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Fixed point theorems for cyclic self-maps involving weaker Meir-Keeler functions in complete metric spaces and applications

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Abstract

We obtain fixed point theorems for cyclic self-maps on complete metric spaces involving Meir-Keeler and weaker Meir-Keeler functions, respectively. In this way, we extend several well-known fixed point theorems and, in particular, improve some very recent results on weaker Meir-Keeler functions. Fixed point results for well-posed property and for limit shadowing property are also deduced. Finally, an application to the study of existence and uniqueness of solutions for a class of nonlinear integral equations is presented.

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Keywords: fixed point; cyclic map; weaker Meir-Keeler function; complete metric space; integral equation

1 Introduction

In their paper [1], Kirk, Srinavasan and Veeramani started the fixed point theory for cyclic self-maps on (complete) metric spaces. In particular, they obtained, among others, cyclic versions of the Banach contraction principle [2], of the Boyd and Wong fixed point theorem [3] and of the Caristi fixed point theorem [4]. From then, several authors have contributed to the study of fixed point theorems and best proximity points for cyclic contractions (see, *e.g.*, [5–13]). Very recently, Chen [14] (see also [15]) introduced the notion of a weaker Meir-Keeler function and obtained some fixed point theorems for cyclic contractions involving weaker Meir-Keeler functions.

In this paper we obtain a fixed point theorem for cyclic self-maps on complete metric spaces involving Meir-Keeler functions and deduce a variant of it for weaker Meir-Keeler functions. In this way, we extend in several directions and improve, among others, the main fixed point theorem of Chen's paper [14, Theorem 3]. Some consequences are given after the main results. Fixed point results for well-posedness property and for limit shadowing property in complete metric spaces are also given. Finally, an application to the study of existence and uniqueness of solution for a class of nonlinear integral equations is presented.

We recall that a self-map f of a (non-empty) set X is called a cyclic map if there exists $m \in \mathbb{N}$ such that $X = \bigcup_{i=1}^{m} A_i$, with A_i non-empty and $f(A_i) \subseteq A_{i+1}$, $i = 1, \ldots, m$, where $A_{m+1} = A_1$.

In this case, we say that $X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to f.

2 Fixed point results

In the sequel, the letters $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ will denote the set of real numbers, the set of non-negative real numbers and the set of positive integer numbers, respectively.

Meir and Keeler proved in [16] that if f is a self-map of a complete metric space (X,d) satisfying the condition that for each $\varepsilon > 0$ there is $\delta > 0$ such that, for any $x, y \in X$, with $\varepsilon \leq d(x,y) < \varepsilon + \delta$, we have $d(fx,fy) < \varepsilon$, then f has a unique fixed point $z \in X$ and $f^n x \to z$ for all $x \in X$.

This important result suggests the notion of a Meir-Keeler function:

A function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a Meir-Keeler function if for each $\varepsilon > 0$, there exists $\delta > 0$ such that for $t > 0$ with $\varepsilon \leq t < \varepsilon + \delta$, we have $\phi(t) < \varepsilon$.

Remark 1 It is obvious that if ϕ is a Meir-Keeler function, then $\phi(t) < t$ for all $t > 0$.

In [14], Chen introduced the following interesting generalization of the notion of a Meir-Keeler function.

Definition 1 [14, Definition 3] A function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a weaker Meir-Keeler function if for each $\varepsilon > 0$, there exists $\delta > 0$ such that for $t > 0$ with $\varepsilon \leq t < \varepsilon + \delta$, there exists $n_0 \in \mathbb{N}$ such that $\phi^{n_0}(t) < \varepsilon$.

Now let $\phi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$. According to Chen [14, Section 2], consider the following conditions for ϕ and φ, respectively.

(ϕ_1) $\phi(t) = 0 \Leftrightarrow t = 0$;
(ϕ_2) for all $t > 0$, the sequence $\{\phi^n(t)\}_{n\in\mathbb{N}}$ is decreasing;
(ϕ_3) for $t_n > 0$,
  (a) if $\lim_{n\to\infty} t_n = \gamma > 0$, then $\lim_{n\to\infty} \phi(t_n) < \gamma$, and
  (b) if $\lim_{n\to\infty} t_n = 0$, then $\lim_{n\to\infty} \phi(t_n) = 0$;
(φ_1) φ is non-decreasing and continuous with $\varphi(t) = 0 \Leftrightarrow t = 0$;
(φ_2) φ is subadditive, that is, for every $t_1, t_2 \in \mathbb{R}^+$, $\varphi(t_1 + t_2) \leq \varphi(t_1) + \varphi(t_2)$;
(φ_3) for $t_n > 0$, $\lim_{n\to\infty} t_n = 0$ if and only if $\lim_{n\to\infty} \varphi(t_n) = 0$.

Definition 2 [14, Definition 4] Let (X,d) be a metric space. A self-map f of X is called a cyclic weaker $(\phi \circ \varphi)$-contraction if there exist $m \in \mathbb{N}$, for which $X = \bigcup_{i=1}^m A_i$ (each A_i a non-empty closed set), and two functions $\phi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying conditions (ϕ_i), $i = 1, 2, 3$, and (φ_i), $i = 1, 2, 3$, respectively, with ϕ a weaker Meir-Keeler function such that
(1) $X = \bigcup_{i=1}^m A_i$ is a cyclic representation of X with respect to f;
(2) for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$,

$$\varphi\big(d(fx,fy)\big) \leq \phi\big(\varphi\big(d(x,y)\big)\big),$$

where $A_{m+1} = A_1$.

By using the above concept, Chen established the following fixed point theorem.

Theorem 1 [14, Theorem 3] *Let (X,d) be a complete metric space. Then every cyclic weaker $(\phi \circ \varphi)$-contraction f of X has a unique fixed point z. Moreover, $z \in \bigcap_{i=1}^m A_i$, where $X = \bigcup_{i=1}^m A_i$ is the cyclic representation of X with respect to f of Definition* 2.

We shall establish fixed point theorems which improve in several directions the preceding theorem. To this end, we start by obtaining a fixed point theorem for cyclic contractions involving Meir-Keeler functions.

Theorem 2 *Let f be a self-map of a complete metric space (X,d), and let $X = \bigcup_{i=1}^{m} A_i$ be a cyclic representation of X with respect to f, with A_i non-empty and closed, $i = 1,\dots,m$. If $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Meir-Keeler function such that for any $x \in A_i$, $y \in A_{i+1}$, $i = 1,2,\dots,m$,*

$$d(fx,fy) \le \phi\big(d(x,y)\big),$$

where $A_{m+1} = A_1$, then f has a unique fixed point z. Moreover, $z \in \bigcap_{i=1}^{m} A_i$.

Proof Let $x_0 \in A_m$. For each $n \in \mathbb{N} \cup \{0\}$, put $x_n = f^n x_0$. Note that $x_{nm+i} \in A_i$ whenever $n \in \mathbb{N} \cup \{0\}$ and $i = 1,2,\dots,m$.

If $x_{n_0} = x_{n_0+1}$ for some n_0, then x_n is a fixed point of f. So, we assume that $x_n \ne x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. By Remark 1 and the contraction condition, it follows that $\{d(x_n,x_{n+1})\}_{n\in\mathbb{N}}$ is a strictly decreasing sequence in $\mathbb{R}^+$, so there exists $r \in \mathbb{R}^+$ such that $\lim_{n\to\infty} d(x_n,x_{n+1}) = r$. If $r > 0$, there is $n_0 \in \mathbb{N}$ such that $\phi(d(x_n,x_{n+1})) < r$ for all $n \ge n_0$ by our assumption that ϕ is a Meir-Keeler function. Hence, $d(x_{n+1},x_{n+2}) < r$ for all $n \ge n_0$, a contradiction. Therefore $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$.

Next we prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,d). Choose an arbitrary $\varepsilon > 0$. Then, there is $\delta \in (0,\varepsilon)$ such that for $t > 0$ with $\varepsilon \le t < \varepsilon + \delta$, we have $\phi(t) < \varepsilon$. Let $k_0 \in \mathbb{N}$ be such that $d(x_k,x_{k+1}) < \delta/2$, $d(x_k,x_{k+m-1}) < \varepsilon/2$ and $d(x_k,x_{k+m+1}) < \delta/2$ for all $k \ge k_0$.

Take any $k > k_0$. Then $k = nm + i$ for some $n \in \mathbb{N}$ and some $i \in \{1,2,\dots,m\}$. By induction we shall show that $d(x_{nm+i},x_{(n+j)m+i+1}) < \varepsilon$ for all $j \in \mathbb{N}$.

Indeed, for $j = 1$, we have

$$d(x_{nm+i},x_{nm+i+m+1}) = d(x_k,x_{k+m+1}) < \frac{\delta}{2} < \varepsilon.$$

Now, assume that $d(x_{nm+i},x_{(n+j)m+i+1}) < \varepsilon$ for some $j \in \mathbb{N}$. Thus

$$\begin{aligned} d(x_{nm+i-1},x_{(n+j+1)m+i}) &\le d(x_{nm+i-1},x_{nm+i}) + d(x_{nm+i},x_{(n+j)m+i+1}) \\ &\quad + d(x_{(n+j)m+i+1},x_{(n+j+1)m+i}) \\ &< \frac{\delta}{2} + \varepsilon + \frac{\delta}{2} = \delta + \varepsilon. \end{aligned}$$

If $\varepsilon \le d(x_{nm+i-1},x_{(n+j+1)m+i})$, then $\phi(d(x_{nm+i-1},x_{(n+j+1)m+i})) < \varepsilon$, and, by the contraction condition,

$$d(x_{nm+i},x_{(n+j+1)m+i+1}) < \varepsilon.$$

If $d(x_{nm+i-1},x_{(n+j+1)m+i}) < \varepsilon$, we deduce

$$\begin{aligned} d(x_{nm+i},x_{(n+j+1)m+i+1}) &\le \phi\big(d(x_{nm+i-1},x_{(n+j+1)m+i})\big) \\ &< d(x_{nm+i-1},x_{(n+j+1)m+i}) < \varepsilon. \end{aligned}$$

It immediately follows that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,d). Hence, there exists $z \in X$ such that $x_n \to z$. Since each A_i is closed, we deduce that $z \in \bigcap_{i=1}^m A_i$.

Moreover, $z = fz$. Indeed, let $i_0 \in \{1,\dots,m\}$ be such that $fz \in A_{i_0+1}$. Then

$$\begin{aligned} d(z,fz) &\leq d(z,x_{nm+i_0}) + d(x_{nm+i_0},fz) \leq d(z,x_{nm+i_0}) + \phi\big(d(x_{nm+i_0-1},z)\big) \\ &< d(z,x_{nm+i_0}) + d(x_{nm+i_0-1},z), \end{aligned}$$

for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} d(z,x_{nm+i_0}) = \lim_{n\to\infty} d(z,x_{nm+i_0-1}) = 0$, it follows that $d(z,fz) = 0$, *i.e.*, $z = fz$.

Finally, let $u \in X$ with $u = fu$ and $u \neq z$. Since $z \in \bigcap_{i=1}^m A_i$, we have $d(fz,fu) \leq \phi(d(z,u))$, so $d(z,u) < d(z,u)$, a contradiction. Hence $u = z$, and thus z is the unique fixed point of f. □

Next we analyze some relations between Chen's conditions (ϕ_i), $i = 1,2,3$.

Lemma 1 *If $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies (ϕ_3)(a), then ϕ is a Meir-Keeler function that satisfies conditions (ϕ_2) and (ϕ_3)(b).*

Proof Suppose that ϕ is not a Meir-Keeler function. Then there exists $\varepsilon > 0$ such that for each $n \in \mathbb{N}$ we can find a $t_n > 0$ with $\varepsilon \leq t_n < \varepsilon + 1/n$ and $\phi(t_n) \geq \varepsilon$. Then $\lim_{n\to\infty} t_n = \varepsilon > 0$, but $\phi(t_n) \geq \varepsilon$ for all n, so condition (ϕ_3)(a) is not satisfied. We conclude that condition (ϕ_3)(a) implies that ϕ is a Meir-Keeler function. Hence, by Remark 1, $\phi(t) < t$ for all $t > 0$, so the sequence $\{\phi^n(t)\}_{n\in\mathbb{N}}$ is (strictly) decreasing for all $t > 0$, and thus condition (ϕ_2) is satisfied. Finally, if $\lim_{n\to\infty} t_n = 0$, with $t_n > 0$, we deduce that $\lim_{n\to\infty} \phi(t_n) = 0$ because $\phi(t_n) < t_n$ for all n, so condition (ϕ_3)(b) also holds. □

Proposition 1 *Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function satisfying conditions (φ_1) and (φ_2). If (X,d) is a metric space, then the function $p : X \times X \to \mathbb{R}^+$, given by*

$$p(x,y) = \varphi\big(d(x,y)\big),$$

is a metric on X. If, in addition, (X,d) is complete and φ satisfies condition (φ_3), then the metric space (X,p) is complete.

Proof We first show that p is a metric on X. Let $x,y,z \in X$:

- Suppose $p(x,y) = 0$. Then $\varphi(d(x,y)) = 0$, so $d(x,y) = 0$ by (φ_1). Hence $x = y$.
- Clearly, $p(x,y) = p(y,x)$.
- Since $d(x,y) \leq d(x,z) + d(z,y)$, and φ is non-decreasing and subadditive, we deduce that $\varphi(d(x,y)) \leq \varphi(d(x,z)) + \varphi(d(z,y))$, *i.e.*, $p(x,y) \leq p(x,z) + p(z,y)$.

Finally, suppose that (X,d) is complete with φ satisfying (φ_i), $i = 1,2,3$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in (X,p). If $\{x_n\}_{n\in\mathbb{N}}$ is not a Cauchy sequence in (X,d), there exist $\varepsilon > 0$ and sequences $\{n_k\}_{k\in\mathbb{N}}$ and $\{m_k\}_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $k < n_k < m_k < n_{k+1}$ and $d(x_{n_k},x_{m_k}) \geq \varepsilon$ for all $k \in \mathbb{N}$. By (φ_3), the sequence $\{p(x_{n_k},x_{m_k})\}_{k\in\mathbb{N}}$ does not converge to zero, which contradicts the fact that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,p). Consequently, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,d), so it converges in (X,d) to some $x \in X$. From (φ_3) we deduce that $\{x_n\}_{n\in\mathbb{N}}$ converges to x in (X,p). Therefore (X,p) is a complete metric space. □

Remark 2 Note that the continuity of φ is not used in the preceding proposition.

Now we easily deduce the following improvement of Chen's theorem.

Theorem 3 *Let f be a self-map of a complete metric space (X,d), and let $X = \bigcup_{i=1}^{m} A_i$ be a cyclic representation of X with respect to f, with A_i non-empty and closed, $i = 1,\ldots,m$. If $\phi,\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy conditions (ϕ_3)(a) and (φ_i), $i = 1,2,3$, respectively, and for any $x \in A_i$, $y \in A_{i+1}$, $i = 1,2,\ldots,m$, it follows*

$$\varphi\big(d(fx,fy)\big) \leq \phi\big(\varphi\big(d(x,y)\big)\big),$$

where $A_{m+1} = A_1$, then f has a unique fixed point z. Moreover, $z \in \bigcap_{i=1}^{m} A_i$.

Proof Define $p : X \times X \to \mathbb{R}^+$ by $p(x,y) = \varphi(d(x,y))$ for all $x,y \in X$. By Proposition 1, (X,p) is a complete metric space. Moreover, from the condition

$$\varphi\big(d(fx,fy)\big) \leq \phi\big(\varphi\big(d(x,y)\big)\big),$$

for all $x \in A_i$, $y \in A_{i+1}$, $i = 1,\ldots,m$, it follows that

$$p(fx,fy) = \varphi\big(d(fx,fy)\big) \leq \phi\big(\varphi\big(d(x,y)\big)\big) = \phi\big(p(x,y)\big)$$

for all $x \in A_i$, $y \in A_{i+1}$, $i = 1,\ldots,m$.

Finally, since by Lemma 1 ϕ is a Meir-Keeler function, we can apply Theorem 2, so there exists $z \in \bigcap_{i=1}^{m} A_i$, which is the unique fixed point of f. □

Note that the continuity of φ can be omitted in Theorem 3. Moreover, the condition that ϕ is a weaker Meir-Keeler function turns out to be irrelevant by virtue of Lemma 1. This fact suggests the question of obtaining a fixed point theorem for cyclic contractions involving explicitly weaker Meir-Keeler functions. In particular, it is natural to wonder if Theorem 2 remains valid when we replace 'Meir-Keeler function' by 'weaker Meir-Keeler function'. In the sequel we answer this question. First we give an easy example which shows that it has a negative answer in general, but the answer is positive whenever the weaker Meir-Keeler function is non-decreasing as Theorem 5 below shows.

Example 1 Let $X = \{0,1\}$ and let d be the discrete metric on X, *i.e.*, $d(0,0) = d(1,1) = 0$ and $d(x,y) = 1$ otherwise. Of course (X,d) is a complete metric space. Define $f : X \to X$ by $f0 = 1$ and $f1 = 0$, and consider the function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\phi(t) = t/2$ for all $t \in [0,1)$, $\phi(1) = 2$ and $\phi(t) = 1/2$ for all $t > 1$. Clearly, ϕ is a weaker Meir-Keeler function (note, in particular, that $\phi^2(1) = 1/2 < 1$), but it is not a Meir-Keeler function because $\phi(1) > 1$. Finally, since $d(f0,f1) = 1$ and $\phi(d(0,1)) = 2$, we deduce that $d(fx,fy) \leq \phi(d(x,y))$ for all $x,y \in X$. However, f has no fixed point.

The function ϕ of the preceding example is not non-decreasing. This fact is not casual as Theorem 5 below shows.

Lemma 2 *Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing weaker Meir-Keeler function. Then the following hold*:

(i) $\phi(t) < t$ *for all* $t > 0$;
(ii) $\lim_{n\to\infty} \phi^n(t) = 0$ *for all* $t > 0$.

Proof (i) Suppose that there exists $t_0 > 0$ such that $t_0 \le \phi(t_0)$. Since ϕ is non-decreasing, we deduce that $\{\phi^n(t_0)\}_{n\in\mathbb{N}\cup\{0\}}$ is a non-decreasing sequence in $\mathbb{R}^+$, so, in particular, $t_0 \le \phi^n(t_0)$ for all $n \in \mathbb{N}$. Finally, since ϕ is a weaker Meir-Keeler function, there exists $n_0 \in \mathbb{N}$ such that $\phi^{n_0}(t_0) < t_0$, which yields a contradiction.

(ii) Fix $t > 0$. By (i) the sequence $\{\phi^n(t)\}_{n\in\mathbb{N}}$ is (strictly) decreasing, so there exists $r \ge 0$ such that $r = \lim_{n\to\infty} \phi^n(t)$. If $r > 0$, there is $\delta > 0$ such that for $s > 0$ with $r \le s < r + \delta$, there exists $n_s \in \mathbb{N}$ with $\phi^{n_s}(s) < r$. Let $n_r \in \mathbb{N}$ be such that $r < \phi^n(t) < r + \delta$ for all $n \ge n_r$. Putting $s = \phi^{n_r}(t)$, we deduce that $\phi^{n_s}(s) < r$, *i.e.*, $\phi^{n_s+n_r}(t) < r$, a contradiction. We conclude that $\lim_{n\to\infty} \phi^n(t) = 0$. □

Remark 3 Observe that, as a partial converse of the above lemma, if $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t > 0$, then ϕ is a weaker Meir-Keeler function. Indeed, otherwise, there exist $\varepsilon > 0$ and a sequence $\{t_n\}_{n\in\mathbb{N}}$ with $t_n \ge \varepsilon$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty} t_n = \varepsilon$ but $\phi^k(t_n) \ge \varepsilon$ for all $k, n \in \mathbb{N}$, a contradiction.

We also will use the following cyclic extension of the celebrated Matkowski fixed point theorem [17, Theorem 1.2], where for a self-map f of a metric space (X, d), we define

$$M_d(x,y) = \max\left\{d(x,y), d(x,fx), d(y,fy), \frac{1}{2}\left[d(x,fy) + d(fx,y)\right]\right\}$$

for all $x, y \in X$.

Theorem 4 (*cf.* [18, Corollary 2.14]) *Let f be a self-map of a complete metric space (X,d), and let $X = \bigcup_{i=1}^m A_i$ be a cyclic representation of X with respect to f, with A_i non-empty and closed, $i = 1,\dots,m$. If $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing function such that* $\lim_{n\to\infty} \phi^n(t) = 0$ *for all $t > 0$, and for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \dots, m$,*

$$d(fx,fy) \le \phi\bigl(M_d(x,y)\bigr),$$

where $A_{m+1} = A_1$, then f has a unique fixed point z. Moreover, $z \in \bigcap_{i=1}^m A_i$.

Then from Lemma 2 and Theorem 4 we immediately deduce the following theorem.

Theorem 5 *Let f be a self-map of a complete metric space (X,d), and let $X = \bigcup_{i=1}^m A_i$ be a cyclic representation of X with respect to f, with A_i non-empty and closed, $i = 1,\dots,m$. If $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing weaker Meir-Keeler function such that for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \dots, m$,*

$$d(fx,fy) \le \phi\bigl(M_d(x,y)\bigr),$$

where $A_{m+1} = A_1$, then f has a unique fixed point z. Moreover, $z \in \bigcap_{i=1}^m A_i$.

Corollary *Let f be a self-map of a complete metric space (X,d), and let $X = \bigcup_{i=1}^m A_i$ be a cyclic representation of X with respect to f, with A_i non-empty and closed, $i = 1,\dots,m$.*

If $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing weaker Meir-Keeler function such that for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$,

$$d(fx, fy) \le \phi\bigl(d(x, y)\bigr),$$

where $A_{m+1} = A_1$, then f has a unique fixed point z. Moreover, $z \in \bigcap_{i=1}^m A_i$.

Proof Since ϕ is non-decreasing, we deduce that for each $x, y \in X$, $\phi(d(x, y)) \le \phi(M_d(x, y))$, so $d(fx, fy) \le \phi(M_d(x, y))$. Hence, by Theorem 5, f has a unique fixed point z and $z \in \bigcap_{i=1}^m A_i$. □

Theorem 5 can be generalized according to the style of Chen's theorem as follows.

Theorem 6 *Let f be a self-map of a complete metric space (X, d), and let $X = \bigcup_{i=1}^m A_i$ be a cyclic representation of X with respect to f, with A_i non-empty and closed, $i = 1, \ldots, m$. If $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing weaker Meir-Keeler function, $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a function satisfying conditions (φ_i), $i = 1, 2, 3$, and for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$, it follows*

$$\varphi\bigl(d(fx, fy)\bigr) \le \phi\bigl(\varphi\bigl(M_d(x, y)\bigr)\bigr),$$

where $A_{m+1} = A_1$, then f has a unique fixed point z. Moreover, $z \in \bigcap_{i=1}^m A_i$.

Proof Construct the complete metric space (X, p) of Proposition 1, and observe that from the well-known fact that for $a_i \in \mathbb{R}^+$, $i = 1, \ldots, k$, one has $\phi(\max_i a_i) = \max_i \phi(a_i)$, one has

$$M_p(x, y) = \varphi\bigl(M_d(x, y)\bigr)$$

for all $x, y \in X$. Therefore, for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$, we deduce that

$$p(fx, fy) \le \phi\bigl(M_p(x, y)\bigr).$$

Theorem 5 concludes the proof. □

We finish this section with two examples illustrating Theorem 5 and its corollary.

Example 2 Let $A = \{n \in \mathbb{N} : n \text{ is even}\} \cup \{0\}$, $B = \{n \in \mathbb{N} : n \text{ is odd}\} \cup \{0\}$, $X = A \cup B = \mathbb{N}$, and let d be the complete metric on X defined by $d(x, x) = 0$ for all $x \in X$ and $d(x, y) = x + y$ otherwise. Since d induces the discrete topology on X, we deduce that A and B are closed subsets of (X, d).

Let f be the self-map of X defined by $f0 = 0$ and $fx = x - 1$ otherwise. It is clear that $X = A \cup B$ is a cyclic representation of X with respect to f.

Now we define the function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\phi(0) = 0$, and $\phi(t) = n - 1$ if $t \in (n - 1, n]$, $n \in \mathbb{N}$. It is immediate to check that ϕ is a non-decreasing weaker Meir-Keeler function which is not a Meir-Keeler function.

Furthermore, we have:

- For $x = 0$ and $y = 1$, $d(fx, fy) = d(0, 0) = 0$.

- For $x = 0$ and $y = n \in \mathbb{N}\backslash\{1\}$,

$$d(fx, fy) = d(0, n-1) = n-1 = \phi(n) = \phi\bigl(d(x,y)\bigr).$$

- For $x = n \in A\backslash\{0\}$ and $y = m \in B\backslash\{0\}$,

$$\begin{aligned} d(fx, fy) &= d(n-1, m-1) = n+m-2 < n+m-1 \\ &= \phi(n+m) = \phi\bigl(d(x,y)\bigr). \end{aligned}$$

Consequently, the conditions of the corollary of Theorem 5 are verified; in fact, $z = 0 \in A \cap B$ is the unique fixed point of f.

Example 3 Let $A = [0, 1/2] \cup \{1\}$, $B = [1, 2]$, $X = A \cup B$ and let d be the restriction to X of the Euclidean metric on $\mathbb{R}$. Obviously, (X, d) is a complete metric space (in fact, it is compact), with A and B closed subsets of (X, d).

Let f be the self-map of X defined by $fx = 2 - x$ if $x \in A$, and $fx = 1$ if $x \in B$. It is clear that $X = A \cup B$ is a cyclic representation of X with respect to f.

Now we define the function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\phi(t) = t/2$ if $t \in [0, 1]$, and $\phi(t) = 1$ if $t > 1$. (Notice that ϕ is a non-decreasing weaker Meir-Keeler function which is not a Meir-Keeler function.)

Furthermore, we have:

- For $x = 1 \in A$ and $y \in B$, $d(fx, fy) = d(1, 1) = 0$.
- For $x = 1/2 \in A$ and $y \in B$,

$$d(fx, fy) = d(3/2, 1) = 1/2 = \phi(1) = \phi\bigl(d(x, fx)\bigr).$$

- For $x \in A\backslash\{1, 1/2\}$ and $y \in B$,

$$d(fx, fy) = d(2-x, 1) = 1-x \le 1 = \phi(2-2x) = \phi\bigl(d(x, fx)\bigr).$$

Consequently, the conditions of Theorem 5 are verified; in fact, $z = 1 \in A \cap B$ is the unique fixed point of f.

Finally, observe that the corollary of Theorem 5 cannot be applied in this case because for $x = 1/2 \in A$ and $y = 1 \in B$, we have

$$d(fx, fy) = 1/2 > \phi(1/2) = \phi\bigl(d(x,y)\bigr).$$

3 Applications to well-posedness and limit shadowing property of a fixed point problem

The notion of well-posedness of a fixed point problem has evoked much interest to several mathematicians, for example, De Blasi and Myjak [19], Lahiri and Das [20], Popa [21, 22] and others.

Definition 3 [19] Let f be a self-map of a metric space (X, d). The fixed point problem of f is said to be well posed if:

(i) f has a unique fixed point $z \in X$;
(ii) for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that $\lim_{n\to\infty} d(fx_n, x_n) = 0$, we have $\lim_{n\to\infty} d(x_n, z) = 0$.

Definition 4 [22] Let f be a self-map of a metric space (X, d). The fixed point problem of f is said to have limit shadowing property in X if for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in X satisfying $\lim_{n\to\infty} d(fx_n, x_n) = 0$, it follows that there exists $z \in X$ such that $\lim_{n\to\infty} d(f^n z, x_n) = 0$.

Concerning the well-posedness and limit shadowing of the fixed point problem for a self-map of a complete metric space satisfying the conditions of Theorem 5, we have the following results.

Theorem 7 *Let (X, d) be a complete metric space. If f is a self-map of X and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing weaker Meir-Keeler function satisfying the conditions of Theorem* 5, *then the fixed point problem of f is well posed.*

Proof Owing to Theorem 5, we know that f has a unique fixed point $z \in X$. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} d(x_n, fx_n) = 0$. Then

$$\begin{aligned} d(x_n, z) &\le d(x_n, fx_n) + d(fx_n, fz) \\ &\le d(x_n, fx_n) \\ &\quad + \phi\left(\max\left\{d(x_n, z), d(x_n, x_{n+1}), d(z, fz), \frac{d(x_n, fz) + d(z, x_{n+1})}{2}\right\}\right). \end{aligned}$$

Passing to the limit as $n \to \infty$ in the above inequality, it follows that $\lim_{n\to\infty} d(x_n, z) = 0$. □

Theorem 8 *Let (X, d) be a complete metric space. If f is a self-map of X and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing weaker Meir-Keeler function satisfying the conditions of Theorem* 5, *then f has the limit shadowing property.*

Proof From Theorem 5, we know that f has a unique fixed point $z \in X$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X such that $\lim_{n\to\infty} d(x_n, fx_n) = 0$. Then, as in the proof of the previous theorem,

$$\begin{aligned} d(x_n, z) &\le d(x_n, fx_n) \\ &\quad + \phi\left(\max\left\{d(x_n, z), d(x_n, x_{n+1}), d(z, fz), \frac{d(x_n, fz) + d(z, x_{n+1})}{2}\right\}\right). \end{aligned}$$

Passing to the limit as $n \to \infty$ in the above inequality, it follows that $\lim_{n\to\infty} d(x_n, f^n z) = d(x_n, z) = 0$. □

4 An application to integral equations

In this section we apply Theorem 5 to study the existence and uniqueness of solutions for a class of nonlinear integral equations.

We consider the nonlinear integral equation

$$u(t) = \int_0^T G(t,s)K\bigl(s,u(s)\bigr)\,ds \quad \text{for all } t \in [0,T], \tag{1}$$

where $T > 0$, $K : [0,T] \times \mathbb{R}^+ \to \mathbb{R}^+$ and $G : [0,T] \times [0,T] \to \mathbb{R}^+$ are continuous functions, and $M := \max_{(s,x)\in[0,T]^2} K(s,x)$.

We shall suppose that the following four conditions are satisfied:

(I) $\int_0^T G(t,s)\,ds \le 1$ for all $t \in [0,T]$.

(II) $K(s,\cdot)$ is a non-increasing function for any fixed $s \in [0,1]$, that is,

$$x,y \in \mathbb{R}^+, \quad x \ge y \quad \Longrightarrow \quad K(s,x) \le K(s,y).$$

(III) There exists a Meir-Keeler function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ that is non-decreasing on $[0,2M]$ and such that

$$\bigl|K(s,x) - K(s,y)\bigr| \le \psi\bigl(|x-y|\bigr)$$

for all $s,x \in [0,T]$ and $y \in \mathbb{R}^+$ with $|x-y| \le 2M$.

(IV) There exists a continuous function $\alpha : [0,T] \to [0,T]$ such that:
For all $t \in [0,T]$, we have

$$\alpha(t) \le \int_0^T G(t,s)K(s,T)\,ds$$

and

$$T \ge \int_0^T G(t,s)K\bigl(s,\alpha(s)\bigr)\,ds.$$

Now denote by $C([0,T],\mathbb{R}^+)$ the set of non-negative real continuous functions on $[0,T]$. We endow $C([0,T],\mathbb{R}^+)$ with the supremum metric

$$d_\infty(u,v) = \max_{t\in[0,T]} \bigl|u(t) - v(t)\bigr|, \quad \text{for all } u,v \in C\bigl([0,T],\mathbb{R}^+\bigr).$$

It is well known that $(C([0,T],\mathbb{R}^+), d_\infty)$ is a complete metric space.

Consider the self-map $f : C([0,T],\mathbb{R}^+) \to C([0,T],\mathbb{R}^+)$ defined by

$$fu(t) = \int_0^T G(t,s)K\bigl(s,u(s)\bigr)\,ds \quad \text{for all } t \in [0,T].$$

Clearly, u is a solution of (1) if and only if u is a fixed point of f.

In order to prove the existence of a (unique) fixed point of f, we construct the closed subsets A_1 and A_2 of $C([0,T],\mathbb{R}^+)$ as follows:

$$A_1 = \bigl\{u \in C\bigl([0,T],\mathbb{R}^+\bigr) : u(s) \le T \text{ for all } s \in [0,T]\bigr\},$$

and

$$A_2 = \bigl\{u \in C\bigl([0,T],\mathbb{R}^+\bigr) : u \ge \alpha\bigr\}.$$

We shall prove that

$$f(A_1) \subseteq A_2 \quad \text{and} \quad f(A_2) \subseteq A_1. \tag{2}$$

Let $u \in A_1$, that is,

$$u(s) \leq T \quad \text{for all } s \in [0, T].$$

Since $G(t, s) \geq 0$ for all $t, s \in [0, T]$, we deduce from (II) and (IV) that

$$\int_0^T G(t,s)K\bigl(s, u(s)\bigr)\,ds \geq \int_0^T G(t,s)K(s,T)\,ds \geq \alpha(t)$$

for all $t \in [0, T]$. Then we have $fu \in A_2$.

Similarly, let $u \in A_2$, that is,

$$u(s) \geq \alpha(s) \quad \text{for all } s \in [0, T].$$

Again, from (II) and (IV), we deduce that

$$\int_0^T G(t,s)K\bigl(s, u(s)\bigr)\,ds \leq \int_0^T G(t,s)K\bigl(s,\alpha(s)\bigr)\,ds \leq T$$

for all $t \in [0, T]$. Then we have $fu \in A_1$. Thus, we have shown that (2) holds.

Hence, if $X := A_1 \cup A_2$, we have that X is closed in $C([0, T], \mathbb{R}^+)$, so the metric space (X, d_∞) is complete.

Moreover, $X := A_1 \cup A_2$ is a cyclic representation of the restriction of f with respect to X, which will be also denoted by f.

Now construct the function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ given by

$$\phi(t) = \psi(t) \quad \text{if } t \in [0, 2M],$$

and

$$\phi(t) = 2M \quad \text{if } t > 2M.$$

Since ψ is a Meir-Keeler function that is non-decreasing on $[0, 2M]$, it immediately follows that ϕ is a non-decreasing weaker Meir-Keeler function. Note also that ϕ is not continuous at $t = 2M$ (in fact, it is not a Meir-Keeler function).

Finally we shall show that for each $u \in A_1$ and $v \in A_2$, one has $d_\infty(fu, fv) \leq \phi(d_\infty(u, v))$.

To this end, let $u \in A_1$ and $v \in A_2$. Since $u(s) \in [0, T]$ for each $s \in [0, T]$, we have that

$$\begin{aligned} fu(t) &= \int_0^T G(t,s)K\bigl(s, u(s)\bigr)\,ds \\ &\leq M \int_0^T G(t,s)\,ds \leq M \end{aligned}$$

for all $t \in [0, T]$.

Similarly, since $v \geq \alpha$ and $\alpha(s) \in [0, T]$ for each $s \in [0, T]$, we deduce that

$$fv(t) \leq \int_0^T G(t,s)K\big(s,\alpha(s)\big)\,ds \leq M$$

for all $t \in [0, T]$. Therefore

$$\big|fu(t) - f(v(t)\big| \leq fu(t) + fv(t) \leq 2M$$

for all $t \in [0, T]$. So,

$$d_\infty(fu, fv) \leq 2M.$$

If $d_\infty(u, v) > 2M$, we have $\phi(d_\infty(u, v)) = 2M$, so

$$d_\infty(fu, fv) \leq \phi\big(d_\infty(u, v)\big).$$

If $d_\infty(u, v) \leq 2M$, then $|u(s) - v(s)| \leq 2M$ for all $s \in [0, T]$, so by (III), we deduce that for each $t \in [0, T]$,

$$\begin{aligned}
\big|fu(t) - f(v(t)\big| &\leq \int_0^T G(t,s)\big|K\big(s,u(s)\big) - K\big(s,v(s)\big)\big|\,ds \\
&\leq \int_0^T G(t,s)\psi\big(\big|u(s) - v(s)\big|\big)\,ds \\
&\leq \psi\big(d_\infty(u,v)\big)\int_0^T G(t,s)\,ds \\
&\leq \psi\big(d_\infty(u,v)\big) \\
&= \phi\big(d_\infty(u,v)\big).
\end{aligned}$$

Consequently, by the corollary of Theorem 5, f has a unique fixed point $u^* \in A_1 \cap A_2$, that is, $u^* \in \mathcal{C}$ is the unique solution to (1) in $A_1 \cup A_2$.

Remark 4 The first author studied in [9, Section 3] a variant of the problem discussed above for the case that ψ is the non-decreasing Meir-Keeler function given by $\psi(t) = (\ln(t^2 + 1))^{1/2}$ for all $t \in \mathbb{R}^+$.

The next example illustrates the preceding development.

Example 4 Consider the integral equation

$$u(t) = \int_0^T G(t,s)K\big(s,u(s)\big)\,ds \quad \text{for all } t \in [0, T],$$

where $T = 1$, $G(t, s) = t$ for all $t, s \in [0, 1]$, and

$$K(s,x) = \frac{\cos s}{1 + x}$$

for all $s \in [0, 1]$ and $x \geq 0$.

Hence, $M = \max_{(s,x)\in[0,1]^2} K(s,x) = K(0,0) = 1$.

Furthermore, it is obvious that G satisfies condition (I), whereas K satisfies condition (II).

Now construct a Meir-Keeler function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ as

$$\psi(t) = t/(1+t) \quad \text{if } t \in [0,2],$$

and

$$\psi(t) = 0 \quad \text{if } t > 2.$$

Note that ψ is non-decreasing on $[0,2]$ and not continuous at $t = 2$.

Moreover, for each $s, x \in [0,1]$ and each $y \in \mathbb{R}^+$ with $|x-y| \le 2$, we have

$$\left|K(s,x) - K(s,y)\right| = \cos s \left|\frac{1}{1+x} - \frac{1}{1+y}\right| \le \frac{|x-y|}{1+|x-y|} = \psi\left(|x-y|\right),$$

so condition (III) is also satisfied.

Finally, define $\alpha : [0,1] \to [0,1]$ as $\alpha(t) = t/3$ for all $t \in [0,1]$. It is not hard to check that α verifies condition (IV), and consequently the integral equation has a unique solution u^* in $A_1 \cup A_2$, where $A_1 = \{u \in C([0,1],\mathbb{R}^+) : u(s) \le 1 \text{ for all } s \in [0,1]\}$ and $A_2 = \{u \in C([0,1],\mathbb{R}^+) : u(s) \ge s/3 \text{ for all } s \in [0,1]\}$. In fact $u^* \in A_1 \cap A_2$, *i.e.*, $t/3 \le u^*(t) \le 1$ for all $t \in [0,1]$.

Note that, according to our constructions, for each pair $u, v \in C([0,1],\mathbb{R}^+)$ with $u \le 1$ and $v \ge \alpha$, we have $d_\infty(fu,fv) \le \phi(d_\infty(u,v))$, where ϕ is the non-decreasing weaker Meir-Keeler function defined as $\phi(t) = t/(t+1)$ if $t \in [0,2]$ and $\phi(t) = 2$ if $t > 2$.

In particular, we can deduce the following approximation to the value of $u^*(t)$ for each $t \in [0,1]$:

$$\begin{aligned}
\left|u^*(t) - \frac{\sin 1}{2} t\right| &= \left|u^*(t) - \int_0^1 t \frac{\cos s}{2}\, ds\right| = \left|fu^*(t) - \int_0^1 G(t,s)K(s,1)\, ds\right| \\
&\le \phi\bigl(d_\infty\bigl(u^*,1\bigr)\bigr) = \frac{\max_{t\in[0,1]}(1-u^*(t))}{1+\max_{t\in[0,1]}(1-u^*(t))} \\
&= \frac{1-\min_{t\in[0,1]} u^*(t)}{2-\min_{t\in[0,1]} u^*(t)} \\
&\le \frac{1}{2}.
\end{aligned}$$

Note also that the contraction inequality $d_\infty(fu,fv) \le \phi(d_\infty(u,v))$ does not follow when the weaker Meir-Keeler function ϕ is replaced by our initial Meir-Keeler function ψ: Take, for instance, the constant functions $u = 0$ and $v = 3$; then $u \le 1$, $v \ge \alpha$, and

$$\psi\left(d_\infty(u,v)\right) = \psi(3) = 0 < d_\infty(fu,fv).$$

Remark 5 In Example 4 above, the inequality $|K(s,x) - K(s,y)| \le \psi(|x-y|)$ is not globally satisfied, *i.e.*, there exist $s, x \in [0,1]$ and $y \in \mathbb{R}^+$ such that $|K(s,x) - K(s,y)| > \psi(|x-y|)$. In fact, this happens for all $x, y \in \mathbb{R}^+$ with $y > x + 2$. However, it is clear that for each $s \in [0,1]$, and $x, y \in \mathbb{R}^+$, one has $|K(s,x) - K(s,y)| \le \psi_1(|x-y|)$ for all $s \in [0,1]$, and $x, y \in \mathbb{R}^+$, where $\psi_1(t) = t/(t+1)$ for all $t \in \mathbb{R}^+$.

We conclude the paper with an example where conditions (I)-(IV) also hold (in particular, (III) for the function ψ_1 defined above) but the inequality $|K(s,x)-K(s,y)| \leq \psi_1(|x-y|)$ is not globally satisfied.

Example 5 We modify Example 4 as follows. Consider the integral equation

$$u(t) = \int_0^T G(t,s)K\big(s,u(s)\big)\,ds \quad \text{for all } t \in [0,T],$$

where $T = 2$, $G(t,s) = t/2$ for all $t,s \in [0,2]$, and

$$K(s,x) = e^{-s}/(1+x) \quad \text{if } s \in [0,2], x \in [0,1];$$
$$K(s,x) = e^{-s}/\big(1+x^{1/2}\big) \quad \text{if } s \in [0,2], x \in (1,4];$$
$$K(s,x) = e^{-s}/(4x-13) \quad \text{if } s \in [0,2], x > 4.$$

Clearly K is continuous on $[0,2] \times \mathbb{R}^+$. Moreover, $M = 1$, and G and K satisfy conditions (I) and (II), respectively.

Now, construct a Meir-Keeler function $\psi_1 : \mathbb{R}^+ \to \mathbb{R}^+$ as $\psi_1(t) = t/(1+t)$ for all $t \in \mathbb{R}^+$.

By discussing the different cases, it is routine to show that for each $s,x \in [0,2]$ and each $y \in \mathbb{R}^+$ with $|x-y| \leq 2$, we have

$$\big|K(s,x) - K(s,y)\big| \leq \psi_1\big(|x-y|\big),$$

so condition (III) is also satisfied.

Finally, define $\alpha : [0,2] \to [0,2]$ as $\alpha(t) = 6t/35$ for all $t \in [0,2]$. Then, for each $t \in [0,2]$, we have

$$\int_0^2 G(t,s)K(s,2)\,ds = \frac{t}{2}\int_0^2 \frac{e^{-s}}{1+\sqrt{2}}\,ds = t\frac{(1-e^{-2})}{2(1+\sqrt{2})} > \frac{6t/7}{5} = \alpha(t).$$

Now observe that $\alpha(s) < 1$ for all $s \in [0,2]$, so $K(s,\alpha(s)) = e^{-s}/(1+\alpha(s))$. Hence, for each $t \in [0,2]$,

$$\begin{aligned}\int_0^2 G(t,s)K\big(s,\alpha(s)\big)\,ds &= \frac{t}{2}\int_0^2 \frac{e^{-s}}{1+(6s/35)}\,ds = \frac{t}{2}\int_0^2 \frac{35e^{-s}}{35+6s}\,ds \\ &\leq \frac{t}{2}\int_0^2 ds = t \leq 2.\end{aligned}$$

Therefore α verifies condition (IV), and consequently the integral equation has a unique solution u^* in $A_1 \cup A_2$, where $A_1 = \{u \in C([0,1],\mathbb{R}^+) : u(s) \leq 2 \text{ for all } s \in [0,2]\}$ and $A_2 = \{u \in C([0,1],\mathbb{R}^+) : u(s) \geq 6s/35 \text{ for all } s \in [0,2]\}$. In fact $u^* \in A_1 \cap A_2$, *i.e.*, $6t/35 \leq u^*(t) \leq 2$ for all $t \in [0,2]$.

It is interesting to observe that the Meir-Keeler function ψ_1 is continuous on $\mathbb{R}^+$ but condition (III) is not globally satisfied: Indeed, take $x = 0$ and $y > 14/3$. Then, for each

$s \in [0,1]$, we obtain

$$K(s,x) - K(s,y) = e^{-s}\left(1 - \frac{1}{4y-13}\right) > e^{-s}\frac{y}{1+y}.$$

Hence, $K(0,0) - K(0,y) > \psi_1(y)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The two authors contributed equally in writing this article. They read and approved the final manuscript.

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