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New iterative scheme with strict pseudo-contractions and multivalued nonexpansive mappings for fixed point problems and variational inequality problems

J Vahidi[1], A Latif[2*] and M Eslamian[1]

[*]Correspondence: alatif@kau.edu.sa
[2]Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

In this paper, we introduce an iterative scheme for finding a common element of the sets of fixed points for multivalued nonexpansive mappings, strict pseudo-contractive mappings and the set of solutions of an equilibrium problem for a pseudomonotone, Lipschitz-type continuous bifunctions. We prove the strong convergence of the sequence, generated by the proposed scheme, to the solution of the variational inequality. Our results generalize and improve some known results.

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1 Introduction

In 1967, Browder and Petryshyn [1] introduced a concept of strict pseudo-contractive in a real Hilbert space. Let C be a nonempty subset of a real Hilbert space H, and let $T : C \to C$ be a single-valued mapping. A mapping T is called a β-strict pseudo-contractive on C [1] if there exists a constant $\beta \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \beta \|(x - Tx) - (y - Ty)\|^2, \quad \forall x, y \in C.$$

We use $F(T)$ to denote the set of all fixed points of T; $F(T) = \{x \in C : x = T(x)\}$. Note that the class of strictly pseudo-contractive mappings strictly includes the class of nonexpansive mappings, which are the mappings T on C such that

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$ (see [2]). Strictly pseudocontractive mappings have more powerful applications than nonexpansive mappings in solving inverse problems, see Scherzer [3]. In the literature, many interesting and important results have been appeared to approximate the fixed points of pseudo-contractive mappings. For example, see [4–6] and the references therein.

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A subset $C \subset H$ is called proximal if for each $x \in H$, there exists an element $y \in C$ such that

$$\|x - y\| = \operatorname{dist}(x, C) = \inf\{\|x - z\| : z \in C\}.$$

We denote by $CB(C)$, $K(C)$ and $P(C)$ the collection of all nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximal bounded subsets of C, respectively. The Hausdorff metric H on $CB(H)$ is defined by

$$H(A, B) := \max\Big\{\sup_{x\in A} \operatorname{dist}(x, B), \sup_{y\in B} \operatorname{dist}(y, A)\Big\}$$

for all $A, B \in CB(H)$.

Let $T : H \to 2^H$ be a multivalued mapping. An element $x \in H$ is said to be a fixed point of T if $x \in Tx$. A multivalued mapping $T : H \to CB(H)$ is called nonexpansive if

$$H(Tx, Ty) \le \|x - y\|, \quad x, y \in H.$$

Much work has been done on the existence of common fixed points for a pair consisting of a single-valued and a multivalued mapping, see, for instance [7–14]. Let f be a bifunction from $C \times C$ into $\mathbb{R}$, such that $f(x, x) = 0$ for all $x \in C$. Consider the classical Ky Fan inequality. Find a point $x^* \in C$ such that

$$f\big(x^*, y\big) \ge 0, \quad \forall y \in C,$$

where $f(x, \cdot)$ is convex and subdifferentiable on C for every $x \in C$. The set of solutions for this problem is denoted by $\mathrm{Sol}(f, C)$. In fact, the Ky Fan inequality can be formulated as an equilibrium problem. Further, if $f(x, y) = \langle Fx, y - x\rangle$ for every $x, y \in C$, where F is a mapping from C into H, then the Ky Fan inequality problem (equilibrium problem) becomes the classical variational inequality problem, which is formulated as finding a point $x^* \in C$ such that

$$\big\langle Fx^*, y - x^*\big\rangle \ge 0, \quad \forall y \in C.$$

Such problems arise frequently in mathematics, physics, engineering, game theory, transportation, economics and network. Due to importance of the solutions of such problems, many researchers are working in this area and studying on the existence of the solutions of such problems, see, *e.g.*, [15–20]. Further, in the recent years, iterative algorithms for finding a common element of the set of solutions of equilibrium problem and the set of fixed points of nonexpansive mappings in a real Hilbert space have been studied by many authors (see, *e.g.*, [21–33]).

Definition 1.1 Let C be a nonempty closed convex subset of a Hilbert space H. The bifunction $f : C \times C \to \mathbb{R}$ is said to be

(i) strongly monotone on C with $\alpha > 0$ if

$$f(x, y) + f(y, x) \le -\alpha\|x - y\|^2, \quad \forall x, y \in C;$$

(ii) monotone on C if

$$f(x,y)+f(y,x)\leq 0,\quad \forall x,y\in C;$$

(iii) pseudomonotone on C if

$$f(x,y)\geq 0\quad\Rightarrow\quad f(y,x)\leq 0,\quad \forall x,y\in C;$$

(iv) Lipschitz-type continuous on C with constants $c_1>0$ and $c_2>0$ (in the sense of Mastroeni [34]) if

$$f(x,y)+f(y,z)\geq f(x,z)-c_1\|x-y\|^2-c_2\|y-z\|^2,\quad \forall x,y,z\in C.$$

Recently, Anh [35, 36] introduced some methods for finding a common element of the set of solutions of monotone Lipschitz-type continuous equilibrium problem and the set of fixed points of a nonexpansive mapping T in a Hilbert space H. In [35], he proved the following theorem.

Theorem 1.2 *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $f:C\times C\to\mathbb{R}$ be a monotone, continuous, and Lipschitz-type continuous bifunction, and let $f(x,\cdot)$ be convex and subdifferentiable on C for every $x\in C$. Let h be a contraction of C into itself with constant $k\in(0,1)$, let S be a nonexpansive mapping of C into itself, and let $F(S)\cap\operatorname{Sol}(f,C)\neq\emptyset$. Let $\{x_n\}$, $\{w_n\}$ and $\{z_n\}$ be sequences generated by $x_0\in C$ and by*

$$\begin{cases} w_n=\arg\min\{\lambda_n f(x_n,w)+\frac{1}{2}\|w-x_n\|^2:w\in C\},\\ z_n=\arg\min\{\lambda_n f(w_n,z)+\frac{1}{2}\|z-x_n\|^2:z\in C\},\\ x_{n+1}=\alpha_n h(x_n)+\beta_n x_n+\gamma_n(\mu S(x_n)+(1-\mu)z_n),\quad \forall n\geq 0,\end{cases}$$

where $\mu\in(0,1)$, and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\lambda_n\}$ satisfy the following conditions:

(i) $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^{\infty}\alpha_n=\infty$,

(ii) $\lim_{n\to\infty}|\lambda_{n+1}-\lambda_n|=0$, $\{\lambda_n\}\subset[a,b]\subset(0,\frac{1}{L})$, *where* $L=\max\{2c_1,2c_2\}$,

(iii) $\alpha_n+\beta_n+\gamma_n=1$ *and* $\alpha_n(2-\alpha_n-2\beta_n k-2\gamma_n)\in(0,1)$,

(iv) $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$.

Then, the sequences $\{x_n\}$, $\{w_n\}$ and $\{z_n\}$ converge strongly to $q\in F(S)\cap\operatorname{Sol}(f,C)$ which solves the variational inequality

$$\langle (I-h)q,x-q\rangle\geq 0,\quad \forall x\in F(S)\cap\operatorname{Sol}(f,C).$$

In this paper, we introduce an iterative algorithm for finding a common element of the sets of fixed points for multivalued nonexpansive mappings, strict pseudo-contractive mappings and the set of solutions of an equilibrium problem for a pseudomonotone, Lipschitz-type continuous bifunctions. We prove the strong convergence of the sequence generated by the proposed algorithm to the solution of the variational inequality. Our results generalize and improve a number of known results including the results of Anh [35].

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let $\{x_n\}$ be a sequence in H, and let $x \in H$. Weak convergence of $\{x_n\}$ to x is denoted by $x_n \rightharpoonup x$, and strong convergence by $x_n \to x$. Let C be a nonempty closed convex subset of H. The nearest point projection from H to C, denoted by Proj_C, assigns to each $x \in H$ the unique point $\mathrm{Proj}_C x \in C$ with the property

$$\|x - \mathrm{Proj}_C x\| := \inf\bigl\{\|x - y\|, \forall y \in C\bigr\}.$$

It is known that Proj_C is a nonexpansive mapping, and for each $x \in H$,

$$\langle x - \mathrm{Proj}_C x, y - \mathrm{Proj}_C x\rangle \le 0, \quad \forall y \in C.$$

Definition 2.1 Let C be a nonempty, closed and convex subset of a Hilbert space H. Denote by $N_C(v)$ the normal cone of C at $v \in C$, *i.e.*,

$$N_C(v) := \bigl\{z \in H : \langle z, y - v\rangle \le 0, \forall y \in C\bigr\}.$$

Definition 2.2 Let C be a nonempty, closed and convex subset of a Hilbert space H, and let $f : C \times C \to \mathbb{R}$ be a bifunction. For each $z \in C$, by $\partial_2 f(z,u)$ we denote the subgradient of the function $f(z,\cdot)$ at u, *i.e.*,

$$\partial_2 f(z,u) = \bigl\{\xi \in H : f(z,t) - f(z,u) \ge \langle\xi, t - u\rangle, \forall t \in C\bigr\}.$$

The following lemmas are crucial for the proofs of our results.

Lemma 2.3 *In a Hilbert space H, the following inequality holds*:

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H.$$

Lemma 2.4 [37] *Let $\{a_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence in $(0,1)$ with $\sum_{n=1}^{\infty}\alpha_n = \infty$, let $\{\gamma_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty}\gamma_n < \infty$, and let $\{\beta_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty}\beta_n \le 0$. Suppose that the following inequality holds*:

$$a_{n+1} \le (1 - \alpha_n)a_n + \alpha_n\beta_n + \gamma_n, \quad n \ge 0.$$

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5 [38] *Let H be a real Hilbert space. Then for all $x, y, z \in H$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|z - y\|^2.$$

Lemma 2.6 [39] *Let $\{t_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $t_{n_i} < t_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence*

$\{\tau(n)\} \subset \mathbb{N}$ *such that* $\tau(n) \to \infty$, *and the following properties are satisfied by all* (*sufficiently large*) *numbers* $n \in \mathbb{N}$:

$$t_{\tau(n)} \leq t_{\tau(n)+1}, \qquad t_n \leq t_{\tau(n)+1}.$$

In fact,

$$\tau(n) = \max\{k \leq n : t_k < t_{k+1}\}.$$

Lemma 2.7 [36] *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $f : C \times C \to \mathbb{R}$ be a pseudomonotone and Lipschitz-type continuous bifunction. For each $x \in C$, let $f(x, \cdot)$ be convex and subdifferentiable on C. Let $\{x_n\}$, $\{z_n\}$ and $\{w_n\}$ be the sequences, generated by $x_0 \in C$ and by*

$$\begin{cases} w_n = \arg\min\{\lambda_n f(x_n, w) + \frac{1}{2}\|w - x_n\|^2 : w \in C\}, \\ z_n = \arg\min\{\lambda_n f(w_n, z) + \frac{1}{2}\|z - x_n\|^2 : z \in C\}. \end{cases}$$

Then for each $x^* \in \operatorname{Sol}(f, C)$,

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - (1 - 2\lambda_n c_1)\|x_n - w_n\|^2 - (1 - 2\lambda_n c_2)\|w_n - z_n\|^2, \quad \forall n \geq 0.$$

Lemma 2.8 [5] *Let C be nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be β-pseudo-contraction mapping. Then $I - T$ is demiclosed at 0. That is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then $x = Tx$.*

Lemma 2.9 [5] *Let C be a closed convex subset of a Hilbert space H, and let $T : C \to C$ be a β-strict pseudo-contraction on C and the fixed-point set $F(T)$ of T is nonempty, then $F(T)$ is closed and convex.*

Lemma 2.10 [40] *Let C be a closed convex subset of a real Hilbert space H. Let $T : C \to CB(C)$ be a nonexpansive multivalued mapping. Assume that $T(p) = \{p\}$ for all $p \in F(T)$. Then $F(T)$ is closed and convex.*

Lemma 2.11 [25] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to K(C)$ be a nonexpansive multivalued mapping. If $x_n \rightharpoonup v$ and $\lim_{n\to\infty} \operatorname{dist}(x_n, Tx_n) = 0$, then $v \in Tv$.*

3 Main results

Now, we are in a position to give our main results.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $f : C \times C \to \mathbb{R}$ be a monotone, continuous, and Lipschitz-type continuous bifunction. Suppose that $f(x, \cdot)$ is convex and subdifferentiable on C for all $x \in C$. Let, $T : C \to CB(C)$ be a multivalued nonexpansive mapping, and let $S : C \to C$ be a β-strict pseudo-contraction mapping. Assume that $\mathcal{F} = F(T) \cap F(S) \cap \operatorname{Sol}(f, C) \neq \emptyset$ and $T(p) = \{p\}$ for each $p \in \mathcal{F}$. Let h*

be a k-contraction of C into itself. Let $\{x_n\}$, $\{w_n\}$ and $\{z_n\}$ be sequences generated by $x_0 \in C$ and by

$$\begin{cases} w_n = \arg\min\{\lambda_n f(x_n, w) + \frac{1}{2}\|w - x_n\|^2 : w \in C\}, \\ z_n = \arg\min\{\lambda_n f(w_n, z) + \frac{1}{2}\|z - x_n\|^2 : z \in C\}, \\ y_n = \alpha_n z_n + \beta_n u_n + \gamma_n S z_n, \\ x_{n+1} = \vartheta_n h(x_n) + (1 - \vartheta_n) y_n, \quad \forall n \geq 0, \end{cases} \tag{1}$$

where $u_n \in T z_n$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ and $\{\vartheta_n\}$ satisfy the following conditions:

(i) $\{\vartheta_n\} \subset (0,1)$, $\lim_{n\to\infty} \vartheta_n = 0$, $\sum_{n=1}^{\infty} \vartheta_n = \infty$,

(ii) $\{\lambda_n\} \subset [a,b] \subset (0, \frac{1}{L})$, *where* $L = \max\{2c_1, 2c_2\}$,

(iii) $\{\alpha_n\}, \{\gamma_n\} \subset [a,1) \subset (0,1)$, $\alpha_n > \beta$ *and* $\alpha_n + \beta_n + \gamma_n = 1$.

Then, the sequence $\{x_n\}$ converges strongly to $q \in \mathcal{F}$, which solves the variational inequality

$$\langle q - hq, x - q\rangle \geq 0, \quad \forall x \in \mathcal{F}. \tag{2}$$

Proof Let $Q = \mathrm{Proj}_{\mathcal{F}}$. It easy to see that Qh is a contraction. By the Banach contraction principle, there exists a $q \in \mathcal{F}$ such that $q = (Qh)(q)$. Applying Lemma 2.7, we have

$$\|z_n - q\|^2 \leq \|x_n - q\|^2 - (1 - 2\lambda_n c_1)\|x_n - w_n\|^2 - (1 - 2\lambda_n c_2)\|w_n - z_n\|^2. \tag{3}$$

This implies that

$$\|z_n - q\| \leq \|x_n - q\|. \tag{4}$$

Since T is nonexpansive and $Tq = \{q\}$, by (4) we have

$$\|u_n - q\| = \mathrm{dist}(u_n, Tq) \leq H(Tz_n, Tq) \leq \|z_n - q\| \leq \|x_n - q\|. \tag{5}$$

We show that $\{x_n\}$ is bounded. Indeed, using inequality (4), (5) and Lemma 2.5, we have

$$\begin{aligned} \|y_n - q\|^2 &= \|\alpha_n z_n + \beta_n u_n + \gamma_n S z_n - q\|^2 \\ &\leq \alpha_n\|z_n - q\|^2 + \beta_n\|u_n - q\|^2 + \gamma_n\|Sz_n - q\|^2 \\ &\quad - \alpha_n\beta_n\|u_n - z_n\|^2 - \alpha_n\gamma_n\|z_n - Sz_n\|^2 \\ &\leq \alpha_n\|x_n - q\|^2 + \beta_n\|x_n - q\|^2 + \gamma_n\big(\|z_n - q\|^2 + \beta\|z_n - Sz_n\|^2\big) \\ &\quad - \alpha_n\beta_n\|u_n - z_n\|^2 - \alpha_n\gamma_n\|z_n - Sz_n\|^2 \\ &\quad - \alpha_n(1 - 2\lambda_n c_1)\|x_n - w_n\|^2 - \alpha_n(1 - 2\lambda_n c_2)\|w_n - z_n\|^2 \\ &\leq \|x_n - q\|^2 - \alpha_n\beta_n\|u_n - z_n\|^2 - \gamma_n(\alpha_n - \beta)\|z_n - Sz_n\|^2 \\ &\quad - \alpha_n(1 - 2\lambda_n c_1)\|x_n - w_n\|^2 - \alpha_n(1 - 2\lambda_n c_2)\|w_n - z_n\|^2. \end{aligned} \tag{6}$$

It follows that

$$\|y_n - q\|^2 \leq \|x_n - q\|^2 - \gamma_n(\alpha_n - \beta)\|z_n - Sz_n\|^2.$$

Since $\alpha_n > \beta$, we get that $\|y_n - q\| \leq \|x_n - q\|$. This implies that

$$\begin{aligned}
\|x_{n+1} - q\| &= \big\|\vartheta_n h x_n + (1-\vartheta_n) y_n - q\big\| \\
&\leq \vartheta_n \|hx_n - q\| + (1-\vartheta_n)\|y_n - q\| \\
&\leq \vartheta_n\big(\|hx_n - hq\| + \|hq - q\|\big) + (1-\vartheta_n)\|x_n - q\| \\
&\leq \vartheta_n k \|x_n - q\| + \vartheta_n \|hq - q\| + (1-\vartheta_n)\|x_n - q\| \\
&= \big(1 - \vartheta_n(1-k)\big)\|x_n - q\| + \vartheta_n \|hq - q\| \\
&\leq \max\left\{\|x_n - q\|, \frac{\|hq - q\|}{1-k}\right\}.
\end{aligned}$$

By induction, we get

$$\|x_n - q\| \leq \max\left\{\|x_0 - q\|, \frac{\|hq - q\|}{1-k}\right\}$$

for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is bounded, and we also obtain that $\{u_n\}$, $\{z_n\}$, $\{hx_n\}$ and $\{Sz_n\}$ are bounded. Next, we show that

$$\lim_{n\to\infty} \|z_n - Sz_n\| = \lim_{n\to\infty} \|z_n - u_n\| = \lim_{n\to\infty} \|z_n - x_n\| = 0.$$

Indeed, using inequality (6), we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \big\|\vartheta_n h x_n + (1-\vartheta_n) y_n - q\big\|^2 \\
&\leq \vartheta_n \|hx_n - q\|^2 + (1-\vartheta_n)\|y_n - q\|^2 \\
&\leq \vartheta_n \|hx_n - q\|^2 + (1-\vartheta_n)\|x_n - p\|^2 \\
&\quad - (1-\vartheta_n)\alpha_n\beta_n\|u_n - z_n\|^2 - (1-\vartheta_n)\gamma_n(\alpha_n - \beta)\|z_n - Sz_n\|^2 \\
&\quad - (1-\vartheta_n)\alpha_n(1 - 2\lambda_n c_1)\|x_n - w_n\|^2 - (1-\vartheta_n)\alpha_n(1 - 2\lambda_n c_2)\|w_n - z_n\|^2.
\end{aligned}$$

Therefore, we have

$$(1-\vartheta_n)\gamma_n(\alpha_n - \beta)\|z_n - Sz_n\|^2 \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + \vartheta_n \|hx_n - q\|. \tag{7}$$

In order to prove that $x_n \to q$ as $n \to \infty$, we consider the following two cases.

Case 1. Suppose that there exists n_0 such that $\{\|x_n - q\|\}$ is nonincreasing, for all $n \geq n_0$. Boundedness of $\{\|x_n - q\|\}$ implies that $\|x_n - q\|$ is convergent. Since $\{hx_n\}$ is bounded and $\lim_{n\to\infty} \vartheta_n = 0$, from (7) and our assumption that $\alpha_n > \beta$, we obtain that

$$\lim_{n\to\infty} \|z_n - Sz_n\| = 0.$$

By similar argument we can obtain that

$$\lim_{n\to\infty} \|u_n - z_n\| = \lim_{n\to\infty} \|x_n - w_n\| = \lim_{n\to\infty} \|w_n - z_n\| = 0. \tag{8}$$

From this with inequality $\|x_n - z_n\| \leq \|x_n - w_n\| + \|w_n - z_n\|$, it follows that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{9}$$

Next, we show that

$$\limsup_{n\to\infty} \langle q - hq, q - x_n \rangle \leq 0,$$

where $q = (Qh)(q)$. To show this inequality, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty} \langle q - hq, q - x_{n_i} \rangle = \limsup_{n\to\infty} \langle q - hq, q - x_n \rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$, which converges weakly to x^*. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup x^*$. From inequality (9), we have $z_{n_i} \rightharpoonup x^*$. Now, since $\lim_{n\to\infty} \|z_n - Sz_n\| = 0$, from Lemma 2.8, we have $x^* \in F(S)$. Also from (8), we have

$$\operatorname{dist}(z_n, Tz_n) \leq \|u_n - z_n\| \to 0 \quad \text{as } n \to \infty.$$

It follows from Lemma 2.9 that $x^* \in F(T)$. Now, we show that $x^* \in \operatorname{Sol}(f, C)$. Since $f(x, \cdot)$ is convex on C for each $x \in C$, we see that

$$w_n = \arg\min\left\{\lambda_n f(x_n, y) + \frac{1}{2}\|y - x_n\|^2 : y \in C\right\}$$

if and only if

$$o \in \partial_2\left(f(x_n, y) + \frac{1}{2}\|y - x_n\|^2\right)(w_n) + N_C(w_n),$$

where $N_C(x)$ is the (outward) normal cone of C at $x \in C$. This follows that

$$0 = \lambda_n v + w_n - x_n + u_n,$$

where $v \in \partial_2 f(x_n, w_n)$ and $u_n \in N_C(w_n)$. By the definition of the normal cone N_C, we have

$$\langle w_n - x_n, y - w_n \rangle \geq \lambda_n \langle v, w_n - y \rangle, \quad \forall y \in C. \tag{10}$$

Since $f(x_n, \cdot)$ is subdifferentiable on C, there exists $v \in \partial_2 f(x_n, w_n)$ such that

$$f(x_n, y) - f(x_n, w_n) \geq \langle v, y - w_n \rangle, \quad \forall y \in C$$

(see, [41, 42]). Combining this with (10), we have

$$\lambda_n\big(f(x_n, y) - f(x_n, w_n)\big) \geq \langle w_n - x_n, w_n - y \rangle, \quad \forall y \in C.$$

Hence

$$f(x_{n_i},y)-f(x_{n_i},w_{n_i})\geq\frac{1}{\lambda_{n_i}}\langle w_{n_i}-x_{n_i},w_{n_i}-y\rangle,\quad\forall y\in C.$$

From (8), we have that $w_{n_i}\rightharpoonup x^*$. Now by continuity of f and assumption that $\{\lambda_n\}\subset[a,b]\subset]0,\frac{1}{L}[$, we have

$$f\bigl(x^*,y\bigr)\geq 0,\quad\forall y\in C.$$

This implies that $x^*\in\mathrm{Sol}(f,C)$, and hence $x^*\in\mathcal{F}$. Since $q=(Qh)(q)$ and $x^*\in\mathcal{F}$, it follows that

$$\limsup_{n\to\infty}\langle q-hq,q-x_n\rangle=\lim_{i\to\infty}\langle q-hq,q-x_{n_i}\rangle=\bigl\langle q-hq,q-x^*\bigr\rangle\leq 0.$$

By using Lemma 2.3 and inequality (6), we have

$$\begin{aligned}\|x_{n+1}-q\|^2&\leq\bigl\|(1-\vartheta_n)(y_n-q)\bigr\|^2+2\vartheta_n\langle hx_n-q,x_{n+1}-q\rangle\\&\leq(1-\vartheta_n)^2\|y_n-q\|^2+2\vartheta_n\langle hx_n-hq,x_{n+1}-q\rangle+2\vartheta_n\langle hq-q,x_{n+1}-q\rangle\\&\leq(1-\vartheta_n)^2\|x_n-q\|^2+2\vartheta_nk\|x_n-q\|\|x_{n+1}-q\|+2\vartheta_n\langle hq-q,x_{n+1}-q\rangle\\&\leq(1-\vartheta_n)^2\|x_n-q\|^2+\vartheta_nk\bigl(\|x_n-q\|^2+\|x_{n+1}-q\|^2\bigr)\\&\quad+2\vartheta_n\langle hq-q,x_{n+1}-q\rangle\\&\leq\bigl((1-\vartheta_n)^2+\vartheta_nk\bigr)\|x_n-q\|^2+\vartheta_nk\|x_{n+1}-q\|^2+2\vartheta_n\langle hq-q,x_{n+1}-q\rangle.\end{aligned}$$

This implies that

$$\begin{aligned}\|x_{n+1}-q\|^2&\leq\left(1-\frac{2(1-k)\vartheta_n}{1-\vartheta_nk}\right)\|x_n-q\|^2+\frac{\vartheta_n^2}{1-\vartheta_nk}\|x_n-q\|^2\\&\quad+\frac{2\vartheta_n}{1-\vartheta_nk}\langle hq-q,x_{n+1}-q\rangle.\end{aligned}$$

From Lemma 2.4, we conclude that the sequence $\{x_n\}$ converges strongly to q.

Case 2. Assume that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\|x_{n_j}-q\|<\|x_{n_{j+1}}-q\|,$$

for all $j\in\mathbb{N}$. In this case from Lemma 2.6, there exists a nondecreasing sequence $\{\tau(n)\}$ of $\mathbb{N}$ for all $n\geq n_0$ (for some n_0 large enough) such that $\tau(n)\to\infty$ as $n\to\infty$, and the following inequalities hold for all $n\geq n_0$,

$$\|x_{\tau(n)}-q\|<\|x_{\tau(n)+1}-q\|,\qquad\|x_n-q\|<\|x_{\tau(n)+1}-q\|.$$

From (3), we obtain $\lim_{n\to\infty}\|z_{\tau(n)}-Sz_{\tau(n)}\|=0$, and similarly we obtain

$$\lim_{n\to\infty}\|x_{\tau(n)}-z_{\tau(n)}\|=\lim_{n\to\infty}\|u_{\tau(n)}-z_{\tau(n)}\|=0.$$

Following an argument similar to that in Case 1, we have

$$\lim_{n\to\infty} \|x_{\tau(n)} - q\| = 0, \qquad \lim_{n\to\infty} \|x_{\tau(n)+1} - q\| = 0.$$

Thus, by Lemma 2.6, we have

$$0 \le \|x_n - q\| \le \max\{\|x_{\tau(n)} - q\|, \|x_n - q\|\} \le \|x_{\tau(n)+1} - q\|.$$

Therefore, $\{x_n\}$ converges strongly to $q \in \mathcal{F}$. This completes the proof. □

Now, let $T : C \to P(C)$ be a multivalued mapping, and let

$$P_T(x) = \{y \in Tx : \|x - y\| = \operatorname{dist}(x, Tx)\}, \quad x \in C.$$

Then, we have $F(T) = F(P_T)$. Indeed, if $p \in F(T)$, then $P_T(p) = \{p\}$, hence $p \in F(P_T)$. On the other hand, if $p \in F(P_T)$, since $P_T(p) \subset Tp$, we have $p \in F(T)$. Now, using the similar arguments as in the proof of Theorem 3.1, we obtain the following result by replacing T by P_T, and removing the strict condition $T(p) = \{p\}$ for all $p \in F(T)$.

Theorem 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $f : C \times C \to \mathbb{R}$ be a monotone, continuous, and Lipschitz-type continuous bifunction. Suppose that $f(x, \cdot)$ is convex and subdifferentiable on C for all $x \in C$. Let $T : C \to P(C)$ be a multivalued mapping such that P_T is nonexpansive, and let $S : C \to C$ be a β-strict pseudo-contraction mapping. Assume that $\mathcal{F} = F(T) \cap F(S) \cap \operatorname{Sol}(f, C) \neq \emptyset$. Let h be a k-contraction of C into itself. Let $\{x_n\}$, $\{w_n\}$ and $\{z_n\}$ be sequences generated by $x_0 \in C$ and by*

$$\begin{cases} w_n = \arg\min\{\lambda_n f(x_n, w) + \frac{1}{2}\|w - x_n\|^2 : w \in C\}, \\ z_n = \arg\min\{\lambda_n f(w_n, z) + \frac{1}{2}\|z - x_n\|^2 : z \in C\}, \\ y_n = \alpha_n z_n + \beta_n u_n + \gamma_n S z_n, \\ x_{n+1} = \vartheta_n h(x_n) + (1 - \vartheta_n) y_n, \quad \forall n \ge 0, \end{cases} \tag{11}$$

where $u_n \in P_T(z_n)$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ and $\{\vartheta_n\}$ satisfy the following conditions:

(i) $\{\vartheta_n\} \subset (0,1)$, $\lim_{n\to\infty} \vartheta_n = 0$, $\sum_{n=1}^{\infty} \vartheta_n = \infty$,

(ii) $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{L})$, *where* $L = \max\{2c_1, 2c_2\}$,

(iii) $\{\alpha_n\}, \{\gamma_n\} \subset [a, 1) \subset (0,1)$, $\alpha_n > \beta$ *and* $\alpha_n + \beta_n + \gamma_n = 1$.

Then, the sequence $\{x_n\}$ converges strongly to $q \in \mathcal{F}$, which solves the variational inequality

$$\langle q - hq, x - q\rangle \ge 0, \quad \forall x \in \mathcal{F}. \tag{12}$$

As a consequence, we obtain the following result for single-valued mappings.

Corollary 3.3 *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $f : C \times C \to \mathbb{R}$ be a monotone, continuous, and Lipschitz-type continuous bifunction. Suppose that $f(x, \cdot)$ is convex and subdifferentiable on C for all $x \in C$. Let $T : C \to C$ be a nonexpansive mapping, and let $S : C \to C$ be a β-strict pseudo-contraction mapping.*

Assume that $\mathcal{F} = F(T) \cap F(S) \cap \mathrm{Sol}(f, C) \neq \emptyset$. Let h be a k-contraction of C into itself. Let $\{x_n\}$, $\{w_n\}$ and $\{z_n\}$ be sequences generated by $x_0 \in C$ and by

$$\begin{cases} w_n = \arg\min\{\lambda_n f(x_n, w) + \frac{1}{2}\|w - x_n\|^2 : w \in C\}, \\ z_n = \arg\min\{\lambda_n f(w_n, z) + \frac{1}{2}\|z - x_n\|^2 : z \in C\}, \\ y_n = \alpha_n z_n + \beta_n T z_n + \gamma_n S z_n, \\ x_{n+1} = \vartheta_n h(x_n) + (1 - \vartheta_n) y_n, \quad \forall n \geq 0. \end{cases} \tag{13}$$

Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ and $\{\vartheta_n\}$ satisfy the following conditions:

(i) $\{\vartheta_n\} \subset (0, 1)$, $\lim_{n\to\infty} \vartheta_n = 0$, $\sum_{n=1}^{\infty} \vartheta_n = \infty$,

(ii) $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{L})$, *where* $L = \max\{2c_1, 2c_2\}$,

(iii) $\{\alpha_n\}, \{\gamma_n\} \subset [a, 1) \subset (0, 1)$, $\alpha_n > \beta$ *and* $\alpha_n + \beta_n + \gamma_n = 1$.

Then, the sequence $\{x_n\}$ converges strongly to $q \in \mathcal{F}$, which solves the variational inequality

$$\langle q - hq, x - q\rangle \geq 0, \quad \forall x \in \mathcal{F}. \tag{14}$$

4 Application to variational inequalities

In this section, we consider the particular Ky Fan inequality, corresponding to the function f, defined by $f(x, y) = \langle F(x), y - x\rangle$ for every $x, y \in C$ with $F : C \to H$. Then, we obtain the classical variational inequality

$$\text{find } z \in C \text{ such that } \langle F(z), y - z\rangle \geq 0, \forall y \in C.$$

The set of solutions of this problem is denoted by $VI(F, C)$. In that particular case, the solution y_n of the minimization problem

$$\arg\min\left\{\lambda_n f(x_n, y) + \frac{1}{2}\|y - x_n\|^2 : y \in C\right\}$$

can be expressed as

$$y_n = \mathrm{Proj}_C\bigl(x_n - \lambda_n F(x_n)\bigr).$$

Let F be L-Lipschitz continuous on C. Then

$$f(x, y) + f(y, z) - f(x, z) = \bigl\langle F(x) - F(y), y - z\bigr\rangle, \quad x, y, z \in C.$$

Therefore,

$$\bigl|\bigl\langle F(x) - F(y), y - z\bigr\rangle\bigr| \leq L\|x - y\|\|y - z\| \leq \frac{L}{2}\bigl(\|x - y\|^2 + \|y - z\|^2\bigr),$$

and, hence, f satisfies Lipschitz-type continuous condition with $c_1 = c_2 = \frac{L}{2}$. Now, using Theorem 3.1, we obtain the following convergence theorem for finding a common element of the set of common fixed points of a strict pseudo-contractive mapping and a multivalued nonexpansive mapping and the solution set of the variational inequality problem.

Theorem 4.1 *Let C be a nonempty closed convex subset of a real Hilbert space H, and let F be a function from C to H such that F is monotone and L-Lipschitz continuous on C. Let, $T : C \to CB(C)$ be a multivalued nonexpansive mapping, and let $S : C \to C$ be a β-strict pseudo-contraction mapping. Assume that $\mathcal{F} = F(T) \cap F(S) \cap VI(F,C) \neq \emptyset$ and $T(p) = \{p\}$ for each $p \in \mathcal{F}$. Let h be a k-contraction of C into itself. Let $\{x_n\}$, $\{w_n\}$, and let $\{z_n\}$ be sequences generated by $x_0 \in C$ and by*

$$\begin{cases} w_n = \mathrm{Proj}_C(x_n - \lambda_n F(x_n)), \\ z_n = \mathrm{Proj}_C(x_n - \lambda_n F(w_n)), \\ y_n = \alpha_n z_n + \beta_n u_n + \gamma_n S z_n, \\ x_{n+1} = \vartheta_n h(x_n) + (1 - \vartheta_n) y_n, \quad \forall n \geq 0, \end{cases} \tag{15}$$

where $u_n \in T z_n$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$ and $\{\vartheta_n\}$ satisfy the following conditions:

(i) $\{\vartheta_n\} \subset (0,1)$, $\lim_{n\to\infty} \vartheta_n = 0$, $\sum_{n=1}^{\infty} \vartheta_n = \infty$,

(ii) $\{\lambda_n\} \subset [a,b] \subset (0, \frac{1}{L})$,

(iii) $\{\alpha_n\}, \{\gamma_n\} \subset [a,1) \subset (0,1)$, $\alpha_n > \beta$ *and* $\alpha_n + \beta_n + \gamma_n = 1$.

Then, the sequence $\{x_n\}$ converges strongly to $q \in \mathcal{F}$, which solves the variational inequality

$$\langle q - hq, x - q\rangle \geq 0, \quad \forall x \in \mathcal{F}. \tag{16}$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Author details

[1]Department of Applied Mathematics, Mazandaran University of Science and Technology, Behshahr, Iran. [2]Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia.

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