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Strong convergence theorems for fixed point problems of infinite family of asymptotically quasi-ϕ-nonexpansive mappings and a system of equilibrium problems

Xin Wang, Changsong Hu* and Jinlin Guan

*Correspondence:
huchang1004@yahoo.com.cn
School of Mathematics and Statistics, Hubei Normal University, Huangshi, 435002, China

Abstract

In this paper, we introduce a general iterative algorithm for finding a common element of the set of common fixed points of infinite family of asymptotically quasi-ϕ-nonexpansive mappings and of the set of solutions for finite equilibrium problems in a real Banach space. Our results are the generalization of the results (Shehu in Comput. Math. Appl. 63:1089-1103, 2012; Kim in Fixed Point Theory Appl., 2011, doi:10.1186/1687-1812-2011-10) and (Kim and Buong in Fixed Point Theory Appl., 2011, doi:10.1155/2011/780764), and improvement of the result (Yang *et al.* in Appl. Math. Comput. 218:6072-6082, 2012).

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1 Introduction

Let C be a nonempty, closed and convex subset of a real Banach space E. A mapping $T : C \to C$ is called to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

A mapping $T : C \to C$ is called to be quasi-nonexpansive if

$$\|Tx - x^*\| \leq \|x - x^*\|, \quad \forall x \in C, x^* \in F(T).$$

Let F be a bifunction of $C \times C$ into $\mathbb{R}$. The equilibrium problem is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of solutions to equilibrium problem (1.2) is denoted by $\mathrm{EP}(F)$. That is,

$$\mathrm{EP}(F) := \{x \in C : F(x, y) \geq 0, \forall y \in C\}.$$

Recently, Yang *et al.* [1] proved strong convergence theorems for approximation of common fixed points of countably infinite family of asymptotically quasi-ϕ-nonexpansive mappings in a uniformly smooth and strictly convex real Banach space, which has the Kadec-Klee property. More precisely, they proved the following theorem.

Theorem 1.1 *Let E be a uniformly smooth and strictly convex Banach space, which has the Kadec-Klee property, and let C be a nonempty closed convex subset of E. Let G be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4), *and let $T_i : C \to C$, $\forall i \in \mathbb{N}$ be an infinite family of closed and asymptotically quasi-ϕ-nonexpansive mapping with $\{k_{ni}\} \subset [1, \infty)$, $k_{ni} \to 1$ as $n \to \infty$, where $T_0 = I$. Assume that T_i, $\forall i \in \mathbb{N}$ is asymptotically regular on C and $\Im = \bigcap_{i=0}^{\infty} F(T_i) \cap \mathrm{EP}(G)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence, generated by*

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}\{\sum_{i=0}^{\infty} \alpha_{ni} J T_i^n x_n\}, \\ u_n \in C \text{ such that } G(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + (k_n - 1)M_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases}$$

where J is the duality mapping on E, $M_n = \sup\{\phi(z, x_n) : z \in \Im\}$ for each $n \geq 1$, $k_n = \sup_{i\geq 0}\{k_{ni}\}$, $\{r_n\}$ is real sequence in $[a, \infty)$, where a is some positive real number, $\{\alpha_{ni}\}$ is a real sequence in $[0, 1]$ satisfying the following conditions: (a) $\sum_{i=0}^{\infty} \alpha_{ni} = 1$, $\forall n \geq 1$, (b) $\liminf_{n\to\infty} \alpha_{n0}\alpha_{ni} > 0$, $\forall i \in \mathbb{N}$. *Then the sequence $\{x_n\}$ converges strongly to $\Pi_{\Im} x_0$.*

In [2], Shehu introduced the following hybrid iterative scheme for approximating a common element of the set of fixed points of relatively quasi-nonexpansive mappings and the set of solutions to an equilibrium problem in a uniformly smooth and uniformly convex real Banach space: $x_0 \in C$, $C_1 = C$, $x_1 = \Pi_{C_1}^f x_0$,

$$\begin{cases} y_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T_n x_n), \quad n \geq 1, \\ u_n = T_{r_{m,n}}^{F_m} T_{r_{m-1,n}}^{F_{m-1}} \cdots T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1} y_n, \\ C_{n+1} = \{w \in C_n : G(w, Ju_n) \leq G(w, Jx_n)\}, \quad n \geq 1, \\ x_{n+1} = \Pi_{C_{n+1}}^f x_0, \quad n \geq 1. \end{cases}$$

Motivated by the facts above, the purpose of this paper is to prove a strong convergence theorem for finding a common element of the set of fixed points of asymptotically quasi-ϕ-nonexpansive mappings and the set of solutions to a system of equilibrium problems in a uniformly smooth and uniformly convex real Banach space, which has the Kadec-Klee property.

2 Preliminaries

Let E be a real Banach space, and let E^* be the dual space of E. The duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}. \tag{2.1}$$

By Hahn-Banach theorem, $J(x)$ is nonempty.

The modulus of smoothness of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(\tau) := \sup\left\{\frac{1}{2}\big(\|x+y\| + \|x-y\|\big) - 1 : \|x\| \le 1, \|y\| \le \tau\right\}. \tag{2.2}$$

E is said to be uniformly smooth if $\lim_{\tau\to 0} \frac{\rho_E(\tau)}{\tau} = 0$.

Let $\dim E \ge 2$. The modulus of convexity of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) := \inf\left\{1 - \left\|\frac{x-y}{2}\right\| : \|x\| = \|y\| = 1; \epsilon = \|x-y\|\right\}. \tag{2.3}$$

E is said to be uniformly convex if $\forall\epsilon \in (0,2]$, there exists a $\delta = \delta(\epsilon) > 0$ such that for $x, y \in E$ with $\|x\| \le 1$, $\|y\| \le 1$ and $\|x-y\| \ge \epsilon$, then $\|\frac{x+y}{2}\| \le 1 - \delta$. Equivalently, E is uniformly convex if and only if $\delta_E(\epsilon) > 0$, $\forall\epsilon \in (0,2]$. E is strictly convex if for all $x, y \in E$, $x \ne y$, $\|x\| = \|y\| = 1$, we have $\|\lambda x + (1-\lambda)y\| < 1$, $\forall\lambda \in (0,1)$.

It is well known that if E is uniformly smooth, then J is norm-to-norm uniformly continuous on each bounded subset of E. If E is smooth, then J is single-valued.

Recall that a Banach space E has the Kadec-Klee property if for any sequence $\{x_n\} \subset E$ and $x \in E$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$, as $n \to \infty$. It is well known that if E is a uniformly convex Banach space, then E has the Kadec-Klee property.

We denoted by ϕ the Lyapunov function from $E \times E$ to $\mathbb{R}$ defined by

$$\phi(x, y) := \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \tag{2.4}$$

It follows from the definition of ϕ that

$$\big(\|x\| - \|y\|\big)^2 \le \phi(x, y) \le \big(\|x\| + \|y\|\big)^2, \quad \forall x, y \in E. \tag{2.5}$$

Let E be a reflexive strictly convex and smooth Banach space. Then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$ (see [3, 4]).

Definition 2.1 Let C be a nonempty closed convex subset of E, and let T be a mapping from C into itself. A point $p \in C$ is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}$, which converges weakly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T is denoted by $\widetilde{F}(T)$.

We say that T is a relatively nonexpansive mapping [5–8] if the following conditions are satisfied:

(R1) $F(T) \ne \emptyset$;

(R2) $\phi(p, Tx) \le \phi(p, x)$, $\forall x \in C$, $p \in F(T)$;

(R3) $F(T) = \widetilde{F}(T)$.

If T satisfies (R1) and (R2), then T is said to be relatively quasi-nonexpansive [9–11].

Definition 2.2 We say that T is an asymptotically ϕ-nonexpansive mapping if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that $\phi(T^n x, T^n y) \leq k_n \phi(x,y)$, $\forall x,y \in C$. We say that T is an asymptotically quasi-ϕ-nonexpansive [11, 12] mapping if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1,\infty)$ as $n \to \infty$ such that $\phi(p, T^n x) \leq k_n \phi(p,x)$, $\forall x \in C$, $p \in F(T)$.

It is easy to see that the class of relatively quasi-nonexpansive mappings and asymptotically quasi-ϕ-nonexpansive mappings contains the class of relatively nonexpansive mappings. The class of asymptotically quasi-ϕ-nonexpansive mappings is more general than the class of relatively quasi-nonexpansive mappings.

Following Alber [13], the generalized projection $\Pi_C : E \to C$ is defined by

$$\Pi_C(x) = \Big\{ u \in C : \phi(u,x) = \min_{y \in C} \phi(y,x) \Big\}, \quad \forall x \in E.$$

The existence and uniqueness of the operator Π_C follows from the properties of the function $\phi(y,x)$ and strict monotonicity of mapping J (see, for example, [3, 4, 13, 14]). If E is a Hilbert space, then $\phi(y,x) = \|y - x\|^2$, $x, y \in E$ and Π_C is the metric projection P_C of E onto C.

Next, we recall the concept and properties of generalized f-projector operator. Let $G : C \times E^* \to \mathbb{R} \cup \{+\infty\}$ be a function defined as follows:

$$G(\xi, \varphi) = \|\xi\|^2 - 2\langle \xi, \varphi \rangle + \|\varphi\|^2 + 2\rho f(\xi),$$

where $\xi \in C$, $\varphi \in E^*$, ρ is a positive number, and $f : C \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semi-continuous. From the definitions of G and f, it is easy to see that the following properties hold:

(i) $G(\xi, \varphi)$ is convex and continuous with respect to φ when ξ is fixed;

(ii) $G(\xi, \varphi)$ is convex and lower semi-continuous with respect to ξ when φ is fixed.

Definition 2.3 [15] Let E be a real Banach space with its dual E^*. Let C be a nonempty closed convex subset of E. We say that $\Pi_C^f : E^* \to 2^C$ is a generalized f-projection operator if

$$\Pi_C^f \varphi = \Big\{ u \in C : G(u, \varphi) = \inf_{\xi \in C} G(\xi, \varphi) \Big\}, \quad \forall \varphi \in E^*.$$

Lemma 2.4 [15] *Let E be a reflexive Banach space with its dual E^*. Let C be a nonempty closed convex subset of E. Then the following statements hold*:

(i) $\Pi_C^f \varphi$ *is a nonempty closed convex subset of C for all $\varphi \in E^*$*;

(ii) *If E is smooth, then for all $\varphi \in E^*$, $x \in \Pi_C^f \varphi$ if and only if*

$$\langle x - y, \varphi - Jx \rangle + \rho f(y) - \rho f(x) \geq 0, \quad \forall y \in C;$$

(iii) [16] *If E is strictly convex, then Π_C^f is a single-valued mapping.*

Recall that J is a single-valued mapping when E is a smooth Banach space. There exists a unique element $\varphi \in E^*$ such that $\varphi = Jx$ for each $x \in E$. This substitution in (2.3) gives

$$G(\xi, Jx) = \|\xi\|^2 - 2\langle \xi, Jx\rangle + \|x\|^2 + 2\rho f(\xi).$$

Now, we consider the second generalized f-projection operator in Banach space.

Definition 2.5 Let E be a real Banach space and C be a nonempty closed convex subset of E. We say that $\Pi_C^f : E \to 2^C$ is a generalized f-projection operator if

$$\Pi_C^f x = \Bigl\{u \in C : G(u, Jx) = \inf_{\xi \in C} G(\xi, Jx)\Bigr\}, \quad \forall x \in E.$$

Obviously, the definition of relatively quasi-nonexpansive mapping T is equivalent to

(R′1) $F(T) \neq \emptyset$;
(R′2) $G(p, JTx) \leq G(p, Jx)$, $\forall x \in C$, $p \in F(T)$.

Lemma 2.6 [17] *Let C be a nonempty, closed and convex subset of a smooth and reflexive Banach space E. Then the following statements hold*:

(i) $\Pi_C^f x$ *is a nonempty closed convex subset of C for all $x \in E$*;
(ii) *For all $x \in E$, $\hat{x} \in \Pi_C^f x$ if and only if*

$$\langle \hat{x} - y, Jx - J\hat{x}\rangle + \rho f(y) - \rho f(\hat{x}) \geq 0, \quad \forall y \in C;$$

(iii) [16] *If E is strictly convex, then $\Pi_C^f x$ is a single-valued mapping.*

Lemma 2.7 [18] *Let E be a Banach space, and $f : E \to \mathbb{R} \cup \{+\infty\}$ is convex and lower semi-continuous. Then there exists $x^* \in E^*$ and $\alpha \in \mathbb{R}$ such that*

$$f(x) \geq \langle x, x^*\rangle + \alpha, \quad \forall x \in E.$$

Lemma 2.8 [17] *Let C be a nonempty closed convex subset of a smooth and reflexive Banach space E. Let $x \in E$ and $\hat{x} \in \Pi_C^f x$. Then*

$$\phi(y, \hat{x}) + G(\hat{x}, Jx) \leq G(y, Jx), \quad \forall y \in C.$$

Lemma 2.9 [1, 19] *Let E be a uniformly smooth and strictly convex Banach space, which has the Kadec-Klee property, and let C be a nonempty closed convex subset of E. Let T be a closed and asymptotically quasi-ϕ-nonexpansive mapping. Then $F(T)$ is a closed and convex subset of C.*

Lemma 2.10 [1] *Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : \|x\| \leq r\}$. Then, for any given sequence $\{x_n\}_{n=1}^\infty \subset B_r(0)$ and for any given sequence $\{\lambda_n\}_{n=1}^\infty$ of positive numbers such that $\sum_{i=1}^\infty \lambda_i = 1$, there exists a continuous strictly increasing convex function $g : [0, 2r] \to \mathbb{R}$, $g(0) = 0$ such that for any positive integers i, j with $i < j$, the following inequality holds*

$$\Biggl\|\sum_{n=1}^\infty \lambda_n x_n\Biggr\|^2 \leq \sum_{n=1}^\infty \lambda_n \|x_n\|^2 - \lambda_i \lambda_j g\bigl(\|x_i - x_j\|\bigr).$$

Lemma 2.11 [17] *Let E be a Banach space and $y \in E$. Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semi-continuous mapping with convex domain $D(f)$. If $\{x_n\}$ is a sequence in $D(f)$ such that $x_n \rightharpoonup x \in \operatorname{int}(D(f))$ and $\lim_{n\to\infty} G(x_n, Jy) = G(x, Jy)$, then $\lim_{n\to\infty} \|x_n\| = \|x\|$.*

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t\to 0} F(tz + (1-t)x, y) \leq F(x,y)$;

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous.

Lemma 2.12 [20] *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4). *Let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.13 [9, 21] *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4). *Let $r > 0$ and $x \in E$. Define a mapping $T_r^F : E \to C$ as follows*:

$$T_r^F(x) = \left\{z \in C : F(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \forall y \in C\right\}$$

for all $z \in E$. Then, the following hold:

1. T_r^F *is single-valued*;
2. T_r^F *is firmly nonexpansive mapping, i.e., for any $x, y \in E$,*

$$\left\langle T_r^F x - T_r^F y, JT_r^F x - JT_r^F y\right\rangle \leq \left\langle T_r^F x - T_r^F y, Jx - Jy\right\rangle;$$

3. $F(T_r^F) = \mathrm{EP}(F)$;
4. $T_r^F x$ *is relatively quasi-nonexpansive*;
5. $\mathrm{EP}(F)$ *is closed and convex.*

Lemma 2.14 [21] *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4). *Let $r > 0$. Then for each $x \in E$ and $q \in F(T_r^F)$,*

$$\phi\left(q, T_r^F x\right) + \phi\left(T_r^F x, x\right) \leq \phi(q,x).$$

An operator T in a Banach space E is said to be closed if $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

3 Main result

Theorem 3.1 *Let E be a uniformly smooth and strictly convex Banach space, which has the Kadec-Klee property, and let C be a nonempty closed convex subset of E. For each $k =$*

$1, 2, \dots, m$, *let* F_k *be a bifunction from* $C \times C$ *satisfying* (A1)-(A4), *and let* $\{T_i\}_{i=0}^{\infty} : C \to C$, $\forall i \in \mathbb{N}$ *be an infinite family of closed and asymptotically quasi-*ϕ*-nonexpansive mappings with sequence* $\{k_{ni}\} \subset [1, \infty)$, $k_{ni} \to 1$ *as* $n \to \infty$, *where* $T_0 = I$. *Assume that* T_i, $\forall i \in \mathbb{N}$ *is asymptotically regular on* C *and* $\Omega = (\bigcap_{i=0}^{\infty} F(T_i)) \cap (\bigcap_{k=1}^{m} \mathrm{EP}(F_k))$ *is nonempty and bounded. Let* $f : E \to \mathbb{R}$ *be a convex and lower semicontinuous mapping with* $C \subset \operatorname{int}(D(f))$, *and suppose that* $\{x_n\}_{n=0}^{\infty}$ *is a sequence generated by* $x_0 \in C$, $C_1 = C$, $x_1 = \Pi_{C_1}^f x_0$,

$$\begin{cases} y_n = J^{-1}\{\sum_{i=0}^{\infty} \alpha_{ni} J T_i^n x_n\}, \\ u_n = T_{r_{m,n}}^{F_m} T_{r_{m-1,n}}^{F_{m-1}} \cdots T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1} y_n, \\ C_{n+1} = \{z \in C_n : G(z, Ju_n) \le G(z, Jx_n) + (k_n - 1)M_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}^f x_0, \end{cases} \tag{3.1}$$

where J *is the duality mapping on* E, $M_n = \sup\{\phi(z, x_n) : z \in \Omega\}$ *for each* $n \ge 1$, $k_n = \sup_{i \ge 0}\{k_{ni}\}$, $\{\alpha_{ni}\}$ *is a real sequence in* $[0,1]$ *and* $\{r_{k,n}\}_{n=1}^{\infty} \subset (0, \infty)$, $k = 1, 2, \dots, m$, *satisfying the following conditions*:

$$\text{(a)} \quad \sum_{i=0}^{\infty} \alpha_{ni} = 1, \quad \forall n \ge 1;$$

$$\text{(b)} \quad \liminf_{n\to\infty} \alpha_{n0}\alpha_{ni} > 0, \quad \forall i \in \mathbb{N};$$

$$\text{(c)} \quad \liminf_{n\to\infty} r_{k,n} > 0.$$

Then the sequence $\{x_n\}$ *converges strongly to* $\Pi_{\Omega}^f x_0$.

Proof Step 1. We first show that C_n, $\forall n \ge 1$ is nonempty, closed and convex.

Now, we show that C_n, $\forall n \ge 1$ is closed and convex. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_n is closed convex for some $n > 1$. From the definition of C_{n+1}, we have $z \in C_{n+1}$, which implies that $G(z, Ju_n) \le G(z, Jx_n) + (k_n - 1)M_n$. This is equivalent to

$$2\bigl(\langle z, Jx_n\rangle - \langle z, Ju_n\rangle\bigr) \le \|x_n\|^2 - \|u_n\|^2 + (k_n - 1)M_n.$$

This implies that C_{n+1} is closed convex for the same $n > 1$. Hence, C_n is closed and convex $\forall n \ge 1$.

By taking $\theta_n^k = T_{r_{k,n}}^{F_k} T_{r_{k-1,n}}^{F_{k-1}} \cdots T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1}$, $k = 1, 2, \dots, m$ and $\theta_n^0 = I$ for all $n \ge 1$, we obtain $u_n = \theta_n^m y_n$.

We next show that $\Omega \subset C_n$, $\forall n \ge 1$. From Lemma 2.13, we have that $T_{r_{k,n}}^{F_k}$, $k = 1, 2, \dots, m$ is relatively nonexpansive mapping. For $n = 1$, we have $\Omega \subset C_1 = C$. Now, assume that $\Omega \subset C_n$ for some $n > 1$. For each $x^* \in \Omega$, we obtain

$$\begin{aligned} G\bigl(x^*, Ju_n\bigr) &= G\bigl(x^*, J\theta_n^m y_n\bigr) \le G\bigl(x^*, Jy_n\bigr) \\ &= \bigl\|x^*\bigr\|^2 - 2\bigl\langle x^*, Jy_n\bigr\rangle + \|y_n\|^2 + 2\rho f\bigl(x^*\bigr) \\ &\le \bigl\|x^*\bigr\|^2 - 2\sum_{i=0}^{\infty} \alpha_{ni}\bigl\langle x^*, JT_i^n x_n\bigr\rangle + \sum_{i=0}^{\infty} \alpha_{ni}\bigl\|T_i^n x_n\bigr\|^2 + 2\rho f\bigl(x^*\bigr) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=0}^{\infty} \alpha_{ni} \phi\left(x^*, T_i^n x_n\right) + 2\rho f\left(x^*\right) \\
&\leq \sum_{i=0}^{\infty} \alpha_{ni} k_{ni} \phi\left(x^*, x_n\right) + 2\rho f\left(x^*\right) \\
&= \sum_{i=0}^{\infty} \alpha_{ni}\left(1 + (k_{ni} - 1)\right)\phi\left(x^*, x_n\right) + 2\rho f\left(x^*\right) \\
&= G\left(x^*, Jx_n\right) + \sum_{i=0}^{\infty} \alpha_{ni}(k_{ni} - 1)\phi\left(x^*, x_n\right) \\
&\leq G\left(x^*, Jx_n\right) + (k_n - 1)M_n. \qquad (3.2)
\end{aligned}$$

So, $x^* \in C_{n+1}$. It implies that $\Omega \subset C_n$, $\forall n \geq 1$, and the sequence $\{x_n\}_{n=0}^{\infty}$ generated by (3.1) is well defined.

Step 2. We show that $\lim_{n\to\infty} G(x_n, Jx_0)$ exists.

Since $f: E \to \mathbb{R}$ is a convex and lower semi-continuous, applying Lemma 2.7, we see that there exist $u^* \in E^*$ and $\alpha \in \mathbb{R}$ such that

$$f(y) \geq \langle y, u^* \rangle + \alpha, \quad \forall y \in E.$$

It follows that

$$\begin{aligned}
G(x_n, Jx_0) &= \|x_n\|^2 - 2\langle x_n, Jx_0\rangle + \|x_0\|^2 + 2\rho f(x_n) \\
&\geq \|x_n\|^2 - 2\langle x_n, Jx_0\rangle + \|x_0\|^2 + 2\rho\langle x_n, u^*\rangle + 2\rho\alpha \\
&= \|x_n\|^2 - 2\langle x_n, Jx_0 - \rho u^*\rangle + \|x_0\|^2 + 2\rho\alpha \\
&\geq \|x_n\|^2 - 2\|x_n\| \left\|Jx_0 - \rho u^*\right\| + \|x_0\|^2 + 2\rho\alpha \\
&= \left(\|x_n\| - \left\|Jx_0 - \rho u^*\right\|\right)^2 + \|x_0\|^2 - \left\|Jx_0 - \rho u^*\right\|^2 + 2\rho\alpha. \qquad (3.3)
\end{aligned}$$

Since $x_n = \Pi_{C_n}^f x_0$, it follows from (3.3) that

$$G\left(x^*, Jx_0\right) \geq G(x_n, Jx_0) \geq \left(\|x_n\| - \left\|Jx_0 - \rho u^*\right\|\right)^2 + \|x_0\|^2 - \left\|Jx_0 - \rho u^*\right\|^2 + 2\rho\alpha$$

for each $x^* \in F(T)$. This implies that $\{x_n\}_{n=0}^{\infty}$ is bounded and so is $\{G(x_n, Jx_0)\}_{n=0}^{\infty}$. By the construction of C_n, we have that $C_{n+1} \subset C_n$ and $x_{n+1} = \Pi_{C_{n+1}}^f x_0 \in C_n$. It follows from Lemma 2.8 that

$$\phi(x_{n+1}, x_n) + G(x_n, Jx_0) \leq G(x_{n+1}, Jx_0). \qquad (3.4)$$

It is obvious that

$$\phi(x_{n+1}, x_n) \geq \left(\|x_{n+1}\| - \|x_n\|\right)^2 \geq 0,$$

and so, $\{G(x_n, Jx_0)\}_{n=0}^{\infty}$ is nondecreasing. It follows that the limit of $\{G(x_n, Jx_0)\}_{n=0}^{\infty}$ exists.

Step 3. We prove that $\lim_{n\to\infty} \|Jx_n - JT_j^n x_n\| = 0$, $\forall j \in \mathbb{N}$.

Now, since $\{x_n\}_{n=0}^{\infty}$ is bounded in C, and E is reflexive, we may assume that $x_n \rightharpoonup p$, and since C_n is closed and convex for each $n \geq 1$, it is easy to see that $p \in C_n$ for each $n \geq 1$. Again, since $x_n = \Pi_{C_n}^f x_0$, we obtain

$$G(x_n, Jx_0) \leq G(p, Jx_0), \quad \forall n \geq 0.$$

Since

$$\begin{aligned}\liminf_{n\to\infty} G(x_n, Jx_0) &= \liminf_{n\to\infty} \bigl\{ \|x_n\|^2 - 2\langle x_n, Jx_0\rangle + \|x_0\|^2 + 2\rho f(x_n) \bigr\} \\ &\geq \|p\|^2 - 2\langle p, Jx_0\rangle + \|x_0\|^2 + 2\rho f(p) = G(p, Jx_0).\end{aligned}$$

Then, we obtain

$$G(p, Jx_0) \leq \liminf_{n\to\infty} G(x_n, Jx_0) \leq \limsup_{n\to\infty} G(x_n, Jx_0) \leq G(p, Jx_0).$$

This implies that

$$\lim_{n\to\infty} G(x_n, Jx_0) = G(p, Jx_0).$$

By Lemma 2.11, we obtain that $\lim_{n\to\infty} \|x_n\| = \|p\|$. In view of Kadec-Klee property of E, we have that $\lim_{n\to\infty} x_n = p$.

By the construction of C_n, we have that $C_{n+1} \subset C_n$ and $x_{n+1} = \Pi_{C_{n+1}}^f x_0 \in C_{n+1}$. It follows that

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n) + (k_n - 1)M_n.$$

Now, (3.4) implies that

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n) + (k_n - 1)M_n \leq G(x_{n+1}, Jx_0) - G(x_n, Jx_0) + (k_n - 1)M_n. \tag{3.5}$$

Taking the limit as $n \to \infty$ in (3.5), we obtain

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0.$$

Therefore,

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0.$$

It then yields that $\lim_{n\to\infty}(\|x_{n+1}\| - \|u_n\|) = 0$. Since $\lim_{n\to\infty} \|x_{n+1}\| = \|p\|$, we have

$$\lim_{n\to\infty} \|u_n\| = \|p\|. \tag{3.6}$$

Hence,

$$\lim_{n\to\infty} \|Ju_n\| = \|Jp\|.$$

This implies that $\{\|Ju_n\|\}_{n=0}^{\infty}$ is bounded in E^*. Since E is reflexive, and so E^* is reflexive, we can then assume that $Ju_n \rightharpoonup f_0 \in E^*$. In view of reflexivity of E, we see that $J(E) = E^*$.

Hence, there exists $x \in E$ such that $Jx = f_0$. Since

$$\phi(x_{n+1}, u_n) = \|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_n\rangle + \|u_n\|^2 = \|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_n\rangle + \|Ju_n\|^2.$$

Taking $\liminf_{n\to\infty}$ for both sides of the equality above, yields that

$$0 \ge \|p\|^2 - 2\langle p, f_0\rangle + \|f_0\|^2 = \|p\|^2 - 2\langle p, Jx\rangle + \|Jx\|^2 = \|p\|^2 - 2\langle p, Jx\rangle + \|x\|^2 = \phi(p, x).$$

That is, $p = x$. This implies that $f_0 = Jp$, and so, $Ju_n \rightharpoonup Jp$. It follows from $\lim_{n\to\infty}\|Ju_n\| = \|Jp\|$ and Kadec-Klee property of E^* (this is because E^* is uniformly convex) that

$$Ju_n \to Jp.$$

Note that $J^{-1} : E^* \to E$ is hemi-continuous (this is because E is a uniformly smooth and strictly convex Banach space with a strictly convex dual E^*), it follows that $u_n \rightharpoonup p$. Since (3.6) and E have the Kadec-Klee property, we obtain that

$$\lim_{n\to\infty} u_n = p. \tag{3.7}$$

It follows that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.8}$$

Since J is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{3.9}$$

Let $r := \sup_{n,i\ge 0}\{\|T_i^n x_n\|\}$. Since E is uniformly smooth, we know that E^* is uniformly convex. Then from Lemma 2.10, we have

$$\begin{aligned}
G(x^*, Ju_n) &= G(x^*, J\theta_n^m y_n) \\
&\le G(x^*, Jy_n) \\
&= \|x^*\|^2 - 2\left\langle x^*, \sum_{i=0}^{\infty}\alpha_{ni} JT_i^n x_n\right\rangle + \left\|\sum_{i=0}^{\infty}\alpha_{ni} JT_i^n x_n\right\|^2 + 2\rho f(x^*) \\
&\le \|x^*\|^2 - 2\sum_{i=0}^{\infty}\alpha_{ni}\langle x^*, JT_i^n x_n\rangle + \sum_{i=0}^{\infty}\alpha_{ni}\|JT_i^n x_n\|^2 \\
&\quad - \alpha_{nk}\alpha_{nj} g(\|JT_k^n x_n - JT_j^n x_n\|) + 2\rho f(x^*) \\
&= \sum_{i=0}^{\infty}\alpha_{ni}\phi(x^*, T_i^n x_n) + 2\rho f(x^*) - \alpha_{nk}\alpha_{nj} g(\|JT_k^n x_n - JT_j^n x_n\|) \\
&\le \sum_{i=0}^{\infty}\alpha_{ni} k_{ni}\phi(x^*, x_n) + 2\rho f(x^*) - \alpha_{nk}\alpha_{nj} g(\|JT_k^n x_n - JT_j^n x_n\|) \\
&\le G(x^*, Jx_n) + (k_n - 1)M_n - \alpha_{nk}\alpha_{nj} g(\|JT_k^n x_n - JT_j^n x_n\|).
\end{aligned} \tag{3.10}$$

Taking $k = 0$ and for any j in (3.10), we have

$$\alpha_{n0}\alpha_{nj}g\big(\big\|Jx_n - JT_j^n x_n\big\|\big) \le G\big(x^*, Jx_n\big) - G\big(x^*, Ju_n\big) + (k_n - 1)M_n \to 0.$$

It follows from the property of g that

$$\lim_{n\to\infty}\big\|Jx_n - JT_j^n x_n\big\| = 0. \tag{3.11}$$

Step 4. Now we prove that $p \in \Omega$.

(a) First, we prove that $p \in \bigcap_{i=0}^{\infty} F(T_i)$.

Since $x_n \to p$ and J is uniformly norm-to-norm continuous on bounded sets, we see that

$$\lim_{n\to\infty}\|Jx_n - Jp\| = 0. \tag{3.12}$$

We observe from (3.11) and (3.12) that

$$\big\|JT_j^n x_n - Jp\big\| \le \big\|Jx_n - JT_j^n x_n\big\| + \|Jx_n - Jp\| \to 0, \quad n \to \infty.$$

Since J^{-1} is hemi-continuous, it follows that $T_j^n x_n \rightharpoonup p$. On the other hand, since

$$\big|\big\|T_j^n x_n\big\| - \|p\|\big| = \big|\big\|JT_j^n x_n\big\| - \|Jp\|\big| \le \big\|JT_j^n x_n - Jp\big\|,$$

and this implies that $\|T_j^n x_n\| \to \|p\|$ as $n \to \infty$. Since E enjoys the Kadec-Klee property, we obtain that

$$\lim_{n\to\infty}\big\|T_j^n x_n - p\big\| = 0.$$

Note that

$$\big\|T_j^{n+1}x_n - p\big\| \le \big\|T_j^{n+1}x_n - T_j^n x_n\big\| + \big\|T_j^n x_n - p\big\|. \tag{3.13}$$

It follows from the asymptotic regularity of T and (3.13) that

$$\lim_{n\to\infty}\big\|T_j^{n+1}x_n - p\big\| = 0.$$

That is, $T_j T_j^n x_n - p \to 0$ as $n \to \infty$. It follows from the closeness of T_j that $T_j p = p$, $\forall j \in \mathbb{N}$, *i.e.*, $p \in \bigcap_{i=0}^{\infty} F(T_i)$.

(b) Next, we prove that $p \in \bigcap_{k=1}^{m} \mathrm{EP}(F_k)$.

From (3.2), we obtain

$$\begin{aligned}\phi\big(x^*, u_n\big) &= \phi\big(x^*, \theta_n^m y_n\big) = \phi\big(x^*, T_{r_{m,n}}^{F_m}\theta_n^{m-1}y_n\big)\\ &\le \phi\big(x^*, \theta_n^{m-1}y_n\big) \le \cdots \le \phi\big(x^*, y_n\big)\\ &\le \phi\big(x^*, x_n\big) + (k_n - 1)M_n.\end{aligned}$$

Next, we show that $\theta_n^k y_n \to p$ as $n \to \infty$, for each $k \in \{0, 1, \ldots, m\}$.

We have proved that $k = m$, $\theta_n^k y_n = u_n \to p$.

Suppose that $\theta_n^k y_n \to p$ as $n \to \infty$ for some k. Since $x^* \in \bigcap_{k=1}^m \mathrm{EP}(F_k) = \bigcap_{k=1}^m F(T_{r_{k,n}}^{F_k})$ for all $n \geq 1$, it follows from Lemma 2.14 that

$$\begin{aligned}\phi(\theta_n^k y_n, \theta_n^{k-1} y_n) &= \phi(T_{r_{k,n}}^{F_k} \theta_n^{k-1} y_n, \theta_n^{k-1} y_n) \\ &\leq \phi(x^*, \theta_n^{k-1} y_n) - \phi(x^*, \theta_n^k y_n) \\ &\leq \phi(x^*, x_n) - \phi(x^*, \theta_n^k y_n) + (k_n - 1)M_n.\end{aligned}$$

Hence, we have

$$\lim_{n\to\infty} \phi(\theta_n^k y_n, \theta_n^{k-1} y_n) = 0.$$

From (2.5), we see that $\|\theta_n^k y_n\| - \|\theta_n^{k-1} y_n\| \to 0$ as $n \to \infty$. From assumption, we have $\theta_n^k y_n \to p$ as $n \to \infty$, so

$$\|\theta_n^{k-1} y_n\| \to \|p\| \quad \text{as } n \to \infty.$$

It follows that

$$\|J\theta_n^{k-1} y_n\| \to \|Jp\| \quad \text{as } n \to \infty. \tag{3.14}$$

This implies that $\{\|J\theta_n^{k-1} y_n\|\}_{n=0}^\infty$ is bounded in E^*. Since E is reflexive, and so E^* is reflexive, we can then assume that $J\theta_n^{k-1} y_n \rightharpoonup f_{k-1} \in E^*$. In view of reflexivity of E, we see that $J(E) = E^*$. Hence, there exists $x^{k-1} \in E$ such that $Jx^{k-1} = f_{k-1}$. Since

$$\begin{aligned}\phi(\theta_n^k y_n, \theta_n^{k-1} y_n) &= \|\theta_n^k y_n\|^2 - 2\langle \theta_n^k y_n, J\theta_n^{k-1} y_n\rangle + \|\theta_n^{k-1} y_n\|^2 \\ &= \|\theta_n^k y_n\|^2 - 2\langle \theta_n^k y_n, J\theta_n^{k-1} y_n\rangle + \|J\theta_n^{k-1} y_n\|^2.\end{aligned}$$

Taking $\liminf_{n\to\infty}$ for both sides of the equality above, yields that

$$\begin{aligned}0 &\geq \|p\|^2 - 2\langle p, f_{k-1}\rangle + \|f_{k-1}\|^2 \\ &= \|p\|^2 - 2\langle p, Jx^{k-1}\rangle + \|Jx^{k-1}\|^2 \\ &= \|p\|^2 - 2\langle p, Jx^{k-1}\rangle + \|x^{k-1}\|^2 \\ &= \phi(p, x^{k-1}).\end{aligned}$$

That is, $p = x^{k-1}$. This implies that $f_{k-1} = Jp$ and so $J\theta_n^{k-1} y_n \rightharpoonup Jp$. It follows from $\lim_{n\to\infty} \|J\theta_n^{k-1} y_n\| = \|Jp\|$ and Kadec-Klee property of E^* (this is because E^* is uniformly convex) that

$$J\theta_n^{k-1} y_n \to Jp.$$

Note that $J^{-1} : E^* \to E$ is hemi-continuous (this is because E is a uniformly smooth and strictly convex Banach space with a strictly convex dual E^*), it follows that $\theta_n^{k-1} y_n \rightharpoonup p$.

Since (3.14) and E have the Kadec-Klee property, we obtain that

$$\lim_{n\to\infty} \theta_n^{k-1} y_n = p.$$

Hence, $\lim_{n\to\infty} \theta_n^k y_n = p$ and $\lim_{n\to\infty} J\theta_n^k y_n = Jp$, for each $k \in \{0, 1, \ldots, m\}$. That is,

$$\lim_{n\to\infty} \left\| \theta_n^k y_n - \theta_n^{k-1} y_n \right\| = 0, \quad k = 1, 2, \ldots, m$$

and

$$\lim_{n\to\infty} \left\| J\theta_n^k y_n - J\theta_n^{k-1} y_n \right\| = 0, \quad k = 1, 2, \ldots, m.$$

Since $\liminf_{n\to\infty} r_{k,n} > 0$, $k = 1, 2, \ldots, m$,

$$\lim_{n\to\infty} \frac{\| J\theta_n^k y_n - J\theta_n^{k-1} y_n \|}{r_{k,n}} = 0. \tag{3.15}$$

By Lemma 2.13, we have that for each $k = 1, 2, \ldots, m$,

$$F_k\big(\theta_n^k y_n, y\big) + \frac{1}{r_{k,n}} \big\langle y - \theta_n^k y_n, J\theta_n^k y_n - J\theta_n^{k-1} y_n \big\rangle \geq 0, \quad \forall y \in C.$$

Furthermore, using (A2), we obtain

$$\frac{1}{r_{k,n}} \big\langle y - \theta_n^k y_n, J\theta_n^k y_n - J\theta_n^{k-1} y_n \big\rangle \geq F_k\big(y, \theta_n^k y_n\big).$$

By (A4), (3.15) and $\theta_n^k y_n \to p$, we have for each $k = 1, 2, \ldots, m$,

$$F_k(y, p) \leq 0, \quad \forall y \in C.$$

For fixed $y \in C$, let $z_t = ty + (1-t)p$ for all $t \in (0, 1]$. This implies that $z_t \in C$. This yields that $F_k(z_t, p) \leq 0$. It follows from (A1) and (A4) that

$$0 = F_k(z_t, z_t) \leq tF_k(z_t, y) + (1-t)F_k(z_t, p) \leq tF_k(z_t, y),$$

and hence

$$0 \leq F_k(z_t, y).$$

From condition (A3), we obtain

$$F_k(p, y) \geq 0, \quad \forall y \in C.$$

This implies that $p \in \mathrm{EP}(F_k)$, $k = 1, 2, \ldots, m$. Thus, $p \in \bigcap_{k=1}^m \mathrm{EP}(F_m)$.

Hence, we have $p \in \Omega = (\bigcap_{k=1}^m \mathrm{EP}(F_m)) \cap (\bigcap_{i=0}^\infty F(T_i))$.

Step 5. Finally, we prove that $p = \Pi_\Omega^f x_0$.

Since $\Omega = (\bigcap_{k=1}^{m} \mathrm{EP}(F_m)) \cap (\bigcap_{i=0}^{\infty} F(T_i))$ is a closed and convex set, from Lemma 2.6, we know that $\Pi_{\Omega}^{f} x_0$ is single-valued and denoted $\omega = \Pi_{\Omega}^{f} x_0$. Since $x_n = \Pi_{C_n}^{f} x_0$ and $\omega \in \Omega \subset C_n$, we have

$$G(x_n, Jx_0) \leq G(\omega, Jx_0), \quad \forall n \geq 0.$$

We know that $G(\xi, \phi)$ is convex and lower semi-continuous with respect to ξ when ϕ is fixed. This implies that

$$G(p, Jx_0) \leq \liminf_{n\to\infty} G(x_n, Jx_0) \leq \limsup_{n\to\infty} G(x_n, Jx_0) \leq G(\omega, Jx_0).$$

From the definition of $\Pi_{\Omega}^{f} x_0$ and $p \in \Omega$, we see that $p = \omega$. This completes the proof. □

Corollary 3.2 *Let E be a uniformly smooth and strictly convex Banach space, which has the Kadec-Klee property, and let C be a nonempty closed convex subset of E. For each $k = 1, 2, \ldots, m$, let F_k be a bifunction from $C \times C$ satisfying* (A1)-(A4), *and let $\{T_i\}_{i=0}^{\infty} : C \to C$, $\forall i \in \mathbb{N}$ be an infinite family of closed and asymptotically quasi-ϕ-nonexpansive mappings with sequence $\{k_{ni}\} \subset [1, \infty)$, $k_{ni} \to 1$ as $n \to \infty$, where $T_0 = I$. Assume that T_i, $\forall i \in \mathbb{N}$ is asymptotically regular on C, and $\Omega = (\bigcap_{i=0}^{\infty} F(T_i)) \cap (\bigcap_{k=1}^{m} \mathrm{EP}(F_k))$ is nonempty and bounded. Suppose that $\{x_n\}_{n=0}^{\infty}$ is generated by $x_0 \in C$, $C_1 = C$, $x_1 = \Pi_{C_1}^{f} x_0$,*

$$\begin{cases} y_n = J^{-1}\{\sum_{i=0}^{\infty} \alpha_{ni} J T_i^n x\}, \\ u_n = T_{r_{m,n}}^{F_m} T_{r_{m-1,n}}^{F_{m-1}} \cdots T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1} y_n, \\ C_{n+1} = \{z \in C_n : \phi(z, Ju_n) \leq \phi(z, Jx_n) + (k_n - 1)M_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases}$$

where J is the duality mapping on E, $M_n = \sup\{\phi(z, x_n) : z \in \Omega\}$ for each $n \geq 1$, $k_n = \sup_{i\geq 0}\{k_{ni}\}$, $\{\alpha_{ni}\}$ is a real sequence in $[0,1]$ and $\{r_{k,n}\}_{n=1}^{\infty} \subset (0, \infty)$, $k = 1, 2, \ldots, m$, satisfying the following conditions:

(a) $\displaystyle\sum_{i=0}^{\infty} \alpha_{ni} = 1, \quad \forall n \geq 1;$

(b) $\liminf_{n\to\infty} \alpha_{n0}\alpha_{ni} > 0, \quad \forall i \in \mathbb{N};$

(c) $\liminf_{n\to\infty} r_{k,n} > 0.$

Then the sequence $\{x_n\}$ converges strongly to $\Pi_{\Omega} x_0$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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