Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

New viscosity method for hierarchical fixed point approach to variational inequalities

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Abstract

A new viscosity method for hierarchically approximating some common fixed point of an infinite family of nonexpansive mappings is presented; and some strong convergence theorems for solving variational inequality problems and hierarchical fixed point problems are obtained without the aid of the convex linear combination of a countable family of nonexpansive mappings. Solutions are sought in the set of fixed points of another nonexpansive mapping. The results improve those of the authors with the related interest.

MSC: 47H09; 47H10; 65J15; 47J25

Keywords: hierarchical fixed point; variational inequality; nonexpansive mappings; viscosity approximation

1 Introduction and preliminaries

A fairly common method in solving some nonlinear problems is to replace the original problems by a family of regularized (or perturbed) ones. Each of these regularized problems will be obtained as a limit of these unique solutions to the regularized problems. In this paper, we will introduce a new viscosity method for the hierarchical fixed point approach to variational inequality problems.

Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $f : C \to C$ is called a ρ-contraction if there exists a constant $\rho \in (0,1)$ such that $\|f(x) - f(y)\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

Let $\{T_n\} : H \to H$ be a countable family of nonexpansive mappings with $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ (hence, it is a nonempty closed and convex set [1]). To hierarchically find a common fixed point of a countable family of nonexpansive mappings $\{T_n\}$ with respect to another nonexpansive mapping $S : H \to H$ is to find an $x^* \in F$ such that

$$\langle x^* - Sx^*, x^* - x \rangle \leq 0, \quad \forall x \in F. \tag{1.1}$$

It is easy to see that (1.1) is equivalent to the following fixed point problem: finding an $x^* \in F$ such that $x^* = P_F Sx^*$, where P_F is the metric projection from H onto a closed convex subset $F \subset H$.

The normal cone N_F to F is defined by

$$N_F(x) = \begin{cases} \{u \in H : \langle y - x, u \rangle \leq 0, \forall y \in F\}, & x \in F; \\ \emptyset, & x \in F^c. \end{cases}$$

Then (1.1) is equivalent to the following variational inclusion problem: finding an $x^* \in C$ such that

$$\theta \in (I - S)x^* + N_F(x^*). \tag{1.2}$$

The existence problem of hierarchical fixed points for a single nonexpansive mapping and approximation problem in the setting of Hilbert spaces has been studied by several authors (see, *e.g.*, [2–12]).

In 2011, Zhang *et al.* [13] proved a strong convergence theorem by projection method for solving some variational inequality problems; and under suitable conditions on parameters, they also obtained a weak convergence theorem, which can solve the hierarchical fixed point problem (1.1).

However, since the involved mapping T is defined by a convex linear combination of a countable family of nonexpansive mappings $\{T_n\}$, *i.e.*, $T = \sum_{n=1}^{\infty} \lambda_n T_n$, $\lambda_n \geq 0$ ($\forall n \geq 1$) with $\sum_{n=1}^{\infty} \lambda_n = 1$, the accurate computation of Tx_n at each step of the iteration process is not easily attainable. In addition, the weak convergence was obtained on condition that the iteration sequence is bounded.

Inspired and motivated by those studies mentioned above, in this paper, we introduce a new viscosity method for hierarchically approximating some common fixed point of an infinite family of nonexpansive mappings and prove the strong convergence theorems for solving some variational inequality problems and hierarchical fixed point problems.

In what follows, we shall make use of the following definitions and lemmas.

Let H be a real Hilbert space. The function $\phi : H \times H \to R$ is defined by

$$\phi(x, y) := \|x - y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2. \tag{1.3}$$

It is obvious from the definition of the function ϕ that

$$\big(\|x\| - \|y\|\big)^2 \leq \phi(x, y) \leq \big(\|x\| + \|y\|\big)^2. \tag{1.4}$$

The function ϕ also has the following property:

$$\phi(y, x) = \phi(z, x) + \phi(y, z) + 2\langle z - y, x - z\rangle. \tag{1.5}$$

The metric projection from H onto C is the mapping $P_C : H \to C$ for each $x \in H$, there exists a unique point $z = P_C(x)$ such that

$$\|x - z\| = \inf_{y \in C} \|x - y\| = d(x, C).$$

Lemma 1.1 *Let $x \in H$ and $z \in C$ be any points. Then we have*:

(1) *$z = P_C(x)$ if and only if the following relation holds*

$$\langle x - z, y - z\rangle \leq 0, \quad \forall y \in C.$$

(2) *There holds the relation*

$$\big\langle P_C(x) - P_C(y), x - y\big\rangle \geq \big\|P_C(x) - P_C(y)\big\|^2, \quad \forall x, y \in H.$$

This implies that $P_C : H \to C$ is nonexpansive.

Lemma 1.2 [14] *Let H be a Hilbert space. Then for all $x, y \in H$ and $\alpha_i \in [0,1]$ for $i = 0, 1, 2, \dots, n$ such that $\sum_{i=0}^{n} \alpha_i = 1$ the following equality holds*

$$\left\| \sum_{i=0}^{n} \alpha_i x_i \right\|^2 = \sum_{i=0}^{n} \alpha_i \|x_i\|^2 - \sum_{0 \le i,j \le n} \alpha_i \alpha_j \|x_i - x_j\|^2. \tag{1.6}$$

Lemma 1.3 [15] *Let $\{a_n\}$, $\{\delta_n\}$, and $\{b_n\}$ be sequences of nonnegative real numbers satisfying*

$$a_{n+1} \le (1 + \delta_n) a_n + b_n, \quad \forall n \ge 1. \tag{1.7}$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Definition 1.4 [13] (1) Let $\{A_n\} : C \to C$ be a sequence of mappings, and let $A : C \to C$ be a mapping. $\{A_n\}$ is said to be graph convergent to A if $\{\operatorname{graph}(A_n)\}$ (the sequence of graph of A_n) converges to graph A in the sense of Kuratowski-Painlevé, *i.e.*,

$$\limsup_{n\to\infty} \operatorname{graph}(A_n) \subset \operatorname{graph}(A_n) \subset \liminf_{n\to\infty} \operatorname{graph}(A_n).$$

(2) A multi-valued mapping $A : H \to H$ is said to be monotone if $\langle Ax - Ay, x - y\rangle \ge 0$, $\forall x, y \in H$. A mapping $A : H \to H$ is said to be maximal monotone if it is monotone, and for any $x, u \in H$ when

$$\langle u - v, x - y\rangle \ge 0, \quad \forall (y, v) \in \operatorname{graph}(A),$$

we have $u \in Ax$.

Lemma 1.5 [16] (1) *Let $A : H \to H$ be a maximal monotone operator. Then $(t^{-1}A)$ graph converges to $N_{A^{-1}(0)}$ as $t \to 0$, which provide that $A^{-1}(0) \ne \emptyset$.*

(2) *Let $\{B_n : H \to H\}$ be a sequence of maximal monotone operators, whose graph converges to an operator B. If A is a Lipschitz maximal monotone operator, then $\{A + B_n\}$ graph converges to $A + B$, and $A + B$ is maximal monotone.*

Lemma 1.6 *Let $f : H \to H$ be a contractive mapping, and let $T : H \to H$ be a nonexpansive mapping. Then, the following results are obtained*:

(1) *the mapping $(I - f) : H \to H$ is strongly monotone*;
(2) *the mapping $(I - T) : H \to H$ is monotone, so it is maximal monotone.*

2 Main results

Theorem 2.1 *Let H be a real Hilbert space, and let C be a closed convex nonempty subset of H. Let $f : C \to C$ be a contractive mapping with a contractive constant $\rho \in (0,1)$, and let $\{T_i\}_{i=1}^{\infty} : C \to C$ be a sequence of nonexpansive mappings with the interior of $F := \bigcap_{i=1}^{\infty} F(T_i) \ne \emptyset$. Starting from an arbitrary $x_1 \in C$, define $\{x_n\}$ by*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T_n^* x_n, \quad \forall n \ge 1, \tag{2.1}$$

where $\{\alpha_n\}$ is a decreasing sequence in $(0,1)$ satisfying the following conditions:

(1) $\sum_{n=1}^{\infty} \alpha_n < \infty$;
(2) $\sum_{n=1}^{\infty} (\frac{\alpha_{n-1}^2}{\alpha_n^2} - 1) < \infty$;
(3) $\sum_{n=1}^{\infty} \frac{\alpha_{n-1}-\alpha_n}{\alpha_n^2} < \infty$;

and $T_n^* = T_{i_n}$ *with* i_n *satisfying the positive integer equation*: $n = i + \frac{(m-1)m}{2}$ $(m \geq i,\ n = 1, 2, \ldots)$, *that is, for each* $n \geq 1$, *there exists a unique* i_n *such that*

$$i_1 = 1, \quad i_2 = 1, \quad i_3 = 2, \quad i_4 = 1, \quad i_5 = 2, \quad i_6 = 3,$$
$$i_7 = 1, \quad i_8 = 2, \quad i_9 = 3, \quad i_{10} = 4, \quad i_{11} = 1, \quad \ldots.$$

If $f \neq 0$, *then* $\{x_n\}$ *converges strongly to some point* $x^* = P_F f x^*$, *which is the unique solution to the following variational inequality*

$$\langle (I - f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in F. \tag{2.2}$$

Proof We divide the proof into several steps.

(I) $\lim_{n\to\infty} \|x_n - p^*\|$ exists, $\forall p^* \in F$.

For any $p^* \in F$, from (2.1), we have that

$$\begin{aligned} \|x_{n+1} - p^*\| &= \|\alpha_n(f(x_n) - p^*) + (1 - \alpha_n)T_n^*(x_n - p^*)\| \\ &\leq \alpha_n \|f(x_n) - p^*\| + (1 - \alpha_n)\|x_n - p^*\| \\ &\leq \alpha_n \|f(x_n) - f(p^*)\| + \alpha_n \|f(p^*) - p^*\| + (1 - \alpha_n)\|x_n - p^*\| \\ &\leq \alpha_n \rho \|x_n - p^*\| + \alpha_n \|f(p^*) - p^*\| + (1 - \alpha_n)\|x_n - p^*\| \\ &\leq \|x_n - p^*\| + \mu_n, \end{aligned} \tag{2.3}$$

where $\mu_n = \alpha_n \|f(p^*) - p^*\|$, and so $\sum_{n=1}^{\infty} \mu_n < \infty$. So by Lemma 1.3, we conclude that $\lim_{n\to\infty} \|x_n - p^*\|$ exists, and hence $\{x_n\}$, $\{f(x_n)\}$ and $\{T_n^* x_n\}$ are bounded.

(II) $x_n \to x^* \in C$ as $n \to \infty$.

From (2.1) and Lemma 1.2, we also have

$$\begin{aligned} \|x_{n+1} - p^*\|^2 &= \|\alpha_n(f(x_n) - p^*) + (1 - \alpha_n)T_n^*(x_n - p^*)\|^2 \\ &= \alpha_n \|f(x_n) - p^*\|^2 + (1 - \alpha_n)\|T_n^*(x_n - p^*)\|^2 \\ &\quad - \alpha_n(1 - \alpha_n)\|f(x_n) - T_n^* x_n\|^2 \\ &\leq \alpha_n (\|f(x_n) - f(p^*)\| + \|f(p^*) - p^*\|)^2 + (1 - \alpha_n)\|x_n - p^*\|^2 \\ &\leq \alpha_n \rho \|x_n - p^*\|^2 + (1 - \alpha_n)\|x_n - p^*\|^2 \\ &\quad + \alpha_n (2\rho \|f(p^*) - p^*\| \cdot \|x_n - p^*\| + \|f(p^*) - p^*\|^2) \\ &\leq \|x_n - p^*\|^2 + \nu_n, \end{aligned} \tag{2.4}$$

where $\nu_n := \alpha_n(2\rho \|f(p^*) - p^*\| \cdot \|x_n - p^*\| + \|f(p^*) - p^*\|^2)$ and $\sum_{n=1}^{\infty} \nu_n < \infty$, since $\{x_n\}$ is bounded and $\sum_{n=1}^{\infty} \alpha_n < \infty$.

Furthermore, it follows from (1.5) that

$$\phi(p, x_n) = \phi(x_{n+1}, x_n) + \phi(p, x_{n+1}) + 2\langle x_{n+1} - p, x_n - x_{n+1} \rangle, \quad \forall p \in H.$$

This implies that

$$\langle x_{n+1} - p, x_n - x_{n+1}\rangle + \frac{1}{2}\phi(x_{n+1}, x_n) = \frac{1}{2}\big(\phi(p, x_n) - \phi(p, x_{n+1})\big). \tag{2.5}$$

Moreover, since the interior of F is nonempty, there exists a $p^* \in F$ and $r > 0$ such that $(p^* + rh) \in F$, whenever $\|h\| \le 1$. Thus, from (2.4) and (2.5), we obtain that

$$0 \le \big\langle x_{n+1} - \big(p^* + rh\big), x_n - x_{n+1}\big\rangle + \frac{1}{2}\phi(x_{n+1}, x_n) + \frac{1}{2}\nu_n. \tag{2.6}$$

Then from (2.5) and (2.6), we obtain that

$$\begin{aligned} r\langle h, x_n - x_{n+1}\rangle &\le \big\langle x_{n+1} - p^*, x_n - x_{n+1}\big\rangle + \frac{1}{2}\phi(x_{n+1}, x_n) + \frac{1}{2}\nu_n \\ &= \frac{1}{2}\big(\phi\big(p^*, x_n\big) - \phi\big(p^*, x_{n+1}\big)\big) + \frac{1}{2}\nu_n, \end{aligned}$$

and hence

$$\langle h, x_n - x_{n+1}\rangle \le \frac{1}{2r}\big(\phi\big(p^*, x_n\big) - \phi\big(p^*, x_{n+1}\big)\big) + \frac{1}{2r}\nu_n. \tag{2.7}$$

Since h with $\|h\| \le 1$ is arbitrary, we have

$$\|x_n - x_{n+1}\| \le \frac{1}{2r}\big(\phi\big(p^*, x_n\big) - \phi\big(p^*, x_{n+1}\big)\big) + \frac{1}{2r}\nu_n. \tag{2.8}$$

So, if $n > m$, then we have that

$$\begin{aligned} \|x_m - x_n\| &\le \sum_{j=m}^{n-1} \|x_j - x_{j+1}\| \\ &\le \frac{1}{2r}\sum_{j=m}^{n-1}\big(\phi\big(p^*, x_j\big) - \phi\big(p^*, x_{j+1}\big)\big) + \frac{1}{2r}\sum_{j=m}^{n-1}\nu_j \\ &= \frac{1}{2r}\big(\phi\big(p^*, x_m\big) - \phi\big(p^*, x_n\big)\big) + \frac{1}{2r}\sum_{j=m}^{n-1}\nu_j. \end{aligned} \tag{2.9}$$

But we know that $\{\phi(p^*, x_n)\}$ converges, and $\sum_{n=1}^{\infty}\nu_n < \infty$. Therefore, we obtain from (2.9) that $\{x_n\}$ is a Cauchy sequence. Since H is complete, there exists an $x^* \in H$ such that $x_n \to x^* \in H$ as $n \to \infty$. Thus, since $\{x_n\} \subset C$ and C is closed and convex, then $x^* \in C$, that is,

$$x_n \to x^* \in C \quad (n \to \infty). \tag{2.10}$$

(III) $\|x_n - T_i x_n\| \to 0$ for each $i \ge 1$ as $n \to \infty$.

It follows from (2.1) and (2.8) that, as $n \to \infty$,

$$\big\|x_{n+1} - T_n^* x_n\big\| = \alpha_n \big\|f(x_n) - T_n^* x_n\big\| \to 0 \tag{2.11}$$

and

$$\|x_{n+1} - x_n\| \to 0, \tag{2.12}$$

which implies that, by induction, for any nonnegative integer j,

$$\lim_{n\to\infty}\|x_{n+j}-x_n\|=0. \tag{2.13}$$

We then have, as $n\to\infty$,

$$\left\|x_n-T_n^*x_n\right\|\le\|x_n-x_{n+1}\|+\left\|x_{n+1}-T_n^*x_n\right\|\to 0. \tag{2.14}$$

For each $i\ge 1$, since

$$\begin{aligned}\left\|x_n-T_{n+i}^*x_n\right\| &\le \|x_n-x_{n+i}\|+\left\|x_{n+i}-T_{n+i}^*x_n\right\| \\ &\le \|x_n-x_{n+i}\|+\left\|x_{n+i}-T_{n+i}^*x_{n+i}\right\| \\ &\quad+\left\|T_{n+i}^*x_{n+i}-T_{n+i}^*x_n\right\| \\ &\le 2\|x_n-x_{n+i}\|+\left\|x_{n+i}-T_{n+i}^*x_{n+i}\right\|, \end{aligned} \tag{2.15}$$

it follows from (2.13) and (2.14) that

$$\lim_{n\to\infty}\left\|x_n-T_{n+i}^*x_n\right\|=0. \tag{2.16}$$

Now, for each $i\ge 1$, we claim that

$$\lim_{n\to\infty}\|x_n-T_ix_n\|=0. \tag{2.17}$$

As a matter of fact, setting

$$n=N_m+i,$$

where $N_m=(m-1)m/2$, $m\ge i$, we obtain that

$$\begin{aligned}\|x_n-T_ix_n\| &\le \|x_n-x_{N_m}\|+\|x_{N_m}-T_ix_n\| \\ &\le \|x_n-x_{N_m}\|+\left\|x_{N_m}-T_{N_m+i}^*x_{N_m}\right\| \\ &\quad+\left\|T_{N_m+i}^*x_{N_m}-T_ix_n\right\| \\ &= \|x_n-x_{N_m}\|+\left\|x_{N_m}-T_{N_m+i}^*x_{N_m}\right\| \\ &\quad+\|T_ix_{N_m}-T_ix_n\| \\ &\le 2\|x_n-x_{N_m}\|+\left\|x_{N_m}-T_{N_m+i}^*x_{N_m}\right\| \\ &= 2\|x_n-x_{n-i}\|+\left\|x_{N_m}-T_{N_m+i}^*x_{N_m}\right\|. \end{aligned} \tag{2.18}$$

Then, since $N_m\to\infty$ as $n\to\infty$, it follows from (2.13) and (2.16) that (2.17) holds obviously.

(IV) $x_n\to x^*=P_Ffx^*$ as $n\to\infty$, which is the unique solution to the following variational inequality

$$\left\langle (I-f)x^*,x-x^*\right\rangle\ge 0,\quad \forall x\in F.$$

It immediately follows from (2.10) and (2.17) that

$$x_n \to x^* \in F \quad (n \to \infty). \tag{2.19}$$

Next, for each $i \geq 1$, we consider the corresponding subsequence $\{x_k^{(i)}\}_{k\in\mathbb{N}_i}$ of $\{x_n\}$, where $\mathbb{N}_i$ is defined by

$$\mathbb{N}_i = \left\{k \in \mathbb{N} : k = j + \frac{(m-1)m}{2}, m \geq j, m \in \mathbb{N}\right\}.$$

For example, by the definition of $\mathbb{N}_1$, we have $\mathbb{N}_1 = \{1, 2, 4, 7, 11, 16, \ldots\}$ and $i_1 = i_2 = i_4 = i_7 = i_{11} = i_{16} = \cdots = 1$. Note that $(T_k^*)^{(i)} = T_i$, whenever $k \in \mathbb{N}_i$ for each $i \geq 1$. It then follows from (2.1) that

$$\begin{aligned}
\|x_{k+1}^{(i)} - x_k^{(i)}\| &= \|\alpha_k^{(i)}(f(x_k^{(i)}) - f(x_{k-1}^{(i)})) + (1-\alpha_k^{(i)})T_i(x_k^{(i)} - x_{k-1}^{(i)}) \\
&\quad + (\alpha_k^{(i)} - \alpha_{k-1}^{(i)})(f(x_{k-1}^{(i)}) - T_i x_{k-1}^{(i)})\| \\
&\leq \alpha_k^{(i)}\rho\|x_k^{(i)} - x_{k-1}^{(i)}\| + (1-\alpha_k^{(i)})\|x_k^{(i)} - x_{k-1}^{(i)}\| + M|\alpha_k^{(i)} - \alpha_{k-1}^{(i)}| \\
&\leq \|x_k^{(i)} - x_{k-1}^{(i)}\| + M|\alpha_k^{(i)} - \alpha_{k-1}^{(i)}|,
\end{aligned} \tag{2.20}$$

where $M := \sup_{k\in\mathbb{N}_i} \|f(x_{k-1}^{(i)}) - T_i x_{k-1}^{(i)}\| < \infty$.

Thus, we have

$$\begin{aligned}
\frac{\|x_{k+1}^{(i)} - x_k^{(i)}\|}{(\alpha_k^{(i)})^2} &\leq \frac{(\alpha_{k-1}^{(i)})^2}{(\alpha_k^{(i)})^2}\frac{\|x_k^{(i)} - x_{k-1}^{(i)}\|}{(\alpha_{k-1}^{(i)})^2} + \frac{M|\alpha_k^{(i)} - \alpha_{k-1}^{(i)}|}{(\alpha_{k-1}^{(i)})^2} \\
&= (1+\eta_k^{(i)})\frac{\|x_k^{(i)} - x_{k-1}^{(i)}\|}{(\alpha_{k-1}^{(i)})^2} + \gamma_k^{(i)},
\end{aligned} \tag{2.21}$$

where $\eta_k^{(i)} := (\frac{\alpha_{k-1}^{(i)}}{\alpha_k^{(i)}})^2 - 1$, $\gamma_k^{(i)} := \frac{M|\alpha_k^{(i)} - \alpha_{k-1}^{(i)}|}{(\alpha_k^{(i)})^2}$; $\sum_{k\in\mathbb{N}_i}\eta_k^{(i)} < \infty$ and $\sum_{k\in\mathbb{N}_i}\gamma_k^{(i)} < \infty$.

It follows from Lemma 1.3 that $\lim_{k\to\infty}\frac{\|x_{k+1}^{(i)} - x_k^{(i)}\|}{(\alpha_k^{(i)})^2}$ exists, and hence $\{\frac{x_{k+1}^{(i)} - x_k^{(i)}}{(\alpha_k^{(i)})^2}\}$ is bounded. Then there exists an $M_i > 0$ such that $\frac{\|x_{k+1}^{(i)} - x_k^{(i)}\|}{M_i(\alpha_k^{(i)})^2} \leq 1$, $\forall k \in \mathbb{N}_i$.

Taking $h = \frac{x_{k+1}^{(i)} - x_k^{(i)}}{M_i(\alpha_k^{(i)})^2}$, we have, from (2.7),

$$\frac{\|x_k^{(i)} - x_{k+1}^{(i)}\|^2}{(\alpha_k^{(i)})^2} \leq \frac{M_i}{2r}(\phi(p^*, x_k^{(i)}) - \phi(p^*, x_{k+1}^{(i)})) + \frac{M_i}{2r}\nu_k^{(i)}. \tag{2.22}$$

This implies that, as $\mathbb{N}_i \ni k \to \infty$,

$$\frac{x_k^{(i)} - x_{k+1}^{(i)}}{\alpha_k^{(i)}} \to \theta. \tag{2.23}$$

Furthermore, from (2.1), we have

$$\frac{x_k^{(i)} - x_{k+1}^{(i)}}{\alpha_k^{(i)}} = \left((I-f) + \frac{1-\alpha_k^{(i)}}{\alpha_k^{(i)}}(I - T_i)\right)x_k^{(i)}. \tag{2.24}$$

In addition, by Lemmas 1.5 and 1.6, $(I-f)+\frac{1-\alpha_k^{(i)}}{\alpha_k^{(i)}}(I-T_i)$ graph converges to $(I-f)+N_{F(T_i)}$. Since the graph of $(I-f)+N_{F(T_i)}$ is weakly-strongly closed, we obtain that by taking into (2.23) and (2.19),

$$\theta \in (I-f)x^* + N_{F(T_i)}\bigl(x^*\bigr). \tag{2.25}$$

This implies that $\langle (I-f)x^*, x^*-x\rangle \le 0$, $\forall x \in F(T_i)$, that is,

$$\bigl\langle (I-f)x^*, x-x^*\bigr\rangle \ge 0, \quad \forall x \in F, \tag{2.26}$$

since $F \subset F(T_i)$. The proof is complete. □

Theorem 2.2 *Let H be a real Hilbert space, and let C be a closed convex nonempty subset of H. Let $S: C \to C$ be a nonexpansive, and let $f: C \to C$ be a contractive mapping with a contractive constant $\rho \in (0,1)$, and let $\{T_i\}_{i=1}^{\infty}: C \to C$ be a sequence of nonexpansive mappings. Let $\{\beta_i\}$ be a sequence in $[0,1)$ with some $\beta_{i_0}=0$ and $\beta_i \to 0$ as $i \to \infty$. Starting from an arbitrary $x_1 \in C$, define $\{x_n\}$ by*

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\bigl(\beta_n^* S x_n + \bigl(1-\beta_n^*\bigr)T_n^{**} x_n\bigr), \quad \forall n \ge 1, \tag{2.27}$$

where $\{\alpha_n\} \subset (0,1)$ satisfying the same conditions as in Theorem 2.1; $\beta_n^* = \beta_{i_n}$, $T_n^{**} = T_{i_n}^*$ with i_n satisfying the positive integer equation $n = i + \frac{(m-1)m}{2}$ ($m \ge i$, $n = 1,2,\ldots$), and T_n^* denotes the same as that in Theorem 2.1. For each $i \ge 1$, a sequence of nonexpansive mappings $\{G_i\}_{i=1}^{\infty}: C \to C$ is defined by

$$G_i x = \beta_i S x + (1-\beta_i)T_i^* x, \quad \forall i \ge 1. \tag{2.28}$$

If the interior of $\tilde{F} := \bigcap_{i=1}^{\infty} F(G_i) \ne \emptyset$, then $\{x_n\}$ converges strongly to some point $x^* = P_F f x^*$, which is the unique solution to the following variational inequality

$$\bigl\langle (I-f)x^*, x-x^*\bigr\rangle \ge 0, \quad \forall x \in F. \tag{2.29}$$

Proof For each $j \ge 1$, setting $\Gamma_j := \{n \in \mathbb{N} : n = j + \frac{(m-1)m}{2}, m \ge j\}$, then we have

$$G_i x = \beta_i S x + (1-\beta_i)T_j x, \quad \forall i \in \Gamma_j. \tag{2.30}$$

Hence, for any $p \in \tilde{F}$,

$$p = \lim_{\Gamma_j \ni i \to \infty} G_i p = \lim_{\Gamma_j \ni i \to \infty} \bigl(\beta_i S p + (1-\beta_i)T_j p\bigr) = T_j p, \quad \forall j \ge 1, \tag{2.31}$$

which means that $p \in F$, *i.e.*, $\tilde{F} \subset F$. Since there exists some $\beta_{i_0} = 0$, we also have $F \subset F(G_{i_0})$.

Now, for each $n \ge 1$, putting $G_n^* = G_{i_n}$ with i_n satisfying the positive integer equation: $n = i + \frac{(m-1)m}{2}$ ($m \ge i$, $m \in \mathbb{N}$), we have

$$G_n^* x = G_{i_n} x = \beta_{i_n} S x + (1-\beta_{i_n})T_{i_n}^* x = \beta_n^* S x + \bigl(1-\beta_n^*\bigr)T_n^{**} x, \quad \forall n \ge 1. \tag{2.32}$$

It then follows from (2.27) that

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)G_n^* x_n, \quad \forall n \geq 1. \tag{2.33}$$

Therefore, by the assumption that the interior of $\tilde{F} \neq \emptyset$ and Theorem 2.1, $\{x_n\}$ converges strongly to some point $x^* \in \tilde{F} \subset F$ such that $\langle (I-f)x^*, x - x^* \rangle \geq 0$, $\forall x \in F(G_{i_0})$, *i.e.*,

$$\langle (I-f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in F, \tag{2.34}$$

since $F \subset F(G_{i_0})$. This is equivalent to $x^* = P_F f x^*$, which is the unique solution to the variational inequality above. The proof is complete. □

Theorem 2.3 *Let H be a real Hilbert space. Let $S : H \to H$ be a nonexpansive and $f : H \to H$ be a contractive mapping with a contractive constant $\rho \in (0,1)$, and let $\{T_i\}_{i=1}^{\infty} : H \to H$ be a sequence of nonexpansive mappings. Let $\{\beta_i\}$ be a sequence in $(0,1)$ with $\beta_i \to 0$ as $i \to \infty$. Starting from an arbitrary $x_1 \in C$, define $\{x_n\}$ by* (2.27), *where $\{\alpha_n\} \subset (0,1)$ satisfying the same conditions as in Theorem* 2.1; $\beta_n^* = \beta_{i_n}$, $T_n^{**} = T_{i_n}^*$ *with i_n satisfying the positive integer equation*: $n = i + \frac{(m-1)m}{2}$ $(m \geq i, n = 1, 2, \ldots)$ *and T_n^* denotes the same as that in Theorem* 2.1. *If the interior of $\tilde{F} := \bigcap_{i=1}^{\infty} F(G_i) \neq \emptyset$, where $\{G_i\}_{i=1}^{\infty} : H \to H$ is defined the same as that in Theorem* 2.2, *then $\{x_n\}$ converges strongly to some point $x^* \in H$, which is a solution to the hierarchical fixed point problem* (1.1), *i.e.*, *$x^* \in F$ such that*

$$\langle x^* - Sx^*, x^* - x \rangle \leq 0, \quad \forall x \in F. \tag{2.35}$$

Proof Letting Γ_i denotes the same as that in Theorem 2.2, we have,

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\big(\beta_i S x_n + (1-\beta_i)T_i^* x_n\big), \quad \forall n \in \Gamma_i. \tag{2.36}$$

By Theorem 2.2, $x_n \to x^* \in F$ as $n \to \infty$. Taking limit on both sides in the equality above yields that

$$x^* = \beta_i S x^* + (1-\beta_i)x^*, \tag{2.37}$$

that is,

$$x^* = Sx^*. \tag{2.38}$$

This implies that $x^* \in F$ is a solution to the fixed point problem (1.1), *i.e.*, it is a hierarchically common fixed point of a countable family of nonexpansive mappings $\{T_i\}_{i=1}^{\infty}$ with respect to another nonexpansive mapping S. The proof is complete. □

Remark 2.4 Since the strong convergence theorems for solving some variational inequality problems and hierarchical fixed point problems are obtained without the aid of the convex linear combination of a countable family of nonexpansive mappings, the results in this article improve those of the authors with related interest.

Competing interests
The author declares that they have no competing interests.

Author's contributions
The author read and approved the final manuscript.

Acknowledgements
This work is supported by the National Natural Science Foundation of China (Grant No. 11061037).

Received: 21 September 2012 Accepted: 31 July 2013 Published: 16 August 2013

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doi:10.1186/1687-1812-2013-219
Cite this article as: Deng: **New viscosity method for hierarchical fixed point approach to variational inequalities.** *Fixed Point Theory and Applications* 2013 **2013**:219.