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Approximation of the common minimum-norm fixed point of a finite family of asymptotically nonexpansive mappings

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Abstract
We introduce an iterative process which converges strongly to the common minimum-norm fixed point of a finite family of asymptotically nonexpansive mappings. As a consequence, convergence result to a common minimum-norm fixed point of a finite family of nonexpansive mappings is proved.
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1 Introduction

Let K and D be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. The split feasibility problem is formulated as finding a point $\bar{x}$ satisfying

$$\bar{x} \in K \quad \text{and} \quad A\bar{x} \in D, \tag{1.1}$$

where A is bounded linear operator from H_1 into H_2. A split feasibility problem in finite dimensional Hilbert spaces was first studied by Censor and Elfving [1] for modeling inverse problems which arise in medical image reconstruction, image restoration and radiation therapy treatment planing (see, *e.g.*, [1–3]).

It is clear that $\bar{x}$ is a solution to the split feasibility problem (1.1) if and only if $\bar{x} \in K$ and $A\bar{x} - P_D A\bar{x} = 0$, where P_D is the metric projection from H_2 onto D. Set

$$\min_{x\in K} \varphi(x) := \min_{x\in K} \frac{1}{2}\|Ax - P_D Ax\|^2. \tag{1.2}$$

Then $\bar{x}$ is a solution of (1.1) if and only if $\bar{x}$ solves the minimization problem (1.2) with the minimum equal to zero. Now, assume that (1.1) is consistent (*i.e.*, (1.1) has a solution), and let Ω denote the (closed convex) solution set of (1.1) (or equivalently, solution of (1.2)). Then, in this case, Ω has a unique element $\bar{x}$ if and only if it is a solution of the following variational inequality:

$$\bar{x} \in K, \quad \langle \nabla\varphi(\bar{x}), x - \bar{x}\rangle = \langle A^*(I - P_D)A\bar{x}, x - \bar{x}\rangle \geq 0, \quad x \in K, \tag{1.3}$$

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where A^* is the adjoint of A. In addition, inequality (1.3) can be rewritten as

$$\bar{x} \in K, \quad \langle \bar{x} - \gamma A^*(I - P_D)A\bar{x} - \bar{x}, x - \bar{x} \rangle \le 0, \quad x \in K, \tag{1.4}$$

where $\gamma > 0$ is any positive scalar. Using the nature of projection, (1.4) is equivalent to the fixed point equation

$$\bar{x} = P_K(\bar{x} - \gamma A^*(I - P_D)A\bar{x}). \tag{1.5}$$

Recall that a point $\bar{x} \in K$ is said to be a fixed point of T if $T(\bar{x}) = \bar{x}$. We denote the set of fixed points of T by $F(T)$, *i.e.*, $F(T) := \{\bar{x} \in K : T\bar{x} = \bar{x}\}$. Therefore, finding a solution to the split feasibility problem (1.1) is equivalent to finding the minimum-norm fixed point of the mapping $x \mapsto P_K(x - \gamma A^*(I - P_D)Ax)$.

Motivated by the above split feasibility problem, we study the general case of finding the minimum-norm fixed point of an *asymptotically nonexpansive* self-mapping T on K; that is, we find a minimum-norm fixed point of T which satisfies

$$\bar{x} \in F(T) \quad \text{such that } \|\bar{x}\| = \min\{\|x\| : x \in F(T)\}. \tag{1.6}$$

Let K be a nonempty subset of a real Hilbert space H; a mapping $T : K \to K$ is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in K$ and it is called *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$, as $n \to \infty$, such that

$$\|T^n x - T^n y\| \le k_n\|x - y\|, \quad \forall x, y \in K, \text{ and } n \ge 1. \tag{1.7}$$

The class of asymptotically nonexpansive mappings was introduced as a generalization of the class of nonexpansive mappings by Goebel and Kirk [4] who proved that if K is a nonempty closed convex bounded subset of a real uniformly convex Banach spaces which includes Hilbert spaces as a special case and T is an asymptotically nonexpansive self-mapping of K, then T has a fixed point.

Let $T : K \to K$ be a nonexpansive mapping. For a given $u \in K$ and a given $t \in (0, 1)$, define a contraction $T_t : K \to K$ by

$$T_t x = (1 - t)u + tTx, \quad x \in K.$$

By the Banach contraction principle, it yields a fixed point $z_t \in K$ of T_t, *i.e.*, z_t is the unique solution of the equation

$$z_t = (1 - t)u + tTz_t. \tag{1.8}$$

In [5], Browder proved that, as $t \to 1$, z_t converges strongly to the nearest point projection of u onto $F(T)$.

In [6], Halpern introduced an explicit iteration scheme $\{x_n\}$ (which was referred to as *Halpern iteration*) defined by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n. \tag{1.9}$$

He proved that, as $n \to \infty$, $\{x_n\}$ converges strongly to the fixed point of a nonexpansive self-mapping T that is closest to u provided that $\{\alpha_n\}$ satisfies (i) $\lim_{n\to\infty} \alpha_n = 0$, (ii) $\sum \alpha_n = \infty$ and (iii) $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 0$. Wittmann [7] also showed that the sequence $\{x_n\}$ defined by

$$x_0 = u \in K, \qquad x_{n+1} = a_{n+1}u + (1 - a_{n+1})Tx_n, \quad n \geq 1, \tag{1.10}$$

converges strongly to the element of $F(T)$ which is nearest to u under certain conditions on $\{a_n\} \subset (0,1)$.

Moreover, using the idea of Browder [5], Shioji and Takahashi [8] studied the following scheme for an approximating fixed point of an asymptotically nonexpansive mapping. Let T be an asymptotically nonexpansive mapping from K into itself with $F(T)$ nonempty. Then they proved that the sequence generated by

$$x_0 = u \in K, \qquad x_n = a_n u + (1 - a_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j x_n, \quad n \geq 1, \tag{1.11}$$

where $\{a_n\} \subset (0,1)$ satisfies certain conditions, converges strongly to the element of $F(T)$ which is nearest to u. Shioji and Takahashi [8] also studied an explicit scheme for asymptotically nonexpansive mappings. They showed that the sequence $\{x_n\}$ defined by

$$x_0 = u \in K, \qquad x_{n+1} = b_n u + (1 - b_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j x_n, \quad n \geq 1, \tag{1.12}$$

where $\{b_n\} \subset (0,1)$ satisfies certain conditions, converges strongly to the element of $F(T)$ which is nearest to u.

Several authors have extended the above results either to a more general Banach spaces or to a more general class of mappings (see, *e.g.*, [9–18]).

It is worth mentioning that the methods studied above are used to approximate the fixed point of T which is closest to the point $u \in K$. These methods can be used to find the minimum-norm fixed point x^* of T if $0 \in K$. If, however, $0 \notin K$, any of the methods above fails to provide the minimum-norm fixed point of T.

In connection with the iterative approximation of the minimum-norm fixed point of a nonexpansive self-mapping T, Yang *et al.* [19] introduced an explicit scheme given by

$$x_{n+1} = \beta Tx_n + (1 - \beta)P_K\big[(1 - \alpha_n)x_n\big], \quad n \geq 1.$$

They proved that under appropriate conditions on $\{\alpha_n\}$ and β, the sequence $\{x_n\}$ converges strongly to the minimum-norm fixed point of T in real Hilbert spaces.

More recently, Yao and Xu [20] have also shown that the explicit scheme $x_{n+1} = P_K((1 - \alpha_n)Tx_n)$, $n \geq 1$, converges strongly to the minimum-norm fixed point of a nonexpansive self-mapping T provided that $\{\alpha_n\}$ satisfies certain conditions.

A natural question arises whether we can extend the results of Yang et al. [19] *and Yao and Xu* [20] *to a class of mappings more general than nonexpansive mappings or not.*

Let K be a closed convex subset of a real Hilbert space H and let $T_i : K \to K$, $i = 1, 2, \ldots, N$ be a finite family of asymptotically nonexpansive mappings.

It is our purpose in this paper to introduce an explicit iteration process which converges strongly to the common minimum-norm fixed point of $\{T_i : i = 1, 2, \dots, N\}$. Our theorems improve several results in this direction.

2 Preliminaries

In what follows, we shall make use of the following lemmas.

Lemma 2.1 *Let H be a real Hilbert space. Then, for any given $x, y \in H$, the following inequality holds*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle.$$

Lemma 2.2 [21] *Let E be a real Hilbert space and $B_R(0)$ be a closed ball of H. Then, for any given subset $\{x_0, x_1, \dots, x_N\} \subset B_r(0)$ and for any positive numbers $\alpha_0, \alpha_1, \dots, \alpha_N$ with $\sum_{i=0}^{N} \alpha_i = 1$, we have that*

$$\|\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_N x_N\|^2 = \sum_{i=0}^{N} \alpha_i \|x_i\|^2 - \sum_{0 \leq i,j \leq N} \alpha_i \alpha_j \|x_i - x_j\|^2.$$

Lemma 2.3 [22] *Let K be a closed and convex subset of a real Hilbert space H. Let $x \in H$. Then $x_0 = P_K x$ if and only if*

$$\langle z - x_0, x - x_0\rangle \leq 0, \quad \forall z \in K.$$

Lemma 2.4 [23] *Let H be a real Hilbert space, K be a closed convex subset of H and $T : K \to K$ be an asymptotically nonexpansive mapping, then $(I - T)$ is demiclosed at zero, i.e., if $\{x_n\}$ is a sequence in K such that $x_n \rightharpoonup x$ and $Tx_n - x_n \to 0$, as $n \to \infty$, then $x = T(x)$.*

Lemma 2.5 [24] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation*:

$$a_{n+1} \leq (1 - \alpha_n) a_n + \alpha_n \delta_n, \quad n \geq n_0,$$

where $\{\alpha_n\} \subset (0, 1)$, and $\{\delta_n\} \subset R$ satisfying the following conditions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\limsup_{n\to\infty} \delta_n \leq 0$, *as $n \to \infty$. Then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6 [25] *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$*:

$$a_{m_k} \leq a_{m_k+1} \quad \text{and} \quad a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

Proposition 2.7 *Let H be a real Hilbert space, let K be a closed convex subset of H, and let T be an asymptotically nonexpansive mapping from K into itself. Then $F(T)$ is closed and convex.*

Proof Clearly, the continuity of T implies that $F(T)$ is closed. Now, we show that $F(T)$ is convex. For $x, y \in F(T)$ and $t \in (0,1)$, put $z = tx + (1-t)y$. Now, we show that $z = T(z)$. In fact, we have

$$\begin{aligned}\|z - T^n z\|^2 &= \|z\|^2 - 2\langle z, T^n z\rangle + \|T^n z\|^2 \\ &= \|z\|^2 - 2\langle tx + (1-t)y, T^n z\rangle + \|T^n z\|^2 \\ &= \|z\|^2 - 2t\langle x, T^n z\rangle - 2(1-t)\langle y, T^n z\rangle + \|T^n z\|^2 \\ &= \|z\|^2 + t\|x - T^n z\|^2 + (1-t)\|y - T^n z\|^2 - t\|x\|^2 - (1-t)\|y\|^2 \\ &\le \|z\|^2 + tk_n^2\|x - z\|^2 + (1-t)k_n^2\|y - z\|^2 - t\|x\|^2 - (1-t)\|y\|^2 \\ &\le \|z\|^2 + tk_n^2\langle x - z, x - z\rangle + (1-t)k_n^2\langle y - z, y - z\rangle \\ &\quad - t\|x\|^2 - (1-t)\|y\|^2 \\ &\le (k_n^2 - 1)\big[t\|x\|^2 + (1-t)\|y\|^2 + \|z\|^2\big], \end{aligned} \tag{2.1}$$

and hence, since $k_n \to 1$ as $n \to \infty$, we get that $\lim_{n\to\infty} \|z - T^n z\|^2 = 0$, which implies that $\lim_{n\to\infty} T^n z = z$. Now, by the continuity of T, we obtain that $z = \lim_{n\to\infty} T^n z = \lim_{n\to\infty} T(T^{n-1}z) = T(\lim_{n\to\infty} T^{n-1}z) = T(z)$. Hence, $z \in F(T)$ and that $F(T)$ is convex. □

3 Main result

We now state and proof our main theorem.

Theorem 3.1 *Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let $T_i : K \to K$ be asymptotically nonexpansive mappings with sequences $\{k_{n,i}\}$ for each $i = 1, 2, \dots, N$. Assume that $F := \bigcap_{i=1}^N F(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in K, \quad \text{chosen arbitrarily}, \\ y_n = P_K[(1-\alpha_n)x_n], \\ x_{n+1} = \beta_{n,0}x_n + \sum_{i=1}^N \beta_{n,i} T_i^n y_n, \quad n \ge 1, \end{cases} \tag{3.1}$$

where $\alpha_n \in (0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \frac{(k_{n,i}^2 - 1)}{\alpha_n} = 0$, for each $i \in \{1, 2, \dots, N\}$ and $\sum_{n=1}^\infty \alpha_n = \infty$, $\{\beta_{n,i}\} \subset [a, b] \subset (0,1)$ for $i = 1, 2, \dots, N$, satisfying $\beta_{n,0} + \beta_{n,1} + \cdots + \beta_{n,N} = 1$ for each $n \ge 1$. Then $\{x_n\}$ converges strongly to the common minimum-norm point of F.

Proof Let $x^* \in P_F 0$. Let $k_n := \max\{k_{n,i} : i = 1, 2, \dots, N\}$. Then from (3.1) and asymptotical nonexpansiveness of T_i, for each $i \in \{1, 2, \dots, N\}$, we have that

$$\begin{aligned}\|y_n - x^*\| &= \|P_C[(1-\alpha_n)x_n] - P_K x^*\| \\ &\le \|(1-\alpha_n)x_n - x^*\| \\ &= \|\alpha_n(0 - x^*) + (1-\alpha_n)(x_n - x^*)\| \\ &\le \alpha_n\|x^*\| + (1-\alpha_n)\|x_n - x^*\|, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\|x_{n+1}-x^*\| &= \Big\|\beta_{n,0}x_n+\sum_{i=1}^N\beta_{n,i}T_i^ny_n-x^*\Big\|\\ &\le \beta_{n,0}\|x_n-x^*\|+\sum_{i=1}^N\beta_{n,i}\|T_i^ny_n-x^*\|\\ &\le \beta_{n,0}\|x_n-x^*\|+(1-\beta_{n,0})k_n\|y_n-x^*\|\\ &\le \beta_{n,0}\|x_n-x^*\|+(1-\beta_{n,0})k_n\big[\alpha_n\|x^*\|+(1-\alpha_n)\|x_n-x^*\|\big]\\ &\le \big[\beta_{n,0}+(1-\beta_{n,0})k_n(1-\alpha_n)\big]\|x_n-x^*\|+\big[(1-\beta_{n,0})k_n\alpha_n\big]\|x^*\|\\ &\le \delta_n\|x^*\|+\big[1-(1-\epsilon)\delta_n\big]\|x_n-x^*\|, \end{aligned}\tag{3.3}$$

where $\delta_n=(1-\beta_{n,0})k_n\alpha_n$, since there exists $N_0>0$ such that $\frac{(k_n-1)}{\alpha_n}\le\epsilon k_n$ for all $n\ge N_0$ and for some $\epsilon>0$ satisfying $(1-\epsilon)\delta_n\le 1$. Thus, by induction,

$$\|x_{n+1}-x^*\|\le\max\big\{\|x_0-x^*\|,(1-\epsilon)^{-1}\|x^*\|\big\},\quad \forall n\ge N_0,$$

which implies that $\{x_n\}$ and hence $\{y_n\}$ is bounded. Moreover, from (3.2) and Lemma 2.1, we obtain that

$$\begin{aligned}\|y_n-x^*\|^2 &= \|P_K[(1-\alpha_n)x_n]-P_Kx^*\|^2\\ &\le \|\alpha_n(0-x^*)+(1-\alpha_n)(x_n-x^*)\|^2\\ &\le (1-\alpha_n)\|x_n-x^*\|^2-2\alpha_n\langle x^*,y_n-x^*\rangle. \end{aligned}\tag{3.4}$$

Furthermore, from (3.1), Lemma 2.2 and asymptotical nonexpansiveness of T_i, for each $i=1,2,\dots,N$, we have that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \Big\|\beta_{n,0}x_n+\sum_{i=1}^N\beta_{n,i}T_i^ny_n-x^*\Big\|^2\\ &\le \beta_{n,0}\|x_n-x^*\|^2+\sum_{i=1}^N\beta_{n,i}\|T_i^ny_n-x^*\|^2\\ &\quad -\sum_{i=1}^N\beta_{n,0}\beta_{n,i}\|x_n-T_i^ny_n\|^2\\ &\le \beta_{n,0}\|x_n-x^*\|^2+(1-\beta_{n,0})k_n^2\|y_n-x^*\|^2\\ &\quad -\sum_{i=1}^N\beta_{n,0}\beta_{n,i}\|x_n-T_i^ny_n\|^2, \end{aligned}$$

which implies, using (3.4), that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 \le{}& \beta_{n,0}\|x_n-x^*\|^2+(1-\beta_{n,0})k_n^2\big[(1-\alpha_n)\|x_n-x^*\|^2\\ &-2\alpha_n\langle x^*,y_n-x^*\rangle\big]-\sum_{i=1}^N\beta_{n,0}\beta_{n,i}\|x_n-T_i^ny_n\|^2 \end{aligned}$$

$$\leq (1-\theta_n)\|x_n - x^*\|^2 - 2\theta_n\langle x^*, y_n - x^*\rangle + (k_n^2 - 1)M$$

$$- \sum_{i=1}^{N} \beta_{n,0}\beta_{n,i}\|x_n - T_i^n y_n\|^2 \tag{3.5}$$

$$\leq (1-\theta_n)\|x_n - x^*\|^2 - 2\theta_n\langle x^*, y_n - x^*\rangle + (k_n^2 - 1)M \tag{3.6}$$

for some $M > 0$, where $\theta_n := \alpha_n(1-\beta_{n,0})$ for all $n \in N$.

Now, we consider the following two cases.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n - x^*\|\}$ is non-increasing for all $n \geq n_0$. In this situation, $\{\|x_n - x^*\|\}$ is convergent. Then from (3.5), we have that $\sum_{i=1}^{N} \beta_{n,0}\beta_{n,i}\|x_n - T_i^n y_n\|^2 \to 0$, which implies that

$$x_n - T_i^n y_n \to 0, \quad \text{as } n \to \infty, \tag{3.7}$$

for each $i \in \{1, 2, \ldots, N\}$. Moreover, from (3.1) and (3.7) and the fact that $\alpha_n \to 0$, we get that

$$\|x_{n+1} - x_n\| = \beta_{n,1}\|T_1^n y_n - x_n\| + \cdots + \beta_{n,N}\|T_N^n y_n - x_n\| \to 0, \tag{3.8}$$

and

$$\begin{aligned} \|y_n - x_n\| &= \|P_C[(1-\alpha_n)x_n] - P_k x_n\| \\ &\leq \|-\alpha_n x_n\| \to 0, \end{aligned} \tag{3.9}$$

as $n \to \infty$ and hence

$$\|y_{n+1} - y_n\| \leq \|y_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - y_n\| \to 0, \tag{3.10}$$

as $n \to \infty$. Furthermore, from (3.7) and (3.9), we get that

$$\|y_n - T_i^n y_n\| \leq \|y_n - x_n\| + \|x_n - T_i^n y_n\| \to 0, \quad \text{as } n \to \infty. \tag{3.11}$$

Therefore, since

$$\begin{aligned} \|y_n - T_i y_n\| &\leq \|y_n - y_{n+1}\| + \|y_{n+1} - T_i^{n+1} y_{n+1}\| + \|T_i^{n+1} y_{n+1} - T_i^{n+1} y_n\| \\ &\quad + \|T_i^{n+1} y_n - T_i y_n\|, \\ &\leq \|y_n - y_{n+1}\| + \|y_{n+1} - T_i^{n+1} y_{n+1}\| + k_{n+1}\|y_{n+1} - y_n\| \\ &\quad + \|T_i(T_i^n y_n) - T_i y_n\|, \end{aligned} \tag{3.12}$$

we have from (3.10), (3.11), (3.12) and uniform continuity of T_i that

$$\|y_n - T_i y_n\| \to 0, \quad \text{as } n \to \infty, \text{ for each } i = 1, 2, \ldots, N. \tag{3.13}$$

Let $\{y_{n_k}\}$ be a subsequence of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle x^*, y_n - x^*\rangle = \lim_{k\to\infty}\langle x^*, y_{n_k} - x^*\rangle,$$

and $y_{n_k} \rightharpoonup z$. Then from (3.9), we have that $x_{n_k} \rightharpoonup z$. Therefore, by Lemma 2.3, we obtain that

$$\limsup_{n\to\infty}\langle x^*, y_n - x^*\rangle = \lim_{k\to\infty}\langle x^*, y_{n_k} - x^*\rangle = \langle x^*, z - x^*\rangle \geq 0. \tag{3.14}$$

Now, we show that $x_{n+1} \to x^*$, as $n \to \infty$. But from (3.13) and Lemma 2.4, we get that $z \in F(T_i)$ for each $i \in \{1, 2, \dots, N\}$ and hence $z \in \bigcap_{i=1}^N F(T_i)$. Then from (3.6), we get that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq (1-\theta_n)\|x_n - x^*\|^2 - 2\theta_n\langle x^*, y_n - x^*\rangle \\ &\quad + (k_n^2 - 1)M \end{aligned} \tag{3.15}$$

for some $M > 0$. But note that θ_n satisfies $\lim_n \theta_n = 0$ and $\sum_{n=1}^\infty \theta_n = \infty$. Thus, it follows from (3.15) and Lemma 2.5 that $\|x_n - x^*\| \to 0$, as $n \to \infty$. Consequently, $x_n \to x^*$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|x_{n_i} - x^*\| < \|x_{n_i+1} - x^*\|$$

for all $i \in \mathbb{N}$. Then by Lemma 2.6, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, $\|x_{m_k} - x^*\| \leq \|x_{m_k+1} - x^*\|$ and $\|x_k - x^*\| \leq \|x_{m_k+1} - x^*\|$ for all $k \in \mathbb{N}$. Then from (3.5) and the fact that $\theta_n \to 0$, we have

$$\begin{aligned} &\sum_{i=1}^N \beta_{m_k,0}\beta_{m_k,i}\|x_{m_k} - T_i^{m_k} y_{m_k}\|^2 \\ &\quad \leq \|x_{m_k} - x^*\|^2 - \|x_{m_k+1} - x^*\|^2 + \theta_{m_k}\|x_{m_k} - x^*\|^2 \\ &\qquad - 2\theta_{m_k}\langle x^*, y_{m_k} - x^*\rangle + (k_{m_k} - 1)M \to 0, \quad \text{as } k \to \infty. \end{aligned}$$

This implies that $x_{m_k} - T_i^{m_k} y_{m_k} \to 0$, as $k \to \infty$. Thus, following the method of Case 1, we obtain that $x_{m_k} - y_{m_k} \to 0$ and $y_{m_k} - T_i y_{m_k} \to 0$ as $k \to \infty$ for each $i = 1, 2, \dots, N$ and hence there exists $z' \in F$ such that

$$\limsup_{n\to\infty}\langle x^*, y_n - x^*\rangle = \lim_{k\to\infty}\langle x^*, y_{n_k} - x^*\rangle = \langle x^*, z' - x^*\rangle \geq 0. \tag{3.16}$$

Then from (3.6), we get that

$$\begin{aligned} \|x_{m_k+1} - x^*\|^2 &\leq (1-\theta_{m_k})\|x_{m_k} - x^*\|^2 - 2\theta_{m_k}\langle x^*, y_{m_k} - x^*\rangle \\ &\quad + (k_{m_k}^2 - 1)M. \end{aligned} \tag{3.17}$$

Since $\|x_{m_k} - x^*\| \leq \|x_{m_k+1} - x^*\|$, (3.17) implies that

$$\begin{aligned} \theta_{m_k}\|x_{m_k} - x^*\|^2 &\leq \|x_{m_k} - x^*\|^2 - \|x_{m_k+1} - x^*\|^2 - 2\theta_{m_k}\langle x^*, y_{m_k} - x^*\rangle \\ &\quad + (k_{m_k}^2 - 1)M \\ &\leq -2\theta_{m_k}\langle x^*, y_{m_k} - x^*\rangle + (k_{m_k}^2 - 1)M. \end{aligned}$$

In particular, since $\theta_{m_k} > 0$, we have that

$$\|x_{m_k} - x^*\|^2 \leq -2\langle x^*, y_{m_k} - x^*\rangle + \frac{(k_{m_k}^2 - 1)}{\theta_{m_k}} M.$$

Thus, from (3.16) and the fact that $\frac{(k_{m_k}^2 - 1)}{\theta_{m_k}} \to 0$, we obtain that $\|x_{m_k} - x^*\| \to 0$ as $k \to \infty$. This together with (3.17) gives $\|x_{m_k+1} - x^*\| \to 0$ as $k \to \infty$. But $\|x_k - x^*\| \leq \|x_{m_k+1} - x^*\|$ for all $k \in \mathbb{N}$, thus we obtain that $x_k \to x^*$. Therefore, from the above two cases, we can conclude that $\{x_n\}$ converges strongly to a point x^* of F which is the common minimum-norm fixed point of the family $\{T_i, i = 1, 2, \dots, N\}$ and the proof is complete. □

If in Theorem 3.1 we assume that $N = 1$, then we get the following corollary.

Corollary 3.2 *Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : K \to K$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\}$. Assume that $F(T)$ is nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in C, \quad \textit{chosen arbitrarily}, \\ y_n = P_K[(1 - \alpha_n)x_n], \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)T^n y_n, \quad n \geq 1, \end{cases} \tag{3.18}$$

where $\alpha_n \in (0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \frac{(k_n^2 - 1)}{\alpha_n} = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_n\} \subset [a, b] \subset (0, 1)$ for each $n \geq 1$. Then $\{x_n\}$ converges strongly to the minimum-norm fixed point of T.

If in Theorem 3.1 we assume that each T_i is nonexpansive for $i = 1, 2, \dots, N$, then the method of proof of Theorem 3.1 provides the following corollary.

Corollary 3.3 *Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let $T_i : K \to K$ be nonexpansive mappings with $F := \bigcap_{i=1}^{N} F(T_i)$ nonempty. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 \in K, \quad \textit{chosen arbitrarily}, \\ y_n = P_K[(1 - \alpha_n)x_n], \\ x_{n+1} = \beta_{n,0} x_n + \sum_{i=1}^{N} \beta_{n,i} T_i y_n, \quad n \geq 1, \end{cases} \tag{3.19}$$

where $\alpha_n \in (0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_{n,i}\} \subset [a, b] \subset (0, 1)$, for $i = 1, 2, \dots, N$, satisfying $\beta_{n,0} + \beta_{n,1} + \cdots + \beta_{n,N} = 1$ for each $n \geq 1$. Then $\{x_n\}$ converges strongly to the common minimum-norm point of F.

If in Corollary 3.3 we assume that $N = 1$, then we have the following corollary.

Corollary 3.4 *Let K be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : K \to K$ be a nonexpansive mapping with $F(T)$ nonempty. Let $\{x_n\}$ be a sequence* *generated by*

$$\begin{cases} x_1 \in K, & \textit{chosen arbitrarily}, \\ y_n = P_K[(1-\alpha_n)x_n], \\ x_{n+1} = \beta_n x_n + (1-\beta_n)Ty_n, & n \geq 1, \end{cases} \tag{3.20}$$

where $\alpha_n \in (0,1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_n\} \subset [a,b] \subset (0,1)$ *for each* $n \geq 1$. *Then* $\{x_n\}$ *converges strongly to the minimum-norm point of* $F(T)$.

4 Applications

In this section, we study the problem of finding a minimizer of a continuously Fréchet-differentiable convex functional which has the minimum norm in Hilbert spaces.

Let K be a closed convex subset of a real Hilbert space H. Consider the minimization problem given by

$$\min_{x\in K} \varphi(x), \tag{4.1}$$

and $\varphi : K \to \mathbb{R}$ be a continuously Fréchet-differentiable convex functional. Let Ω, the solution set of (4.1), be nonempty; that is,

$$\Omega := \Big\{ z \in K : \varphi(z) = \min_{x\in K} \varphi(x) \Big\} \neq \emptyset. \tag{4.2}$$

It is known that a point $z \in K$ is a solution of (4.1) if and only if the following optimality condition holds:

$$z \in K, \quad \big\langle \nabla\varphi(z), x - z \big\rangle \geq 0, \quad x \in K, \tag{4.3}$$

where $\nabla\varphi(x)$ is the gradient of φ at $x \in K$. It is also known that the optimality condition (4.3) is equivalent to the following fixed point problem:

$$z = T_\gamma(z), \quad \text{where } T_\gamma := P_K(I - \gamma \nabla \varphi), \tag{4.4}$$

for all $\gamma > 0$.

Now, we have the following corollary deduced from Corollary 3.2.

Corollary 4.1 *Let* K *be a closed convex subset of a real Hilbert space* H. *Let* φ *be a continuously Fréchet-differentiable convex functional on* K *such that* $T_\gamma := P_K(I - \gamma \nabla \varphi)$ *is asymptotically nonexpansive with a sequence* $\{k_n\}$ *for some* $\gamma > 0$. *Assume that the solution of the minimization problem* (4.1) *is nonempty. Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} x_1 \in K, & \textit{chosen arbitrarily}, \\ y_n = P_K[(1-\alpha_n)x_n], \\ x_{n+1} = \beta_n x_n + (1-\beta_n)[P_K(I - \gamma \nabla \varphi)]^n y_n, & n \geq 1, \end{cases} \tag{4.5}$$

where $\alpha_n \in (0,1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \frac{(k_n^2-1)}{\alpha_n} = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_n\} \subset [a,b] \subset (0,1)$ *for each* $n \geq 1$. *Then* $\{x_n\}$ *converges strongly to the minimum-norm solution of the minimization problem* (4.1).

Remark 4.2 Our results extend and unify most of the results that have been proved for this important class of nonlinear mappings. In particular, Theorem 3.1 improves Theorem 3.2 of Yang *et al.* [19] and of Yao and Xu [20] to a more general class of a finite family of asymptotically nonexpansive mappings.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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References
1. Censor, Y, Elfving, T: A multiprojection algorithm using Bregman projections in a product space. Numer. Algorithms **8**, 221-239 (1994)
2. Byrne, C: Iterative oblique projection onto convex subsets and the split feasibility problem. Inverse Probl. **18**, 441-453 (2002)
3. Censor, Y, Bortfeld, T, Martin, B, Trofimov, A: A unified approach for inversion problem in intensity-modulated radiation therapy. Phys. Med. Biol. **51**, 2353-2365 (2006)
4. Goebel, K, Kirk, WA: A fixed point theorem for asymptotically nonexpansive mappings. Proc. Am. Math. Soc. **35**, 171-174 (1972)
5. Browder, FE: Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces. Arch. Ration. Mech. Anal. **24**, 82-90 (1967)
6. Halpern, B: Fixed points of nonexpansive maps. Bull. Am. Math. Soc. **73**, 957-961 (1967)
7. Wittmann, R: Approximation of fixed point of nonexpansive mappings. Arch. Math. **58**, 486-491 (1992)
8. Shimizu, T, Takahashi, W: Strong convergence theorems for asymptotically nonexpansive semigroups in Hilbert spaces. Nonlinear Anal. **34**, 87-99 (1998)
9. Bruck, RE, Kuczumow, T, Reich, S: Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property. Colloq. Math. **65**, 169-179 (1993)
10. Lim, TC, Xu, HK: Fixed point theorems for asymptotically nonexpansive mappings. Nonlinear Anal. **22**, 1345-1355 (1994)
11. Morales, CH, Jung, JS: Convergence of paths for pseudo-contractive mappings in Banach spaces. Proc. Am. Math. Soc. **128**, 3411-3419 (2000)
12. Reich, S: Strong convergence theorems for resolvents of accretive operators in Banach spaces. J. Math. Anal. Appl. **75**, 287-292 (1980)
13. Schu, J: Iterative construction of fixed points of asymptotically nonexpansive mappings. J. Math. Anal. Appl. **158**, 407-413 (1991)
14. Schu, J: Weak and strong convergence of fixed points of asymptotically nonexpansive mappings. Bull. Aust. Math. Soc. **43**, 153-159 (1991)
15. Shioji, N, Takahashi, W: A strong convergence theorem for asymptotically nonexpansive mappings in Banach spaces. Arch. Math. **72**, 354-359 (1999)
16. Shioji, N, Takahashi, W: Strong convergence of averaged approximants for asymptotically nonexpansive mappings in Banach spaces. J. Approx. Theory **97**, 53-64 (1999)
17. Takahashi, W, Ueda, Y: On Reich's strong convergence theorems for resolvents of accretive operators. J. Math. Anal. Appl. **104**, 546-553 (1984)
18. Tan, KK, Xu, HK: Fixed point iteration processes for asymptotically nonexpansive mappings. Proc. Am. Math. Soc. **122**, 733-739 (1994)
19. Yang, X, Liou, Y-C, Yao, Y: Finding minimum norm fixed point of nonexpansive mappings and applications. Math. Probl. Eng. **2011**, Article ID 106450 (2011). doi:10.1155/2011/106450
20. Yao, Y, Xu, H-K: Iterative methods for finding minimum-norm fixed points of nonexpansive mappings with applications. Optimization **60**, 645-658 (2011)
21. Zegeye, H: A hybrid iteration method for equilibrium, variational inequality problems and common fixed point problems in Banach spaces. Nonlinear Anal. **72**(3-4), 2136-2146 (2010)
22. Takahashi, W: Nonlinear Functional Analysis-Fixed Point Theory and Applications. Yokohama Publishers, Yokohama (2000)
23. Chang, SS, Cho, YJ, Zhou, H: Demi-closed principle and weak convergence problems for asymptotically nonexpansive mappings. J. Korean Math. Soc. **38**, 1245-1260 (2001)
24. Ohara, JG, Pillay, P, Xu, HK: Iterative approaches to convex feasibility problems in Banach spaces. Nonlinear Anal. **64**, 2022-2042 (2006)

25. Maingé, PE: Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization. Set-Valued Anal. **16**, 899-912 (2008)

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