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# A note on cone $b$-metric and its related results: generalizations or equivalence?

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**Abstract**
Very recently, a notion of cone $b$-metric was introduced as a generalization of $b$-metric, and some related fixed point results were obtained. In this paper, we investigate the answer to the question whether the given results generalize the existing ones or are equivalent to them.
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## 1 Introduction and preliminaries

Topological vector space-valued metric space (or TVS-cone metric space) introduced by Du [1] as a generalization of the Banach-valued/cone metric space [2]. Recently, Du [1] noted that fixed point theorems in generalized cone metric spaces and in usual metric spaces are equivalent. In particular, the author proved that the celebrated fixed point theorems of Banach, Kannan, Chatterjea, *etc.* in both TVS-cone metric can be derived easily from the usual metric space set-up, by a simple manipulation, namely, using a scalarization function. Very recently, a number of publications, dealing with the cone $b$-metric space structure and fixed point theorems on such spaces, appeared. In this paper, we show that fixed point theorems in cone $b$-metric and usual $b$-metric spaces are equivalent. This paper can be considered as a continuation of the report [1].

A topological vector space (t.v.s. for short) is a vector space with a topology such that the vector space operations (addition and scalar multiplication) are continuous. A topological vector space is *locally convex* if its origin has a basis of neighborhoods that are convex. Let $Y$ be a locally convex Hausdorff t.v.s. with its zero vector $\theta$, let $\tau$ denote the topology of $Y$, and let $\mathcal{U}_\tau$ be the base at $\theta$, consisting of all absolutely convex neighborhood of $\theta$. Let

$$\mathcal{L} = \{\ell : \ell \text{ is a Minkowski functional of } U \text{ for } U \in \mathcal{U}_\tau\}.$$

Then $\mathcal{L}$ is a family of seminorms on $Y$. For each $\ell \in \mathcal{L}$, let

$$V(\ell) = \{y \in Y : \ell(y) < 1\},$$

and let

$$\mathcal{U}_{\mathcal{L}} = \{U : U = r_1 V(\ell_1) \cap r_2 V(\ell_2) \cap \cdots \cap r_n V(\ell_n), \\ r_k > 0, \ell_k \in \mathcal{L}, 1 \le k \le n, n \in \mathbb{N}\}.$$

Then $\mathcal{U}_{\mathcal{L}}$ is a base at $\theta$, and the topology $\Gamma_{\mathcal{L}}$ generated by $\mathcal{U}_{\mathcal{L}}$ is the weakest topology for $Y$ such that all seminorms in $\mathcal{L}$ are continuous and $\tau = \Gamma_{\mathcal{L}}$. Moreover, given any neighborhood $\mathcal{O}_\theta$ of $\theta$, there exists $U \in \mathcal{U}_{\mathcal{L}}$ such that $\theta \in U \subset \mathcal{O}_\theta$ (see, *e.g.*, [3, Theorem 12.4 in II.12, p.113]).

Throughout this paper, we follow all notations considered in [1]. Let $E$ be a t.v.s. with its zero vector $\theta_E$. A nonempty subset $K$ of $E$ is called a *cone* if $\lambda K \subseteq K$ for $\lambda \geq 0$. A cone $K$ is said to be *pointed* if $K \cap (-K) = \{\theta_E\}$. For a given cone $K \subseteq E$, we can define a partial ordering $\precsim$ (or $\precsim_K$) with respect to $K$ by

$$x \precsim y \quad \Longleftrightarrow \quad y - x \in K.$$

$x \prec y$ will stand for $x \precsim y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \operatorname{int} K$, where $\operatorname{int} K$ denotes the interior of $K$.

Let $E$ be a t.v.s. and $K$ a convex cone with $\operatorname{int} K \neq \emptyset$ in $E$. Then it is obvious that

$$\operatorname{int} K + \operatorname{int} K \subseteq \operatorname{int} K + K \subseteq \operatorname{int} K$$

and

$$\lambda \operatorname{int} K \subseteq \operatorname{int} K \quad \text{for all } \lambda > 0.$$

In the following, unless otherwise specified, we always assume that $Y$ is a locally convex Hausdorff t.v.s. with its zero vector $\theta$, $K$ a proper, closed and convex pointed cone in $Y$ with $\operatorname{int} K \neq \emptyset$, $e \in \operatorname{int} K$ and $\precsim$ a partial ordering with respect to $K$.

The nonlinear scalarization function [1, 4, 5] $\xi_e : Y \to \mathbb{R}$ is defined as follows:

$$\xi_e(y) = \inf\{r \in \mathbb{R} : y \in re - K\} \quad \text{for all } y \in Y.$$

**Lemma 1.1** (See, *e.g.*, [1, 4, 5]) *For each $r \in \mathbb{R}$ and $y \in Y$, the following statements are satisfied*:

(i) $\xi_e(y) \leq r \iff y \in re - K$,
(ii) $\xi_e(y) > r \iff y \notin re - K$,
(iii) $\xi_e(y) \geq r \iff y \notin re - \operatorname{int} K$,
(iv) $\xi_e(y) < r \iff y \in re - \operatorname{int} K$,
(v) $\xi_e(\cdot)$ *is positively homogeneous and continuous on* $Y$,
(vi) *if* $y_1 \in y_2 + K$ (*i.e.* $y_2 \precsim y_1$), *then* $\xi_e(y_2) \leq \xi_e(y_1)$,
(vii) $\xi_e(y_1 + y_2) \leq \xi_e(y_1) + \xi_e(y_2)$ *for all* $y_1, y_2 \in Y$.

**Remark 1.2**

(a) Clearly, $\xi_e(\theta) = 0$.
(b) It is worth mentioning that the reverse statement of (vi) in Lemma 1.1 (*i.e.*, $\xi_e(y_2) \leq \xi_e(y_1) \Longrightarrow y_2 \precsim y_1$) does not hold in general. For example, let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 : x, y \geq 0\}$, and let $e = (1,1)$. Then $K$ is a proper, closed, convex and pointed cone in $Y$ with $\operatorname{int} K = \{(x,y) \in \mathbb{R}^2 : x, y > 0\} \neq \emptyset$ and $e \in \operatorname{int} K$. For $r = 1$, it is easy to see that $y_1 = (8, -15) \notin re - \operatorname{int} K$, and $y_2 = (0,0) \in re - \operatorname{int} K$. By applying (iii) and (iv) of Lemma 1.1, we have $\xi_e(y_2) < 1 \leq \xi_e(y_1)$, while $y_1 \notin y_2 + K$.

### 1.1 TVS-cone metric spaces

**Definition 1.3** (See [1]) Let $X$ be a nonempty set. Suppose that a vector-valued function $p : X \times X \to Y$ satisfies:

(C1) $\theta \precsim p(x,y)$ for all $x, y \in X$ and $p(x,y) = \theta$ if and only if $x = y$,

(C2) $p(x,y) = p(y,x)$ for all $x, y \in X$,

(C3) $p(x,y) \precsim p(x,z) + p(z,y)$ for all $x, y, z \in X$.

Then, the function $p$ is called a TVS-cone metric on $X$. Furthermore, the pair $(X,p)$ is called a TVS-cone metric space (in short, TVS-CMS).

**Lemma 1.4** (See [1]) *Let $(X,p)$ be a TVS-CMS. Then, $d_p : X \times X \to [0,\infty)$ defined by $d_p = \xi_e \circ p$ is a metric.*

**Remark 1.5** We notice that a cone metric space (in short, CMS), introduced by Huang and Zhang [2], is a special case of TVS-CMS. Indeed, the authors [2] considered $E$ as a real Banach space instead of TVS in Definition 1.3. Further, for a CMS $(X,p)$, the function $d_p : X \times X \to [0,\infty)$ defined by $d_p = \xi_e \circ p$ is also a metric.

**Definition 1.6** (See [1]) Let $(X,p)$ be a TVS-CMS, $x \in X$ and $\{x_n\}_{n\in\mathbb{N}}$ a sequence in $X$.

(i) $\{x_n\}_{n\in\mathbb{N}}$ TVS-cone converges to $x \in X$ whenever for every $\theta \ll c \in Y$, there is a natural number $M$ such that $p(x_n, x) \ll c$ for all $n \geq M$ and denoted by cone-$\lim_{n\to\infty} x_n = x$ (or $x_n \xrightarrow{\text{cone}} x$ as $n \to \infty$),

(ii) $\{x_n\}_{n\in\mathbb{N}}$ TVS-cone Cauchy sequence in $(X,p)$ whenever for every $\theta \ll c \in Y$, there is a natural number $M$ such that $p(x_n, x_m) \ll c$ for all $n, m \geq M$,

(iii) $(X,p)$ is TVS-cone complete if every sequence TVS-cone Cauchy sequence in $X$ is a TVS-cone convergent.

**Theorem 1.7** (See [1, 6]) *Let $(X,p)$ be a TVS-CMS, $x \in X$, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $X$. Set $d_p = \xi_e \circ p$. Then the following statements hold*:

(i) *$\{x_n\}_{n\in\mathbb{N}}$ converges to $x$ in TVS-CMS $(X,p)$ if and only if $d_p(x_n, x) \to 0$ as $n \to \infty$,*

(ii) *$\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in TVS-CMS $(X,p)$ if and only if $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X, d_p)$,*

(iii) *$(X,p)$ is a complete TVS-CMS if and only if $(X, d_p)$ is a complete metric space.*

**Remark 1.8** From Theorem 1.7, we conclude that for every complete TVS-cone metric space, there exists a correspondent isomorphic complete usual metric space. Notice that the cone should have a nonempty interior.

**Proposition 1.9** (See [1]) *Let $(X,p)$ be a complete TVS-CMS and $0 \leq \gamma < 1$. If $T : X \to X$ satisfies the contractive condition*

$$p(Tx, Ty) \precsim \gamma p(x,y) \quad \textit{for all } x, y \in X,$$

*then $T$ has a unique fixed point in $X$. Moreover, for each $x \in X$, the iterative sequence $\{T^n x\}_{n=1}^{\infty}$ converges to the unique fixed point of $T$.*

In particular, if $K$ is a cone of a real Banach space $V$, then it is called *normal* if there is a number $\rho \geq 1$ such that for all $x, y \in V : \theta \precsim x \precsim y \Longrightarrow \|x\| \leq \rho\|y\|$. The least positive integer $\rho$, satisfying this inequality, is called the normal constant of $K$.

### 1.2 *b*-Metric spaces

The notion of a $b$-metric space was considered by Bakhtin [7] and Czerwik [8] as a generalization of metric space.

**Definition 1.10** (See [7–10]) Let $X$ be a nonempty set, and let $s \geq 1$ be a given real number. A function $d : X \times X \to [0,\infty)$ is called a $b$-metric if the following conditions are satisfied:
(1) $d(x,y) = 0$ if and only if $x = y$;
(2) $d(x,y) = d(y,x)$;
(3) $d(x,z) \leq s[d(x,y) + d(y,z)]$ for all $x,y,z \in X$.
A pair $(X,d)$ is called a $b$-metric space.

In this paper, we first introduce the concept of TVS-cone $b$-metric space which generalize the concept of $b$-metric space and cone $b$-metric space.

**Definition 1.11** Let $X$ be a non-empty set and $s \geq 1$ be a given real number. A vector-valued function $p : X \times X \to Y$ is said to be TVS-cone $b$-metric if the following conditions are satisfied:
(BM1) $\theta \precsim p(x,y)$ for all $x,y \in X$ and $p(x,y) = \theta$ if and only if $x = y$;
(BM2) $p(x,y) = p(y,x)$;
(BM3) $p(x,z) \precsim s[p(x,y) + p(y,z)]$ for all $x,y,z \in X$.
The pair $(X,p)$ is called a TVS-cone $b$-metric space.

If we replace $Y$ by a real Banach space in Definition 1.11, we get the cone $b$-metric space in the sense of [11–13]. It is evident that Definition 1.10 coincides with Definition 1.11 if we replace $Y$ by a set of non-negative real numbers.

## 2 Main results

The following theorem is one of main results in this paper. Although it is the mimic of the proof of Lemma 1.4, we give the proof for the sake of completeness and for the readers' convenience.

**Theorem 2.1** *Let $(X,p)$ be a TVS-cone b-metric space. Then, $d_p : X \times X \to [0,\infty)$ defined by $d_p = \xi_e \circ p$ is a b-metric.*

*Proof* Clearly, $d_p(x,y) = d_p(y,x)$ for all $x,y \in X$. By Lemma 1.1, we have $d_p(x,y) \geq 0$ for all $x,y \in X$. If $x = y$, then, by (BM1), $d_p(x,y) = \xi_e(\theta) = 0$. Conversely, if $d_p(x,y) = 0$, then by Lemma 1.1 $p(x,y) \in K \cap (-K) = \{\theta\}$, which implies that $x = y$. Since $s \geq 1$, by applying (v), (vi) and (vii) of Lemma 1.1, we have

$$\xi_e\big(p(x,z)\big) \leq s\big(\xi_e\big(p(x,y)\big) + \xi_e\big(p(y,z)\big)\big)$$

or

$$d_p(x,z) \leq s\big[d_p(x,y) + d_p(y,z)\big] \quad \text{for all } x,y,z \in X.$$

So we prove that $d_p$ is a $b$-metric. □

The following consequence of Theorem 2.1 is evident.

**Corollary 2.2** *Let $(X,p)$ be a cone b-metric space. Then, $d_p : X \times X \to [0,\infty)$ defined by $d_p = \xi_e \circ p$ is a b-metric.*

Following the idea of Du [1], we can define the following.

**Definition 2.3** Let $(X,p)$ be a TVS-cone $b$-metric space, let $x \in X$, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $X$.

(i) $\{x_n\}_{n\in\mathbb{N}}$ TVS-cone converges to $x \in X$ whenever for every $\theta \ll c \in Y$, there is a natural number $M$ such that $p(x_n,x) \ll c$ for all $n \geq M$ and denoted by cone-$\lim_{n\to\infty} x_n = x$ (or $x_n \xrightarrow{\text{cone}} x$ as $n \to \infty$),

(ii) $\{x_n\}_{n\in\mathbb{N}}$ TVS-cone Cauchy sequence in $(X,p)$ whenever for every $\theta \ll c \in Y$, there is a natural number $M$ such that $p(x_n,x_m) \ll c$ for all $n,m \geq M$,

(iii) $(X,p)$ is TVS-cone complete if every sequence TVS-cone Cauchy sequence in $X$ is a TVS-cone convergent.

Using a similar argument as in the proof of [2, Theorem 2.2], we can prove the following result.

**Theorem 2.4** *Let $(X,p)$ be a TVS-cone b-metric space, let $x \in X$, let and $\{x_n\}_{n=1}^{\infty}$ be a sequence in $X$. Set $d_p = \xi_e \circ p$. Then the following statements hold:*

(i) *$\{x_n\}_{n\in\mathbb{N}}$ converges to $x$ in TVS-cone b-metric space $(X,p)$ if and only if $d_p(x_n,x) \to 0$ as $n \to \infty$,*

(ii) *$\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in TVS-cone b-metric space $(X,p)$ if and only if $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $(X,d_p)$,*

(iii) *$(X,p)$ is a complete TVS-cone b-metric space if and only if $(X,d_p)$ is a complete b-metric space.*

**Remark 2.5** From Theorem 2.4, we conclude that for every complete TVS-cone $b$-metric space there exists a correspondent isomorphic complete usual (associated) $b$-metric space.

**Theorem 2.6** *Let $(X,p)$ be a complete TVS-cone b-metric space with $s \geq 1$ and $0 \leq \gamma < 1$. If $T : X \to X$ satisfies the contractive condition*

$$p(Tx,Ty) \precsim \gamma p(x,y) \quad \text{for all } x,y \in X,$$

*then $T$ has a unique fixed point in $X$. Moreover, for each $x \in X$, the iterative sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to the unique fixed point of $T$.*

*Proof* Set $d_p = \xi_e \circ p$. Due to Theorem 2.4, we conclude that $(X,d_p)$ is a complete $b$-metric space. On the other hand, from Lemma 1.1, we derive that

$$p(Tx,Ty) \precsim \gamma p(x,y) \quad \Longrightarrow \quad d_p(Tx,Ty) \leq \gamma d_p(x,y) \quad \text{for all } x,y \in X.$$

We conclude the results from the characterization of the Banach contraction mapping principle in the context of $b$-metric space (see, *e.g.*, [14, Theorem 2]). The proof is completed. □

**Theorem 2.7** *Let $(X,p)$ be a complete TVS-cone b-metric space with $s \geq 1$, and let $T: X \to X$ satisfy the contractive condition*

$$p(Tx, Ty) \precsim \lambda_1 p(x, Tx) + \lambda_2 p(y, Ty) + \lambda_3 p(x, Ty) + \lambda_4 p(y, Tx) \quad \textit{for all } x, y \in X,$$

*where $\lambda_i \in [0,1)$, $i = 1,2,3,4$, and $\lambda_1 + \lambda_2 + s(\lambda_3 + \lambda_4) < \min\{1, \frac{2}{s}\}$. Then $T$ has a unique fixed point in $X$. Moreover, for each $x \in X$, the iterative sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to the unique fixed point of $T$.*

The idea of the proof is the same with the proof of Theorem 2.6. For the sake of completeness, we put it here.

*Proof* Set $d_p = \xi_e \circ p$. Due to Theorem 2.4, we conclude that $(X, d_p)$ is a complete $b$-metric space. On the other hand, from Lemma 1.1, we derive that

$$p(Tx, Ty) \precsim \lambda_1 p(x, Tx) + \lambda_2 p(y, Ty) + \lambda_3 p(x, Ty) + \lambda_4 p(y, Tx)$$

implies that

$$d_p(Tx, Ty) \precsim \lambda_1 d_p(x, Tx) + \lambda_2 d_p(y, Ty) + \lambda_3 d_p(x, Ty) + \lambda_4 d(y, Tx) \quad \text{for all } x, y \in X.$$

We conclude the result from [14, Corollary 4.1] with taking $S = T$. The proof is completed. □

## 3 Conclusion

In this paper, we just show that two fixed point theorems in the setting of cone $b$-metric spaces can be easily derived from the existing result in the context of $b$-metric space. Hence, the notion of 'cone $b$-metric' is not a real generalization of neither $b$-metric nor metric. By using the techniques above, one can easily prove the equivalence of other fixed point results (published, unpublished/that will be published) in the context of cone $b$-metric space. Regarding the published papers on the equivalence of cone metric and usual (associated) metric in the literature, it is natural to conclude that some other techniques can also be developed for the equivalence of the mentioned notions.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
Both authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

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