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Strong convergence of a relaxed three-step iterative algorithm for countable families of pseudocontractions

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Abstract

An up-to-date method for the approximation of common fixed points of countable families of nonlinear operators is introduced, by which a relaxed three-step iterative algorithm is developed for the class of pseudocontractive mappings, and a strong convergence theorem is established in the framework of Hilbert spaces. *Since there is no need to impose uniformity assumption on the involved Lipschitzian and closed mappings*, the results improve and extend those announced by Cheng *et al.* (Fixed Point Theory Appl. 2013:100, 2013) and other authors with the related interest.

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Keywords: Lipschitz pseudocontractions; common fixed points; countable families of nonlinear operators; strong convergence

1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H with the inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. A mapping $T : C \to H$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

A mapping $T : C \to H$ is called pseudocontractive or a pseudocontraction if

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2, \quad \forall x, y \in C. \tag{1.2}$$

Note that inequality (1.2) can be equivalently written as

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \big\|(I - T)x - (I - T)y\big\|^2, \quad \forall x, y \in C, \tag{1.3}$$

where I denotes the identity operator. A mapping A with the domain $D(A)$ and the range $R(A)$ in H is called monotone if the inequality

$$\|x - y\| \leq \big\|(x - y) + s(Ax - Ay)\big\| \tag{1.4}$$

holds for any $x, y \in D(A)$ and for all $s \geq 0$.

Not only from its being an important generalization of nonexpansive mappings, but also from the firm connection with the important class of nonlinear monotone mappings stems the interest in the class of pseudocontractions. We observe that A is monotone if and only if $T := I - A$ is pseudocontractive, and hence a zero x^* of A, that is, $x^* \in \mathcal{N}(A) := \{x \in D(A) : Ax = 0\}$, is just a fixed point of T. It is well known (see, *e.g.*, [1]) that if A is monotone, then the solutions to the equation $Ax = 0$ correspond to the equilibrium points of some evolution systems. Considerable efforts have then been devoted to developing iterative techniques for approximating fixed points of pseudocontractive mappings (see, for example, [2–4] and the references contained therein).

In 2013, Cheng *et al.* [5] constructed the following three-step iteration method and obtain the convergence theorem for a countable family of Lipschitz pseudocontractive mappings in Hilbert space H. For the iteration format,

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T_n z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n T_n x_n, \end{cases} \tag{1.5}$$

they proved that the sequence $\{x_n\}$ generated from above converges strongly to a common fixed point of $\{T_n\}_{n\geq 1}$. *But it is worth mentioning that the involved mappings were assumed to be uniformly closed and uniformly Lipschitz pseudocontractive, which are obviously two quite strong conditions for countable families of nonlinear operators*. Recall that a countable family of mapping $\{T_n\}_{n=1}^{\infty} : C \to H$ is called *uniformly Lipschitz* with Lipschitz constant $L_n \geq 0$, if there exists an $L := \sup_{n\geq 1} L_n > 0$ such that

$$\|T_n x - T_n y\| \leq L\|x - y\|, \quad \forall x, y \in C, n \geq 1. \tag{1.6}$$

A countable family of mapping $\{T_n\}_{n=1}^{\infty} : C \to H$ is called *uniformly closed* if as $n \to \infty$, $x_n \to x^*$ and $\|x_n - T_n x_n\| \to 0$ imply $x^* \in \bigcap_{n=1}^{\infty} F(T_n)$.

Inspired and motivated by the studies mentioned above, in this paper, we introduced an up-to-date method for the approximation of common fixed points of countable families of nonlinear operators, by which a relaxed three-step iterative algorithm is developed for the class of pseudocontractive mappings, and a strong convergence theorem is established in the framework of Hilbert spaces. No compactness assumption is imposed either on the involved mappings or on the set C. The results are more applicable than those of other authors with the related interest.

2 Preliminaries

In the sequel, we shall need the following definitions. Let H be a real Hilbert space. The function $\phi : H \times H \to \mathbb{R}$, defined by

$$\phi(x, y) := \|x - y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2, \tag{2.1}$$

is studied by Alber [6], Kamimura and Takahashi [7] and Reich [8]. It is obvious from the definition of the function ϕ that

$$\big(\|x\| - \|y\|\big)^2 \leq \phi(x, y) \leq \big(\|x\| + \|y\|\big)^2. \tag{2.2}$$

The function ϕ has also the following property

$$\phi(y,x) = \phi(z,x) + \phi(y,z) + 2\langle z - y, x - z\rangle. \tag{2.3}$$

In what follows, we shall make use of the following lemmas.

Lemma 2.1 [9] *Let H be a Hilbert space. Then for all $x, y \in H$ and $\alpha_i \in [0,1]$ for $i = 0,1,2,\dots,n$ such that $\sum_{i=0}^{n} \alpha_i = 1$ the following equality holds*

$$\left\|\sum_{i=0}^{n} \alpha_i x_i\right\|^2 = \sum_{i=0}^{n} \alpha_i \|x_i\|^2 - \sum_{0\le i,j\le n} \alpha_i\alpha_j \|x_i - x_j\|^2. \tag{2.4}$$

Lemma 2.2 [10] *Let $\{a_n\}$, $\{\delta_n\}$, and $\{b_n\}$ be the sequences of nonnegative real numbers satisfying*

$$a_{n+1} \le (1 + \delta_n)a_n + b_n, \quad \forall n \ge 1. \tag{2.5}$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.3 [11] *The unique solutions to the positive integer equation*

$$n = i + \frac{(m-1)m}{2}, \quad m \ge i, n = 1,2,3,\dots \tag{2.6}$$

are

$$i = n - \frac{(m-1)m}{2}, \quad m = -\left[\frac{1}{2} - \sqrt{2n + \frac{1}{4}}\right], n = 1,2,3,\dots, \tag{2.7}$$

where $[x]$ denotes the maximal integer that is not larger than x.

3 Main results

Recall that an operator T on a Hilbert space is *closed* if $x_n \to x$ and $Tx_n \to y$ as $n \to \infty$, then $Tx = y$.

Theorem 3.1 *Let H be a real Hilbert space, and let C be a closed convex nonempty subset of H. Let $\{T_i\}_{i=1}^{\infty} : C \to C$ be a sequence of closed and Lipschitz pseudocontractive mappings with Lipschitzian constants L_i for each $i \ge 1$ and the interior of $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Starting from an arbitrary $x_1 \in C$, define $\{x_n\}$ by*

$$\begin{cases} x_{n+1} = (1 - \alpha_{i_n})x_n + \alpha_{i_n} T_{i_n} y_n, \\ y_n = (1 - \beta_{i_n})x_n + \beta_{i_n} T_{i_n} z_n, \\ z_n = (1 - \gamma_{i_n})x_n + \gamma_{i_n} T_{i_n} x_n, \end{cases} \tag{3.1}$$

where $\{\alpha_i\}, \{\beta_i\}, \{\gamma_i\} \subset (0,1)$ satisfying the following conditions: (i) $\alpha_i \le \beta_i \le \gamma_i$ *and* (ii) $\gamma_i^3 L_i^4 + 2\gamma_i^2 L_i^3 + \gamma_i^2 L_i^2 + \gamma_i L_i^2 + 2\gamma_i < 1$ *for each $i \ge 1$; i_n is the solutions to the positive* *integer equation*: $n = i + \frac{(m-1)m}{2}$ $(m \geq i, n = 1, 2, \ldots)$, *that is*, *for each* $n \geq 1$, *there exists a unique* i_n *such that*

$$\begin{array}{llllll} i_1 = 1, & i_2 = 1, & i_3 = 2, & i_4 = 1, & i_5 = 2, & i_6 = 3, \\ i_7 = 1, & i_8 = 2, & i_9 = 3, & i_{10} = 4, & i_{11} = 1, & \ldots. \end{array}$$

Then $\{x_n\}$ *converges strongly to an* $x^* \in F$.

Proof Let $p \in F$. Using the similar argument presented in the proof of [1, Theorem 3.1], we have from (3.1) and Lemma 2.1,

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq (1 - \alpha_{i_n})\|x_n - p\|^2 + \alpha_{i_n}\|y_n - p\|^2 + \alpha_{i_n}\|y_n - T_{i_n}y_n\|^2 \\ &\quad - \alpha_{i_n}(1 - \alpha_{i_n})\|x_n - T_{i_n}y_n\|^2, \end{aligned} \tag{3.2}$$

$$\begin{aligned} \|y_n - p\|^2 &\leq (1 - \beta_{i_n})\|x_n - p\|^2 + \beta_{i_n}\|z_n - p\|^2 + \beta_{i_n}\|z_n - T_{i_n}z_n\|^2 \\ &\quad - \beta_{i_n}(1 - \beta_{i_n})\|x_n - T_{i_n}z_n\|^2 \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \|z_n - p\|^2 &\leq (1 - \gamma_{i_n})\|x_n - p\|^2 + \gamma_{i_n}\|x_n - p\|^2 + \gamma_{i_n}\|x_n - T_{i_n}x_n\|^2 \\ &\quad - \gamma_{i_n}(1 - \gamma_{i_n})\|x_n - T_{i_n}x_n\|^2 \\ &= \|x_n - p\|^2 + \gamma_{i_n}^2\|x_n - T_{i_n}x_n\|^2. \end{aligned} \tag{3.4}$$

In addition, from (3.1), we also have

$$\begin{aligned} \|z_n - T_{i_n}z_n\|^2 &\leq (1 - \gamma_{i_n})\|x_n - T_{i_n}z_n\|^2 + \gamma_{i_n}L_{i_n}^2\|x_n - z_n\|^2 \\ &\quad - \gamma_{i_n}(1 - \gamma_{i_n})\|x_n - T_{i_n}x_n\|^2 \\ &= (1 - \gamma_{i_n})\|x_n - T_{i_n}z_n\|^2 \\ &\quad + \gamma_{i_n}\big(\gamma_{i_n}^2L_{i_n}^2 + \gamma_{i_n} - 1\big)\|x_n - T_{i_n}x_n\|^2. \end{aligned} \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.3), we obtain that

$$\begin{aligned} \|y_n - p\|^2 &\leq \|x_n - p\|^2 + \beta_{i_n}\gamma_{i_n}\big(\gamma_{i_n}^2L_{i_n}^2 + 2\gamma_{i_n} - 1\big)\|x_n - T_{i_n}x_n\|^2 \\ &\quad + \beta_{i_n}(\beta_{i_n} - \gamma_{i_n})\|x_n - T_{i_n}z_n\|^2. \end{aligned} \tag{3.6}$$

Since

$$\begin{aligned} \|y_n - T_{i_n}y_n\|^2 &\leq (1 - \beta_{i_n})\|x_n - T_{i_n}y_n\|^2 + \beta_{i_n}L_{i_n}^2\|z_n - y_n\|^2 \\ &\quad - \beta_{i_n}(1 - \beta_{i_n})\|x_n - T_{i_n}z_n\|^2 \end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\|z_n - y_n\| &\le (\gamma_{i_n} - \beta_{i_n})\|x_n - T_{i_n}x_n\| + \beta_{i_n}L_{i_n}\|x_n - z_n\| \\ &= (\gamma_{i_n} - \beta_{i_n})\|x_n - T_{i_n}x_n\| + \beta_{i_n}\gamma_{i_n}L_{i_n}\|x_n - T_{i_n}x_n\| \\ &= (\gamma_{i_n} - \beta_{i_n} + \beta_{i_n}\gamma_{i_n}L_{i_n})\|x_n - T_{i_n}x_n\|, \end{aligned}\tag{3.8}$$

it then follows from (3.7) and (3.8) that

$$\begin{aligned}\|y_n - T_{i_n}y_n\|^2 \le{} & (1 - \beta_{i_n})\|x_n - T_{i_n}y_n\|^2 \\ & + \beta_{i_n}L_{i_n}^2(\gamma_{i_n} - \beta_{i_n} + \beta_{i_n}\gamma_{i_n}L_{i_n})^2\|x_n - T_{i_n}x_n\|^2 \\ & - \beta_{i_n}(1 - \beta_{i_n})\|x_n - T_{i_n}z_n\|^2. \end{aligned}\tag{3.9}$$

Substituting (3.6) and (3.9) into (3.2), we obtain that

$$\begin{aligned}\|x_{n+1} - p\|^2 \le{} & \|x_n - p\|^2 + \big[\alpha_{i_n}\beta_{i_n}\gamma_{i_n}\big(\gamma_{i_n}^2L_{i_n}^2 + 2\gamma_{i_n} - 1\big) \\ & + \alpha_{i_n}\beta_{i_n}L_{i_n}^2(\gamma_{i_n} - \beta_{i_n} + \beta_{i_n}\gamma_{i_n}L_{i_n})^2\big]\|x_n - T_{i_n}x_n\|^2 \\ & + \alpha_{i_n}(\alpha_{i_n} - \beta_{i_n})\|x_n - T_{i_n}y_n\|^2 \\ & + \alpha_{i_n}\beta_{i_n}(2\beta_{i_n} - \gamma_{i_n} - 1)\|x_n - T_{i_n}z_n\|^2, \end{aligned}\tag{3.10}$$

which, together with condition (i), that is, $\alpha_{i_n}(\alpha_{i_n} - \beta_{i_n}) \le 0$ and $\alpha_{i_n}\beta_{i_n}(2\beta_{i_n} - \gamma_{i_n} - 1) \le 0$, yields that

$$\|x_{n+1} - p\|^2 \le \|x_n - p\|^2 - \delta_{i_n}\|x_n - T_{i_n}x_n\|^2, \tag{3.11}$$

where $\delta_{i_n} := \alpha_{i_n}\beta_{i_n}\gamma_{i_n}(1 - \gamma_{i_n}^2L_{i_n}^2 - 2\gamma_{i_n}) + \alpha_{i_n}\beta_{i_n}L_{i_n}^2(\beta_{i_n} - \gamma_{i_n} - \beta_{i_n}\gamma_{i_n}L_{i_n})^2$. Noting that, in the light of condition (ii), $\delta_{i_n} > 0$, we have

$$\|x_{n+1} - p\| \le \|x_n - p\|. \tag{3.12}$$

So, by Lemma 2.2, we conclude that $\lim_{n\to\infty}\|x_n - p\|$ exists.

Furthermore, from (2.3), we also have that

$$\phi(p, x_n) = \phi(x_{n+1}, x_n) + \phi(p, x_{n+1}) + 2\langle x_{n+1} - p, x_n - x_{n+1}\rangle, \quad \forall p \in H.$$

This implies that

$$\langle x_{n+1} - p, x_n - x_{n+1}\rangle + \frac{1}{2}\phi(x_{n+1}, x_n) = \frac{1}{2}\big(\phi(p, x_n) - \phi(p, x_{n+1})\big). \tag{3.13}$$

Moreover, since the interior of F is nonempty, there exists a $p^* \in F$ and $r > 0$ such that $(p^* + rh) \in F$ whenever $\|h\| \le 1$. Thus, from (3.12) and (3.13), we obtain that

$$0 \le \big\langle x_{n+1} - \big(p^* + rh\big), x_n - x_{n+1}\big\rangle + \frac{1}{2}\phi(x_{n+1}, x_n). \tag{3.14}$$

Then from (3.13) and (3.14), we obtain that

$$\begin{aligned} r\langle h, x_n - x_{n+1}\rangle &\le \langle x_{n+1} - p^*, x_n - x_{n+1}\rangle + \frac{1}{2}\phi(x_{n+1}, x_n) \\ &= \frac{1}{2}\big(\phi(p^*, x_n) - \phi(p^*, x_{n+1})\big), \end{aligned}$$

and hence

$$\langle h, x_n - x_{n+1}\rangle \le \frac{1}{2r}\big(\phi(p^*, x_n) - \phi(p^*, x_{n+1})\big).$$

Since h with $\|h\| \le 1$ is arbitrary, we have

$$\|x_n - x_{n+1}\| \le \frac{1}{2r}\big(\phi(p^*, x_n) - \phi(p^*, x_{n+1})\big).$$

So, if $n > m$, then we have that

$$\begin{aligned} \|x_m - x_n\| &\le \sum_{j=m}^{n-1} \|x_j - x_{j+1}\| \\ &\le \frac{1}{2r}\sum_{j=m}^{n-1}\big(\phi(p^*, x_j) - \phi(p^*, x_{j+1})\big) \\ &= \frac{1}{2r}\big(\phi(p^*, x_m) - \phi(p^*, x_n)\big). \end{aligned} \tag{3.15}$$

Since $\{\phi(p^*, x_n)\}$ converges, it then follows from (3.15) that $\{x_n\}$ is a Cauchy sequence, and hence there exists an $x^* \in H$ such that

$$x_n \to x^* \in H \quad (n \to \infty). \tag{3.16}$$

Next, set $\mathbb{N}_i = \{k \in \mathbb{N} : k = i + \frac{(m-1)m}{2}, m \ge i, m \in \mathbb{N}\}$ for each $i \ge 1$. For example, by Lemma 2.3 and the definition of $\mathbb{N}_1$, we have $\mathbb{N}_1 = \{1, 2, 4, 7, 11, 16, \ldots\}$ and $i_1 = i_2 = i_4 = i_7 = i_{11} = i_{16} = \cdots = 1$. Note that $T_{i_k} = T_i$, $\delta_{i_k} = \delta_i$ whenever $k \in \mathbb{N}_i$ for each $i \ge 1$. We have, from (3.11),

$$\delta_i \|x_k - T_i x_k\|^2 \le \|x_k - p\|^2 - \|x_{k+1} - p\|^2, \quad \forall k \in \mathbb{N}_i. \tag{3.17}$$

Since $\{x_k\}_{k \in \mathbb{N}_i}$ and $\{x_{k+1}\}_{k \in \mathbb{N}_i}$ are subsequences of $\{x_n\}$, the existence of $\lim_{n\to\infty} \|x_n - p\|$ implies that

$$\lim_{\mathbb{N}_i \ni k \to \infty} \|x_k - T_i x_k\| = 0, \quad \forall i \ge 1. \tag{3.18}$$

Note that, from (3.16), $x_k \to x^*$ as $\mathbb{N}_i \ni k \to \infty$. It immediately follows from (3.18) and the closedness of T_i that $x^* \in F(T_i)$ for each $i \ge 1$, and hence $x^* \in F$. This completes the proof. □

We now give an example, to which the results of Cheng *et al.* [5] cannot be applied.

Example 3.2 Let $H = \mathbb{R}^1$ and $C = [-1,1]$. Let $\{T_i\}_{i=1}^{\infty} : C \to C$ be a sequence of nonlinear mappings defined by

$$T_i x = \begin{cases} (\frac{i}{2i+1} + \frac{1}{2})x, & x \in C_1 := [0,1], \\ x, & x \in C_2 := [-1,0). \end{cases}$$

It is clear that $F := \bigcap_{i=1}^{\infty} F(T_i) = [-1,0]$, and hence the interior of the common fixed points is nonempty. We show that $\{T_i\}_{i=1}^{\infty}$ is a countable family of pseudocontractive mappings. If $x \in C_2$ and $y \in C_1$, then

$$\begin{aligned} \langle T_i x - T_i y, x - y\rangle &= \left\langle x - \left(\frac{i}{2i+1} + \frac{1}{2}\right)y, x - y\right\rangle \\ &= \left\langle (x-y) + \left(\frac{1}{2} - \frac{i}{2i+1}\right)y, x - y\right\rangle \\ &= |x-y|^2 - \left(\frac{i}{2i+1} - \frac{1}{2}\right)(x-y)y. \end{aligned}$$

Noting that $(\frac{i}{2i+1} - \frac{1}{2})(x-y)y > 0$, we have

$$\langle T_i x - T_i y, x - y\rangle \leq |x-y|^2.$$

The rest is trivial, and it is easy to show that each T_i is Lipschitz and closed. *However,* $\{T_i\}_{i=1}^{\infty}$ *is not uniformly closed*. In fact, for any $[0,1] \supset \{x_i\} \to x^* \in (0,1]$ as $i \to \infty$, we have

$$|x_i - T_i x_i| = \left| x_i - \left(\frac{i}{2i+1} + \frac{1}{2}\right)x_i \right| \to 0 \quad (i \to \infty),$$

while x^* is obviously not a member of F.

Remark 3.3 *By using a specific way of choosing the indexes of the involved mappings and parameters*, we propose an up-to-date iterative approach to approximating common fixed points of countable families of pseudocontractive mappings. The results extend previous results, announced by the authors with the related research interest.

Competing interests
The author declares that they have no competing interests.

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