\
&\qquad + (1+\varepsilon)\left\|\lambda u(as_1s_0) + (1-\lambda)f - g\right\| \\
&\quad\le \varepsilon + (1+\varepsilon)\gamma^{-1}(5\varepsilon) + (1+\varepsilon)\left\|\lambda u(as_1s_0) + (1-\lambda)f - g\right\|.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\inf_{s\in G}\sup_{t\in G}\left\|\lambda u(ts) + (1-\lambda)f - g\right\| \\
&\quad\le \varepsilon + (1+\varepsilon)\gamma^{-1}(5\varepsilon) + (1+\varepsilon)\inf_{a\in G}\left\|\lambda u(as_1s_0) + (1-\lambda)f - g\right\| \\
&\quad\le \varepsilon + (1+\varepsilon)\gamma^{-1}(5\varepsilon) + (1+\varepsilon)\sup_{b\in G}\inf_{a\in G}\left\|\lambda u(ab) + (1-\lambda)f - g\right\|.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we can conclude

$$\inf_{s\in G}\sup_{t\in G}\left\|\lambda u(ts) + (1-\lambda)f - g\right\| \le \sup_{s\in G}\inf_{t\in G}\left\|\lambda u(ts) + (1-\lambda)f - g\right\|.$$

This completes the proof. □

4 Main results

Theorem 4.1 *Let X be a uniformly convex Banach space, and let C be a nonempty bounded closed convex subset of X. Let $\Im = \{T(t) : t \in G\}$ be a reversible semigroup of asymptotically nonexpansive mappings on C. If D has a left invariant mean, then there exists a retraction P from $\mathrm{LAO}(\Im)$ onto $F(\Im)$ satisfying the following properties*:

(1) *P is nonexpansive in the sense*

$$\|Pu - Pv\| \le \inf_{s\in G}\sup_{t\in G}\left\|u(st) - v(st)\right\|, \quad \forall u, v \in \mathrm{LAO}(\Im);$$

(2) *$PT(h)u = T(h)Pu = Pu$ for all $u \in \mathrm{AO}(\Im)$ and $h \in G$*;

(3) *$Pu \in \bigcap_{s\in G}\overline{\mathrm{conv}}\{u(t) : t \ge s\}$ for all $u \in \mathrm{LAO}(\Im)$.*

Proof Since D has a left invariant mean, there exists a net $\{\lambda_\alpha : \alpha \in A\}$ of finite means on G such that $\lim_{\alpha\in A}\|\lambda_\alpha - l_s^*\lambda_\alpha\| = 0$ for every $s \in G$, where A is a directed system. Put $I = A \times G = \{\beta = (\alpha, t) : \alpha \in A, t \in G\}$. For $\beta_i = (\alpha_i, t_i) \in I$, $i = 1, 2$, we define $\beta_1 \le \beta_2$ if and only if $\alpha_1 \le \alpha_2$, $t_1 \le t_2$. In this case, I is also a directed system. For each $\beta = (\alpha, t) \in I$, define $P_1\beta = \alpha$, $P_2\beta = t$ and $\lambda_\beta = \lambda_\alpha$. Then for every $s \in G$,

$$\lim_{\beta\in I}\left\|\lambda_\beta - l_s^*\lambda_\beta\right\| = 0. \tag{4.1}$$

Let $\Lambda = \{\{t_\beta\}_{\beta\in I}, t_\beta \ge P_2\beta, \forall\beta \in I\}$. Taking any $\{t_\beta, \beta \in I\} \in \Lambda$, since $r_{t_\beta}^*\lambda_\beta$ is bounded, without loss of generality, suppose that $r_{t_\beta}^*\lambda_\beta$ is weakly* convergent. Then for all $u \in \mathrm{LAO}(\Im)$, $\omega\text{-}\lim_{\beta\in I}\lambda_\beta(t)\langle u(tt_\beta)\rangle$ exists. We define

$$Pu = \omega\text{-}\lim_{\beta\in I}\lambda_\beta(t)\big\langle u(tt_\beta)\big\rangle.$$

It is easy to see that for all $u \in \mathrm{LAO}(\Im)$, $Pu \in \bigcap_{s\in G}\overline{\mathrm{conv}}\{u(t) : t \ge s\}$. Next, we shall show that $Pu \in F(\Im)$. In fact, for any given $\varepsilon \in (0, 1]$, there exists $t_0 \in G$ such that for any $t \ge t_0$, $\varphi(t) < \frac{a(\varepsilon)}{4}$. Also, we can suppose that $P_2\beta \ge t_0$ for all $\beta \in I$, then $t_\beta \ge t_0$, $\{t_\beta\} \in \Lambda$. By Lemma 3.5, for any $h \in G_{a(a(\frac{a(\varepsilon)}{16}))}$, there exists $n \in N$ such that for all $t \in G$ and $\beta \in I$,

$$\frac{1}{n}\sum_{i=1}^{n}T\big(h^i\big)u(tt_\beta) \in F_{\frac{a(\varepsilon)}{4}}\big(T(h)\big).$$

Noting for all $t \in G$,

$$\left\|\frac{1}{n}\sum_{i=1}^{n}T\big(h^i\big)u(tt_\beta) - \frac{1}{n}\sum_{i=1}^{n}u\big(h^i tt_\beta\big)\right\| \le \varphi(tt_\beta) < \frac{a(\varepsilon)}{4},$$

we have for any $h \in G_{a(a(\frac{a(\varepsilon)}{16}))}$,

$$\frac{1}{n}\sum_{i=1}^{n} u\big(h^i tt_\beta\big) \in F_{\frac{a(\varepsilon)}{4}}\big(T(h)\big) + B\left(0, \frac{a(\varepsilon)}{4}\right) \subset F_{a(\varepsilon)}\big(T(h)\big).$$

By (4.1), we obtain

$$\lim_{\beta\in I}\left\| \lambda_\beta(t)\left\langle \frac{1}{n}\sum_{i=1}^{n} u\big(h^i tt_\beta\big)\right\rangle - \lambda_\beta(t)\big\langle u(tt_\beta)\big\rangle \right\| = 0.$$

Combining it with the definition of Pu, we get for all $h \in G_{a(a(\frac{a(\varepsilon)}{16}))}$,

$$Pu = \omega\text{-}\lim_{\beta\in I} \lambda_\beta(t)\left\langle \frac{1}{n}\sum_{i=1}^{n} u\big(h^i tt_\beta\big)\right\rangle \in \overline{\mathrm{co}} F_{a(\varepsilon)}\big(T(h)\big).$$

Thus, by the Lemma 3.3, we can conclude that for all $h \in G_{a(a(\frac{a(\varepsilon)}{16}))}$, $Pu \in F_\varepsilon(T(h))$. The continuity of $T(h)$ then implies that $Pu \in F(\Im)$. Obviously, for any $h \in G$,

$$\begin{aligned} PT(h)u &= \omega\text{-}\lim_{\beta\in I} \lambda_\beta(t)\big\langle T(h)u(tt_\beta)\big\rangle \\ &= \omega\text{-}\lim_{\beta\in I} \lambda_\beta(t)\big\langle u(htt_\beta)\big\rangle \\ &= \omega\text{-}\lim_{\beta\in I} \lambda_\beta(t)\big\langle u(tt_\beta)\big\rangle \quad \text{(by (4.1))} \\ &= Pu \end{aligned}$$

and for any $v \in \mathrm{LAO}(\Im)$ and $s \in G$,

$$\begin{aligned} \|Pu - Pv\| &\le \liminf_{\beta\in I} \big\| \lambda_\beta(t)\big\langle u(tt_\beta)\big\rangle - \lambda_\beta(t)\big\langle v(tt_\beta)\big\rangle \big\| \\ &= \liminf_{\beta\in I} \big\| \lambda_\beta(t)\big\langle u(stt_\beta)\big\rangle - \lambda_\beta(t)\big\langle v(stt_\beta)\big\rangle \big\| \quad \text{(by (4.1))} \\ &\le \liminf_{\beta\in I} \big\| \lambda_\beta(t)\big\| \cdot \sup_{t\in G} \big\| u(stt_\beta) - v(stt_\beta)\big\| \\ &\le \sup_{t\in G} \big\| u(st) - v(st)\big\|. \end{aligned}$$

Thus,

$$\|Pu - Pv\| \le \inf_{s\in G}\sup_{t\in G} \big\| u(st) - v(st)\big\|.$$

This completes the proof. □

Remark 4.1 It should be noted that in Theorem 4.1, we do not assume $F(\Im) \neq \emptyset$. In fact, we can find a fixed point $Pu \in F(\Im)$. It also should be pointed out that in the case of reversible semigroup, if D has a left invariant mean, then $F(\Im) \neq \emptyset$ (see [6, Theorem 3.1] and [10, Lemma 4]).

As in [6], we have the following ergodic theorem.

Theorem 4.2 *Let X be a uniformly convex Banach space and C a nonempty bounded closed convex subset of X. Let $\Im = \{T(t) : t \in G\}$ of a reversible semigroup of asymptotically nonexpansive mappings on C. If D has a left invariant mean and there exists a unique retraction P from $\mathrm{LAO}(\Im)$ onto $F(\Im)$, which satisfies the properties* (1)-(3) *in Theorem* 4.1, *then for every strongly regular net $\{\mu_\alpha : \alpha \in A\}$ on D and $u \in \mathrm{AO}(\Im)$,*

$$\omega\text{-}\lim_{\alpha\in A} \int u(th)\, d\mu_\alpha(t) = p \in F(\Im) \quad \textit{umiforly in } h \in \Lambda(G),$$

where $\Lambda(G) = \{s \in G : st = ts \text{ for all } t \in G\}$.

Remark 4.2 By Theorem 4.1 and Theorem 4.2, we can get many known theorems in [2, 6, 10, 11, 16], such as Theorem 3.1 and Theorem 3.2 in [6], Theorem 1 in [11]. The key assumption in [6] that the almost orbit $u(\cdot)$ is almost asymptotically isometric is not necessary in our theorems.

Theorem 4.3 *Let X be a uniformly convex Banach space, and let C be a nonempty bounded closed convex subset of X. Let $\Im = \{T(t) : t \in G\}$ of a reversible semigroup of asymptotically nonexpansive mappings on C, and let $u(\cdot)$ be an almost orbit of $\Im$. If*

$$\omega\text{-}\lim_{t\in G} u(ht) - u(t) = 0$$

for every $h \in G$, then

$$\omega_\omega(u) \subset F(\Im).$$

Proof For any given $\varepsilon \in (0,1]$, there exists $t_0 \in G$ such that for any $t \geq t_0$, $\varphi(t) < \frac{a(\varepsilon)}{4}$. Let $p \in \omega_\omega(u)$, then there exists a subnet $\{u(t_\alpha)\}_{\alpha\in A}$ in $\{u(t)\}_{t\in G}$ with $\omega\text{-}\lim_{\alpha\in A} u(t_\alpha) = p$ such that for all $\alpha \in A$, $t_\alpha \geq t_0$, where A is a directed system. By Lemma 3.5, for any $h \in G_{a(a(\frac{a(\varepsilon)}{16}))}$, there exists a $n \in N$ such that for all $\alpha \in A$,

$$\frac{1}{n}\sum_{i=1}^{n} T\bigl(h^i\bigr)u(t_\alpha) \in F_{\frac{a(\varepsilon)}{4}}\bigl(T(h)\bigr).$$

Noting for each $\alpha \in A$,

$$\left\| \frac{1}{n}\sum_{i=1}^{n} T\bigl(h^i\bigr)u(t_\alpha) - \frac{1}{n}\sum_{i=1}^{n} u\bigl(h^i t_\alpha\bigr) \right\| \leq \varphi(t_\alpha) < \frac{a(\varepsilon)}{4},$$

we get

$$\frac{1}{n}\sum_{i=1}^{n} u\bigl(h^i t_\alpha\bigr) \in F_{\frac{a(\varepsilon)}{4}}\bigl(T(h)\bigr) + B\left(0, \frac{a(\varepsilon)}{4}\right) \subset F_{a(\varepsilon)}\bigl(T(h)\bigr).$$

Since $u(ht) - u(t) \rightharpoonup 0$ for every $h \in G$, we have $u(h^i t_\alpha) \rightharpoonup p$, $i = 1, \ldots, n$. Then for all $h \in G_{a(a(\frac{a(\varepsilon)}{16}))}$,

$$p = \omega\text{-}\lim_{\alpha\in A} \frac{1}{n}\sum_{i=1}^{n} u\bigl(h^i t_\alpha\bigr) \in \overline{\mathrm{co}} F_{a(\varepsilon)}\bigl(T(h)\bigr).$$

Consequently, from Lemma 3.3, we can conclude that for all $h \in G_{a(a(\frac{a(\varepsilon)}{16}))}$, $p \in F_\varepsilon(T(h))$, which implies $p \in F(\Im)$. This completes the proof. □

Remark 4.3 In Theorem 4.1, Theorem 4.2 and Theorem 4.3, we do not assume that X has a Fréchet differentiable norm.

Theorem 4.4 *Let X be a uniformly convex Banach space whose dual has the Kadec-Klee property, and let C be a nonempty bounded closed convex subset of X. Let $\Im = \{T(t) : t \in G\}$ of a reversible semigroup of asymptotically nonexpansive mappings on C and $u(\cdot)$ be an almost orbit of $\Im$. Then the following statements are equivalent*:

(1) $\omega_\omega(u) \subset F(\Im)$.
(2) $\omega\text{-}\lim_{t\in G} u(t) = p \in F(\Im)$
(3) $\omega\text{-}\lim_{t\in G} u(ht) - u(t) = 0$ *for every* $h \in G$.

Proof (1) ⇒ (2). It suffices to show that $\omega_\omega(u)$ is a singleton. Since X is reflexive, it is nonempty. Let f and g be two elements in $\omega_\omega(u)$, then by (1), we can obtain $f, g \in F(\Im)$. For any $\lambda \in (0,1)$, by Lemma 3.6, $\lim_{t\in G} \|\lambda u(t) + (1-\lambda)f - g\|$ exists. Setting

$$h(\lambda) = \lim_{t\in G} \bigl\|\lambda u(t) + (1-\lambda)f - g\bigr\|,$$

then for a given $\varepsilon > 0$, there exists $t_1 \in G$ such that for all $t \geq t_1$,

$$\bigl\|\lambda u(t) + (1-\lambda)f - g\bigr\| \leq h(\lambda) + \varepsilon.$$

Hence for all $t \geq t_1$,

$$\bigl\langle \lambda u(t) + (1-\lambda)f - g, j(f-g)\bigr\rangle \leq \|f-g\|\bigl(h(\lambda) + \varepsilon\bigr),$$

where $j(f-g) \in J(f-g)$. Let us note $f \in \overline{\mathrm{co}}\{u(t) : t \geq t_1\}$, then

$$\bigl\langle \lambda f + (1-\lambda)f - g, j(f-g)\bigr\rangle \leq \|f-g\|\bigl(h(\lambda) + \varepsilon\bigr),$$

which means $\|f-g\| \leq h(\lambda) + \varepsilon$. Since ε is arbitrary, we get

$$\|f-g\| \leq h(\lambda).$$

It then follows from $g \in \omega_\omega(u)$ that there exists a subnet $\{u(t_\alpha)\}_{\alpha\in A}$ in $\{u(t)\}_{t\in G}$ such that $\omega\text{-}\lim_{\alpha\in A} u(t_\alpha) = g$, where A is a directed system. Putting

$$I = A \times N = \bigl\{\beta = (\alpha, n) : \alpha \in A, n \in N\bigr\},$$

then for $\beta_i = (\alpha_i, n_i) \in I$, $i = 1, 2$, we define $\beta_1 \leq \beta_2$ if and only if $\alpha_1 \leq \alpha_2$, $n_1 \leq n_2$. In this case, I is also a directed system. For arbitrary $\beta = (\alpha, n) \in I$, define $P_1\beta = \alpha$, $P_2\beta = n$, $t_\beta = t_\alpha$, $\varepsilon_\beta = \frac{1}{P_2\beta}$, then $\omega\text{-}\lim_{\beta\in I} u(t_\beta) = g$ and $\lim_{\beta\in I} \varepsilon_\beta = 0$. By Lemma 3.1, we have

$$\bigl\|\lambda u(t_\beta) + (1-\lambda)f - g\bigr\| \leq \|f-g\|^2 + 2\lambda\bigl\langle u(t_\beta) - f, j\bigl(\lambda u(t_\beta) + (1-\lambda)f - g\bigr)\bigr\rangle.$$

Applying Lemma 3.6 and noting the inequality $\|f - g\| \le h(\lambda)$, we obtain

$$\liminf_{\beta \in I} \langle u(t_\beta) - f, j(\lambda u(t_\beta) + (1-\lambda)f - g) \rangle \ge 0.$$

Then for each $\gamma \in I$, there exists $\beta_\gamma \in I$ such that $\beta_\gamma \ge \gamma$ and

$$\langle u(t_{\beta_\gamma}) - f, j(\varepsilon_\gamma u(t_{\beta_\gamma}) + (1-\varepsilon_\gamma)f - g) \rangle \ge -\varepsilon_\gamma. \tag{4.2}$$

Obviously, $\{\beta_\gamma\}$ is also a subnet of I, then $\omega\text{-}\lim_{\gamma \in I} u(t_{\beta_\gamma}) = g$. Putting

$$j_\gamma = j(\varepsilon_\gamma u(t_{\beta_\gamma}) + (1-\varepsilon_\gamma)f - g).$$

In as much as X is reflexive, X^* is also reflexive, we can conclude that the set of all weak limit points of $\{j_\gamma, \gamma \in I\}$ is nonempty. Hence, without loss of generality, we may assume that $\omega\text{-}\lim_{\gamma \in I} j_\gamma = j \in X^*$. Therefore, $\|j\| \le \liminf_{\gamma \in I} \|j_\gamma\| = \|f - g\|$. Since

$$\langle f - g, j_\gamma \rangle = \|\varepsilon_\gamma u(t_{\beta_\gamma}) + (1-\varepsilon_\gamma)f - g\|^2 - \varepsilon_\gamma \langle u(t_{\beta_\gamma}) - f, j_\gamma \rangle,$$

passing the limit for $\gamma \in I$, we get $\langle f - g, j \rangle = \|f - g\|^2$, which implies $\|j\| \ge \|f - g\|$. Hence we can get

$$\langle f - g, j \rangle = \|f - g\|^2 = \|j\|^2,$$

i.e., $j \in J(f - g)$. Therefore, $\omega\text{-}\lim_{\gamma \in I} j_\gamma = j$ and $\lim_{\gamma \in I} \|j_\gamma\| = \|j\|$. Since X^* is reflexive and has Kadec-Klee property, it has the Kadec property, and this implies that $\lim_{\gamma \in I} j_\gamma = j$. Taking the limit for $\gamma \in I$ in (4.2), we obtain $\langle g - f, j \rangle \ge 0$, *i.e.*, $\|f - g\|^2 \le 0$, which implies $f = g$.

(2) $\Rightarrow$ (3). Obviously.

(3) $\Rightarrow$ (1). See Theorem 4.3. This completes the proof. □

Remark 4.4 By Theorem 4.4, we can get many known theorems in [2, 6, 10, 11, 14–16], such as Theorem 4.3 and Theorem 8.1 in [14], Theorem 3.1 and Theorem 3.2 in [15]. And in [6, 10], it is assumed that the almost orbit $u(\cdot)$ is almost asymptotically isometric and the subspace D has a left invariant mean. Those key conditions are not necessary by the theorem above.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Acknowledgements

This research is supported by the National Science Foundation of China (11201410, 11271316 and 11101353), the Natural Science Foundation of Jiangsu Province (BK2012260) and the Natural Science Foundation of Jiangsu Education Committee (10KJB110012 and 11KJB110018).

Received: 21 March 2013 Accepted: 9 August 2013 Published: 29 August 2013

References

1. Baillon, JB: Un théorème de type ergodique pour les contraction non linéaires dans un espace de Hilbert. C. R. Math. Acad. Sci. Paris **280**, 1511-1514 (1976)
2. Takahashi, W: A nonlinear ergodic theorem for a reversible semigroup of nonexpansive mappings in a Hilbert space. Proc. Am. Math. Soc. **17**, 55-58 (1986)
3. Li, G, Kim, JK: Nonlinear ergodic theorems for general curves defined on general semigroups. Nonlinear Anal. **55**, 1-14 (2003)
4. Li, G, Kim, JK: Asymptotic behavior of non-autonomous dissipative systems in Hilbert spaces. Acta Math. Sci. **18**, 25-30 (1998)
5. Lau, AT, Shioji, N, Takahashi, W: Existences of nonexpansive retractions for amenable semigroups of nonexpansive mappings and nonlinear ergodic theorems in Banach spaces. J. Funct. Anal. **161**, 62-75 (1999)
6. Kim, JK, Li, G: Nonlinear ergodic theorems for reversible semigroups of asymptotically nonexpansive mappings in Banach spaces. Dyn. Syst. Appl. **9**, 255-268 (2000)
7. Kaczor, W, Kuczumow, T, Michalska, M: Convergence of ergodic means of orbits of semigroups of nonexpansive mappings in sets with the Γ-Opial property. Nonlinear Anal. **67**, 2122-2130 (2007)
8. Kim, KS: Nonlinear ergodic theorems of nonexpansive type mappings. J. Math. Anal. Appl. **358**, 261-272 (2009)
9. Li, G, Kim, JK: Nonlinear ergodic theorem for commutative semigroups of non-Lipschitzian mappings in Banach space. Houst. J. Math. **29**, 23-36 (2003)
10. Li, G: Weak convergence and nonlinear ergodic theorem for reversible semigroups of non-Lipschitzian mappings. J. Math. Anal. Appl. **206**, 451-464 (1997)
11. Oka, H: Nonlinear ergodic theorems for commutative semigroups of asymptotically nonexpansive mappings. Nonlinear Anal. **7**, 619-635 (1992)
12. Saeidi, S: Ergodic retractions for amenable semigroups in Banach spaces with normal structure. Nonlinear Anal. **71**, 2558-2563 (2009)
13. Saeidi, S: Existence of ergodic retractions for semigroups in Banach spaces. Nonlinear Anal. **69**, 3417-3422 (2008)
14. Falset, JG, Kaczor, W, Kuczumow, T, et al.: Weak convergence theorems for asymptotically nonexpansive mappings and semigroups. Nonlinear Anal. **43**, 377-401 (2001)
15. Kaczor, W: Weak convergence of almost-orbit for asymptotically nonexpansive commutative semigroups. J. Math. Anal. Appl. **272**, 565-574 (2002)
16. Miyadera, I, Kobayasi, K: On the asymptotic behavior of almost-orbits of nonlinear contraction semigroups in Banach space. Nonlinear Anal. **6**, 349-365 (1982)
17. Aksoy, AG, Khamsi, MA: Nonstandard Methods in Fixed Point Theory. Springer, New York (1990)
18. Day, MM: Amenable semigroups. Ill. J. Math. **1**, 509-544 (1957)
19. Holmes, RD, Narayanaswamy, PP: On asymptotically nonexpansive semigroups of mappings. Can. Math. Bull. **13**, 209-214 (1970)
20. Holmes, RD, Lau, AT: Asymptotically non-expansive actions of topological semigroups and fixed points. Bull. Lond. Math. Soc. **3**, 343-347 (1971)
21. Kirk, W, Torrejon, R: Asymptotically nonexpansive semigroup in Banach space. Nonlinear Anal. **3**, 111-121 (1979)
22. Saeidi, S: Strong convergence of Browder's type iterations for left amenable semigroups of Lipschitzian mappings in Banach spaces. J. Fixed Point Theory Appl. **5**, 93-103 (2009)
23. Saeidi, S: Approximating common fixed points of Lipschitzian semigroup in smooth Banach spaces. Fixed Point Theory Appl. **2008**, Article ID 363257 (2008)
24. Chang, SS: Some problems and results in the study of nonlinear analysis. Nonlinear Anal. **30**, 4197-4208 (1997)
25. Bruck, RE: A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces. Isr. J. Math. **32**, 107-116 (1979)

doi:10.1186/1687-1812-2013-231
Cite this article as: Zhu et al.: Nonlinear ergodic theorems and weak convergence theorems for reversible semigroup of asymptotically nonexpansive mappings in Banach spaces. *Fixed Point Theory and Applications* 2013 **2013**:231.