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Common fixed point theorems for nonlinear contractive mappings in fuzzy metric spaces

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Abstract
In this paper, we prove several common fixed point theorems for nonlinear mappings with a function ϕ in fuzzy metric spaces. In these fixed point theorems, very simple conditions are imposed on the function ϕ. Our results improve some recent ones in the literature. Finally, an example is presented to illustrate the main result of this paper.
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1 Introduction

The concept of fuzzy metric spaces was defined in different ways [1–3]. Grabiec [4] presented a fuzzy version of the Banach contraction principle in a fuzzy metric space in Kramosi and Michalek's sense. Fang [5] proved some fixed point theorems in fuzzy metric spaces, which improved, generalized, unified and extended some main results of Edelstein [6], Istratescu [7], Sehgal and Bharucha-Reid [8].

In order to obtain a Hausdorff topology, George and Veeramani [9, 10] modified the concept of fuzzy metric space due to Kramosil and Michalek [11]. Many fixed point theorems in complete fuzzy metric spaces in the sense of George and Veeramani (GV) [9, 10] have been obtained. For example, Singh and Chauhan [12] proved some common fixed point theorems for four mappings in GV fuzzy metric spaces. Gregori and Sapena [13] proved that each fuzzy contractive mapping has a unique fixed point in a complete GV fuzzy metric space, in which fuzzy contractive sequences are Cauchy.

In 2006, Bhaskar and Lakshmikantham [14] introduced the concept of coupled fixed point in metric spaces and obtained some coupled fixed point theorems with the application to a bounded value problem. Based on Bhaskar and Lakshmikantham's work, many researchers have obtained more coupled fixed point theorems in metric spaces and cone metric spaces; see [14, 15]. Recently, the investigation of coupled fixed point theorems has been extended from metric spaces to probabilistic metric spaces and fuzzy metric spaces; see [16–19]. In [18], the authors gave the following results.

Theorem SAS [18, Theorem 2.5] *Let $a * b > ab$ for all $a, b \in [0,1]$ and let $(X, M, *)$ be a complete fuzzy metric space such that M has an n-property. Let $F : X \times X \to X$ and $g : X \to X$ be two functions such that*

$$M\big(F(x,y), F(u,v), kt\big) \geq M(gx, gu, t) * M(gy, gv, t)$$

for all $x, y, u, v \in X$, where $0 < k < 1$, $F(X \times X) \subseteq g(X)$ and g is continuous and commutes with F. Then there exists a unique $x \in X$ such that $x = gx = F(x, x)$.

Let $\Phi = \{\phi : \mathbb{R}^+ \to \mathbb{R}^+\}$, where $\mathbb{R}^+ = [0, +\infty)$ and each $\phi \in \Phi$ satisfies the following conditions:

(ϕ-1) ϕ is nondecreasing,
(ϕ-2) ϕ is upper semicontinuous from the right,
(ϕ-3) $\sum_{n=0}^{\infty} \phi^n(t) < +\infty$ for all $t > 0$, where $\phi^{n+1}(t) = \phi(\phi^n(t))$, $n \in \mathbb{N}$.

In [17], Hu proved the following result.

Theorem of Hu [17, Theorem 1] *Let $(X, M, *)$ be a complete fuzzy metric space, where $*$ is a continuous t-norm of H-type. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings and let there exist $\phi \in \Phi$ such that*

$$M\big(F(x,y), F(u,v), \phi(t)\big) \ge M(gx, gu, t) * M(gy, gv, t)$$

for all $x, y, u, v \in X$, $t > 0$. Suppose that $F(X \times X) \subseteq g(X)$ and that g is continuous, F and g are compatible. Then there exists $x \in X$ such that $x = gx = F(x, x)$, that is, F and g have a unique common fixed point in X.

In this paper, inspired by Sedghi *et al.* and Hu's work mentioned above, we prove some common fixed point theorems for ϕ-contractive mappings in fuzzy metric spaces, in which a very simple condition is imposed on the function ϕ. Our results improve the corresponding ones of Sedghi *et al.* [18] and Hu [17]. Finally, an example is presented to illustrate the main result in this paper.

2 Preliminaries

Definition 2.1 [9] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $*$ satisfies the following conditions:

(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a * 1 = a$ for all $a \in [0,1]$,
(4) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0,1]$.

Two typical examples of the continuous t-norm are $a *_1 b = ab$ and $a *_2 b = \min\{a, b\}$ for all $a, b \in [0,1]$.

Definition 2.2 [20] A t-norm $*$ is said to be of Hadžić type (for short H-type) if the family of functions $\{*^m(t)\}_{m=1}^{\infty}$ is equicontinuous at $t = 1$, where

$$*^1(t) = t * t, \qquad *^{m+1}(t) = t * \big(*^m(t)\big), \quad m = 1, 2, \ldots, t \in [0,1].$$

The t-norm $*_2$ is an example of t-norm of H-type, but t-norm $*_1$ is not of H-type. Some other t-norm of H-type can be found in [20].

Definition 2.3 (Kramosil and Michalek [11]) A fuzzy metric space (in the sense of Kramosil and Michalek) is a triple $(X, M, *)$, where X is a nonempty set, $*$ is a continuous t-norm and $M : X^2 \times [0, \infty)$ is a mapping, satisfying the following:

(KM-1) $M(x,y,0) = 0$ for all $x,y \in X$,
(KM-2) $M(x,y,t) = 1$ for all $t > 0$ if and only if $x = y$,
(KM-3) $M(x,y,t) = M(y,x,t)$ for all $x,y \in X$ and all $t > 0$,
(KM-4) $M(x,y,\cdot) : [0,\infty) \to [0,1]$ is left continuous for all $x,y \in X$,
(KM-5) $M(x,y,t+s) \geq M(x,z,t) * M(y,z,s)$ for all $x,y,z \in X$ and all $t,s > 0$.

In Definition 2.3, if M is a fuzzy set on $X^2 \times (0,\infty)$ and (KM-1), (KM-2), (KM-4) are replaced with the following (GV-1), (GV-2), (GV-4), respectively, then $(X,M,*)$ is called a fuzzy metric space in the sense of George and Veeramani [9]:
(GV-1) $M(x,y,t) > 0$ for all $t > 0$ and all $x,y \in X$,
(GV-2) $M(x,y,t) = 1$ for some $t > 0$ if and only if $x = y$,
(GV-4) $M(x,y,\cdot) : (0,\infty) \to [0,1]$ is continuous.

Lemma 2.1 [4] *Let $(X,M,*)$ be a fuzzy metric space in the sense of GV. Then $M(x,y,\cdot)$ is nondecreasing for all $x,y \in X$.*

Definition 2.4 (George and Veeramani [9]) Let $(X,M,*)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is called an M-Cauchy sequence if for each $\epsilon \in (0,1)$ and $t > 0$, there is $n_0 \in \mathbb{N}$ such that $M(x_n,x_m,t) > 1-\epsilon$ for all $m,n \geq n_0$. The fuzzy metric space $(X,M,*)$ is called M-complete if every M-Cauchy sequence is convergent.

Definition 2.5 [14] An element $(x,y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if

$$F(x,y) = g(x), \qquad F(y,x) = g(y).$$

Here (gx,gy) is called a coupled point of coincidence.

Definition 2.6 [15] An element $x \in X \times X$ is called a common fixed point of the mappings $F : X \times X \to X$ and $g : X \to X$ if

$$F(x,x) = g(x) = x.$$

Definition 2.7 [17] Let $(X,M,*)$ be a fuzzy metric space. The mappings F and g, where $F : X \times X \to X$ and $g : X \to X$, are said to be compatible if for all $t > 0$,

$$\lim_{n\to\infty} M\bigl(g\bigl(F(x_n,y_n)\bigr), F\bigl(g(x_n),g(y_n)\bigr), t\bigr) = 1 \quad \text{and}$$
$$\lim_{n\to\infty} M\bigl(g\bigl(F(y_n,x_n)\bigr), F\bigl(g(y_n),g(x_n)\bigr), t\bigr) = 1$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n,y_n) = \lim_{n\to\infty} g(x_n) = x$ and $\lim_{n\to\infty} F(x_n,y_n) = \lim_{n\to\infty} g(x_n) = y$ for some $x,y \in X$.

In [21], Abbas *et al.* introduced the concept of w-compatible mappings. Here we give a similar concept in fuzzy metric spaces as follows.

Definition 2.8 Let $(X,M,*)$ be a fuzzy metric space, and let $F : X \times X \to X$ and $g : X \to X$ be two mappings. F and g are said to be weakly compatible (or w-compatible) if they commute at their coupled coincidence points, *i.e.*, if (x, y) is a coupled coincidence point of g and F, then $g(F(x, y)) = F(gx, gy)$.

3 Main results

In this section, the fuzzy metric space $(X, M, *)$ is in the sense of GV and the fuzzy metric M is assumed to satisfy the condition $\sup_{t>0} M(x, y, t) = 1$ for all $x, y \in X$.

By using the continuity of $*$ and [22, Lemma 1], we get the following result.

Lemma 3.1 *Let $n \in \mathbb{N}$, let $g_n : (0, \infty) \to (0, \infty)$, and let $F_n : \mathbb{R} \to [0, 1]$. Assume that $\sup\{F(t) : t > 0\} = 1$ and*

$$\lim_{n\to\infty} g_n(t) = 0, \qquad F_n\bigl(g_n(t)\bigr) \geq *^{2n}\bigl(F(t)\bigr), \quad \forall t > 0.$$

If each F_n is nondecreasing, then $\lim_{n\to\infty} F_n(t) = 1$ for any $t > 0$.

Theorem 3.1 *Let $(X, M, *)$ be a fuzzy metric space under a continuous t-norm $*$ of H-type. Let $\phi : (0, \infty) \to (0, \infty)$ be a function satisfying that $\lim_{n\to\infty} \phi^n(t) = 0$ for any $t > 0$. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings with $F(X \times X) \subseteq g(X)$ and assume that for any $t > 0$,*

$$M\bigl(F(x, y), F(u, v), \phi(t)\bigr) \geq M(gx, gu, t) * M(gy, gv, t) \tag{3.1}$$

for all $x, y, u, v \in X$. Suppose that $F(X \times X)$ is complete and that g and F are w-compatible, then g and F have a unique common fixed point $x^* \in X$, that is, $x^* = g(x^*) = F(x^*, x^*)$.

Proof Since $F(X \times X) \subseteq g(X)$, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$gx_{n+1} = F(x_n, y_n) \quad \text{and} \quad gy_{n+1} = F(y_n, x_n) \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.2}$$

From (3.1) and (3.2) we have

$$\begin{aligned} M\bigl(gx_n, gx_{n+1}, \phi(t)\bigr) &= M\bigl(F(x_{n-1}, y_{n-1}), F(x_n, y_n), \phi(t)\bigr) \\ &\geq M(gx_{n-1}, gx_n, t) * M(gy_{n-1}, gy_n, t) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} M\bigl(gy_n, gy_{n+1}, \phi(t)\bigr) &= M\bigl(F(y_{n-1}, x_{n-1}), F(y_n, x_n), \phi(t)\bigr) \\ &\geq M(gy_{n-1}, gy_n, t) * M(gx_{n-1}, gx_n, t). \end{aligned} \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$\begin{aligned} &M\bigl(gx_n, gx_{n+1}, \phi^n(t)\bigr) * M\bigl(gy_n, gy_{n+1}, \phi^n(t)\bigr) \\ &\quad \geq *^2\bigl(M\bigl(gx_{n-1}, gx_n, \phi^{n-1}(t)\bigr) * M\bigl(gy_{n-1}, gy_n, \phi^{n-1}(t)\bigr)\bigr) \\ &\quad \geq \cdots \geq *^{2n}\bigl(M(gx_0, gx_1, t) * M(gy_0, gy_1, t)\bigr). \end{aligned}$$

Let $E_n(t) = M(gx_n, gx_{n+1}, t) * M(gy_n, gy_{n+1}, t)$. Then

$$E_n\bigl(\phi^n(t)\bigr) \geq *^2\bigl(E_{n-1}\bigl(\phi^{n-1}(t)\bigr)\bigr) \geq \cdots \geq *^{2n}\bigl(E_0(t)\bigr).$$

Since $\phi^n(t) \to 0$ and $\sup_{t>0} E_0(t) = 1$, by Lemma 3.1 we have

$$\lim_{n\to\infty} E_n(t) = 1.$$

Noting that $\min\{M(gx_n, gx_{n+1}, t), M(gy_n, gy_{n+1}, t)\} \geq E_n(t)$, we get that

$$\lim_{n\to\infty} M(gx_n, gx_{n+1}, t) = \lim_{n\to\infty} M(gy_n, gy_{n+1}, t) = 1, \quad \forall t > 0. \tag{3.5}$$

For any fixed $t > 0$, since $\lim_{n\to\infty} \phi^n(t) = 0$, there exists $n_0 = n_0(t) \in \mathbb{N}$ such that $\phi^{n_0+1}(t) < \phi^{n_0}(t) < t$. Next we show by induction that for any $k \in \mathbb{N} \cup \{0\}$, there exists $b_k \in \mathbb{N}$ such that

$$\begin{aligned} &M\bigl(gx_n, gx_{n+k}, \phi^{n_0}(t)\bigr) * M\bigl(gy_n, gy_{n+k}, \phi^{n_0}(t)\bigr) \\ &\quad \geq *^{b_k}\bigl(M\bigl(gx_n, gx_{n+1}, \phi^{n_0}(t) - \phi^{n_0+1}(t)\bigr) * M\bigl(gy_n, gy_{n+1}, \phi^{n_0}(t) - \phi^{n_0+1}(t)\bigr)\bigr). \end{aligned} \tag{3.6}$$

It is obvious for $k = 0$ since $M(gx_n, gx_n, \phi^{n_0}(t)) = M(gy_n, gy_n, \phi^{n_0}(t)) = 1$. Assume that (3.6) holds for some $k \in \mathbb{N}$. Since $\phi^{n_0}(t) - \phi^{n_0+1}(t) > 0$, by (KM-5) we have

$$\begin{aligned} &M\bigl(gx_n, gx_{n+k+1}, \phi^{n_0}(t)\bigr) \\ &\quad = M\bigl(gx_n, gx_{n+k+1}, \phi^{n_0}(t) - \phi^{n_0+1}(t) + \phi^{n_0+1}(t)\bigr) \\ &\quad \geq M\bigl(gx_n, gx_{n+1}, \phi^{n_0}(t) - \phi^{n_0+1}(t)\bigr) * M\bigl(gx_{n+1}, gx_{n+k+1}, \phi^{n_0+1}(t)\bigr). \end{aligned} \tag{3.7}$$

It follows from (3.1) and (3.6) that

$$\begin{aligned} &M\bigl(gx_{n+1}, gx_{n+k+1}, \phi^{n_0+1}(t)\bigr) \\ &\quad = M\bigl(F(x_n, y_n), F(x_{n+k}, y_{n+k}), \phi^{n_0+1}(t)\bigr) \\ &\quad \geq M\bigl(gx_n, gx_{n+k}, \phi^{n_0}(t)\bigr) * M\bigl(gy_n, gy_{n+k}, \phi^{n_0}(t)\bigr) \\ &\quad \geq *^{b_k}\bigl(M\bigl(gx_n, gx_{n+1}, \phi^{n_0}(t) - \phi^{n_0+1}(t)\bigr) * M\bigl(gy_n, gy_{n+1}, \phi^{n_0}(t) - \phi^{n_0+1}(t)\bigr)\bigr). \end{aligned} \tag{3.8}$$

Now from (3.7) and (3.8) we get

$$\begin{aligned} &M\bigl(gx_n, gx_{n+k+1}, \phi^{n_0}(t)\bigr) \\ &\quad \geq M\bigl(gx_n, gx_{n+1}, \phi^{n_0}(t) - \phi^{n_0+1}(t)\bigr) * \bigl[*^{b_k}\bigl(M\bigl(gx_n, gx_{n+1}, \phi^{n_0}(t) - \phi^{n_0+1}(t)\bigr) \\ &\qquad * M\bigl(gy_n, gy_{n+1}, \phi^{n_0}(t) - \phi^{n_0+1}(t)\bigr)\bigr)\bigr]. \end{aligned} \tag{3.9}$$

Similarly, we have

$$\begin{aligned} &M\bigl(gy_n, gy_{n+k+1}, \phi^{n_0}(t)\bigr) \\ &\quad \geq M\bigl(gy_n, gy_{n+1}, \phi^{n_0}(t) - \phi^{n_0+1}(t)\bigr) * \bigl[*^{b_k}\bigl(M\bigl(gy_n, gy_{n+1}, \phi^{n_0}(t) - \phi^{n_0+1}(t)\bigr) \\ &\qquad * M\bigl(gx_n, gx_{n+1}, \phi^{n_0}(t) - \phi^{n_0+1}(t)\bigr)\bigr)\bigr]. \end{aligned} \tag{3.10}$$

From (3.9) and (3.10) we conclude that

$$\begin{aligned} & M\big(gx_n, gx_{n+k+1}, \phi^{n_0}(t)\big) * M\big(gy_n, gy_{n+k+1}, \phi^{n_0}(t)\big) \\ & \quad \geq M\big(gx_n, gx_{n+1}, \phi^{n_0}(t) - \phi^{n_0+1}(t)\big) * M\big(gy_n, gy_{n+1}, \phi^{n_0}(t) - \phi^{n_0+1}(t)\big) \\ & \qquad * \big[*^{2b_k}\big(M\big(gx_n, gx_{n+1}, \phi^{n_0}(t) - \phi^{n_0+1}(t)\big) * M\big(gy_n, gy_{n+1}, \phi^{n_0}(t) - \phi^{n_0+1}(t)\big)\big)\big] \\ & \quad = *^{2b_k+1}\big(M\big(gx_n, gx_{n+1}, \phi^{n_0}(t) - \phi^{n_0+1}(t)\big) * M\big(gy_n, gy_{n+1}, \phi^{n_0}(t) - \phi^{n_0+1}(t)\big)\big). \end{aligned}$$

Since $b_{k+1} = 2b_k + 1 \in \mathbb{N}$, this implies that (3.6) holds for $k+1$. Therefore, there exists $b_k \in \mathbb{N}$ such that (3.6) holds for each $k \in \mathbb{N} \cup \{0\}$.

Now we prove that $\{F(x_n, y_n)\}$ and $\{F(y_n, x_n)\}$ are Cauchy sequences in X. Let $t > 0$ and $\epsilon > 0$. Since $\lim_{n\to\infty} \phi^n(t) = 0$, there exists $n_1 = n_1(t) \in \mathbb{N}$ such that $\phi^{n_1+1}(t) < \phi^{n_1}(t) < t$. Since $\{*^n : n \in \mathbb{N}\}$ is equicontinuous at 1 and $*(1) = 1$, there is $\delta > 0$ such that

$$\text{if } s \in (1-\delta, 1], \quad \text{then } *^n(s) > 1 - \epsilon \quad \text{for all } n \in \mathbb{N}. \tag{3.11}$$

By (3.5), one has $\lim_{n\to\infty} M(gx_n, gx_{n+1}, \phi^{n_1}(t) - \phi^{n_1+1}(t)) = \lim_{n\to\infty} M(gy_n, gy_{n+1}, \phi^{n_1}(t) - \phi^{n_1+1}(t)) = 1$. Since $*$ is continuous, there is $N \in \mathbb{N}$ such that for all $n > N$,

$$M\big(gx_n, gx_{n+1}, \phi^{n_1}(t) - \phi^{n_1+1}(t)\big) * M\big(gy_n, gy_{n+1}, \phi^{n_1}(t) - \phi^{n_1+1}(t)\big) > 1 - \delta.$$

Hence, by (3.6) (replacing n_0 with n_1) and (3.11), we get

$$M\big(gx_n, gx_{n+k}, \phi^{n_1}(t)\big) * M\big(gy_n, gy_{n+k}, \phi^{n_1}(t)\big) > 1 - \epsilon$$

for any $k \in \mathbb{N} \cup \{0\}$. Since

$$\begin{aligned} & \min\big\{M\big(gx_n, gx_{n+k}, \phi^{n_1}(t)\big), M\big(gy_n, gy_{n+k}, \phi^{n_1}(t)\big)\big\} \\ & \quad \geq M\big(gx_n, gx_{n+k}, \phi^{n_1}(t)\big) * M\big(gy_n, gy_{n+k}, \phi^{n_1}(t)\big), \end{aligned}$$

one has

$$\min\big\{M\big(gx_n, gx_{n+k}, \phi^{n_1}(t)\big), M\big(gy_n, gy_{n+k}, \phi^{n_1}(t)\big)\big\} > 1 - \epsilon.$$

By monotonicity of M, we have, for any $k \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} & \min\big\{M(gx_n, gx_{n+k}, t), M(gy_n, gy_{n+k}, t)\big\} \\ & \quad \geq \min\big\{M\big(gx_n, gx_{n+k}, \phi^{n_1}(t)\big), M\big(gy_n, gy_{n+k}, \phi^{n_1}(t)\big)\big\} > 1 - \epsilon. \end{aligned}$$

Thus $\{gx_n\}$ and $\{gy_n\}$, *i.e.*, $\{F(x_n, y_n)\}$ and $\{F(y_n, x_n)\}$ are Cauchy sequences in X. Since $F(X \times X)$ is complete and $F(X \times X) \subseteq g(X)$, there exist $\hat{x}, \hat{y} \in X$ such that $\{F(x_n, y_n)\}$ converges to $g\hat{x}$ and $\{F(y_n, x_n)\}$ converges to $g\hat{y}$.

Next we prove that $g\hat{x} = F(\hat{x}, \hat{y})$ and $g\hat{y} = F(\hat{y}, \hat{x})$. Let $t > 0$; since $\lim_{n\to\infty} \phi^n(t) = 0$, there exists $n_2 = n_2(t) \in \mathbb{N}$ such that $\phi^{n_2}(\phi(t)) < \phi(t)$. By (KM-5) and (3.1), we have

$$\begin{aligned}
&M\bigl(F(\hat{x}, \hat{y}), g\hat{x}, \phi(t)\bigr) \\
&\quad \geq M\bigl(F(\hat{x}, \hat{y}), F(x_{n+n_2}, y_{n+n_2}), \phi^{n_2+1}(t)\bigr) \\
&\qquad * M\bigl(F(x_{n+n_2}, y_{n+n_2}), g(\hat{x}), \phi(t) - \phi^{n_2+1}(t)\bigr) \\
&\quad \geq M\bigl(g\hat{x}, gx_{n+n_2}, \phi^{n_2}(t)\bigr) * M\bigl(g\hat{y}, gy_{n+n_2}, \phi^{n_2}(t)\bigr) \\
&\qquad * M\bigl(F(x_{n+n_2}, y_{n+n_2}), g(\hat{x}), \phi(t) - \phi^{n_2+1}(t)\bigr).
\end{aligned} \tag{3.12}$$

Note that $\{gx_n\} \to g\hat{x}$, $\{gy_n\} \to g\hat{y}$ and $\{F(x_{n+n_2}, y_{n+n_2})\} \to g\hat{x}$. Thus, letting $n \to \infty$ in (3.12), we have

$$M\bigl(F(\hat{x}, \hat{y}), g\hat{x}, \phi(t)\bigr) \geq 1 * 1 = 1.$$

By induction we can get

$$M\bigl(F(\hat{x}, \hat{y}), g\hat{x}, \phi^n(t)\bigr) \geq 1.$$

By (GV-2) one has $F(\hat{x}, \hat{y}) = g\hat{x}$. Similarly, we can prove that $F(\hat{y}, \hat{x}) = g\hat{y}$.

Next we prove that if $(x^*, y^*) \in X \times X$ is another coupled coincidence point of g and F, then $g\hat{x} = gx^*$ and $g\hat{y} = gy^*$. In fact, by (3.1) we have

$$\begin{aligned}
&M\bigl(g\hat{x}, gx^*, \phi(t)\bigr) = M\bigl(F(\hat{x}, \hat{y}), F\bigl(x^*, y^*\bigr), \phi(t)\bigr) \geq M\bigl(g\hat{x}, gx^*, t\bigr) * M\bigl(g\hat{y}, gy^*, t\bigr) \quad \text{and} \\
&M\bigl(g\hat{y}, gy^*, \phi(t)\bigr) = M\bigl(F(\hat{y}, \hat{x}), F\bigl(y^*, x^*\bigr), \phi(t)\bigr) \geq M\bigl(g\hat{y}, gy^*, t\bigr) * M\bigl(g\hat{x}, gx^*, t\bigr).
\end{aligned}$$

It follows that

$$M\bigl(g\hat{x}, gx^*, \phi(t)\bigr) * M\bigl(g\hat{y}, gy^*, \phi(t)\bigr) \geq *^2\bigl(M\bigl(g\hat{x}, gx^*, t\bigr) * M\bigl(g\hat{y}, gy^*, t\bigr)\bigr).$$

By induction we get

$$\begin{aligned}
&\min\bigl\{M\bigl(g\hat{x}, gx^*, \phi^n(t)\bigr), M\bigl(g\hat{y}, gy^*, \phi^n(t)\bigr)\bigr\} \\
&\quad \geq M\bigl(g\hat{x}, gx^*, \phi^n(t)\bigr) * M\bigl(g\hat{y}, gy^*, \phi^n(t)\bigr) \geq *^{2n}\bigl(M\bigl(g\hat{x}, gx^*, t\bigr) * M\bigl(g\hat{y}, gy^*, t\bigr)\bigr).
\end{aligned}$$

It follows from Lemma 3.1 and (GV-2) that $g\hat{x} = gx^*$ and $g\hat{y} = gy^*$. This shows that g and F have the unique coupled point of coincidence.

Now we show that $g\hat{x} = g\hat{y}$ and $g\hat{y} = g\hat{x}$. In fact, from (3.1) we get

$$\begin{aligned}
M\bigl(g\hat{x}, gy_n, \phi(t)\bigr) &= M\bigl(F(\hat{x}, \hat{y}), F(y_{n-1}, x_{n-1}), \phi(t)\bigr) \\
&\geq M(g\hat{x}, gy_{n-1}, t) * M(g\hat{y}, gx_{n-1}, t)
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
M\bigl(g\hat{y}, gx_n, \phi(t)\bigr) &= M\bigl(F(\hat{y}, \hat{x}), F(x_{n-1}, y_{n-1}), \phi(t)\bigr) \\
&\geq M(g\hat{y}, gx_{n-1}, t) * M(g\hat{x}, gy_{n-1}, t).
\end{aligned} \tag{3.14}$$

Let $M_n(t) = M(g\hat{y}, gx_n, t) * M(g\hat{x}, gy_n, t)$. From (3.13) and (3.14) it follows that

$$M_n(\phi^n(t)) \geq *^2(M_{n-1}(\phi^{n-1}(t))) \geq \cdots \geq *^{2n}(M_0(t)).$$

By Lemma 3.1 we get $\lim_{n\to\infty} M_n(t) = 1$, which implies that

$$\lim_{n\to\infty} M(g\hat{y}, gx_n, t) = \lim_{n\to\infty} M(g\hat{x}, gy_n, t) = 1.$$

Since $\{gx_n\}$ converges to $g\hat{x}$ and $\{gy_n\}$ converges to $g\hat{y}$, we see that $g\hat{y} = g\hat{x}$.

Now let $u = g\hat{x}$. Then we have $u = g\hat{y}$ since $g\hat{x} = g\hat{y}$. Since g and F are w-compatible, we have

$$gu = g(g\hat{x}) = g(F(\hat{x}, \hat{y})) = F(g\hat{x}, g\hat{y}) = F(u, u),$$

which implies that (u, u) is a coupled coincidence point of g and F. Since g and F have a unique coupled point of coincidence, we can conclude that $gu = g\hat{x}$, *i.e.*, $gu = u$. Therefore, we have $u = gu = F(u, u)$. Finally, we prove the uniqueness of a common fixed point of g and F. Let $v \in X$ be such that $v = gv = F(v, v)$. By (3.1) we have

$$M(u, v, \phi(t)) = M(F(u, u), F(v, v), \phi(t)) \geq M(gu, gv, t) * M(gu, gv, t) = *^2(M(u, v, t)),$$

which implies that

$$M(u, v, \phi^n(t)) \geq *^{2n}(M(u, v, t)).$$

By Lemma 3.1 and (GV-2), we see that $u = v$. This completes the proof. □

Theorem 3.2 *Let $(X, M, *)$ be a fuzzy metric space under a continuous t-norm $*$ of H-type. Let $\phi : (0, \infty) \to (0, \infty)$ be a function satisfying that* $\lim_{n\to\infty} \phi^n(t) = \infty$ *for any $t > 0$. Suppose that $g : X \to X$ and $F : X \times X \to X$ are two mappings such that $F(X \times X) \subseteq g(X)$, and assume that for any $t > 0$,*

$$M(F(x, y), F(p, q), t) \geq M(gx, gp, \phi(t)) * M(gy, gq, \phi(t)) \tag{3.15}$$

for all $x, y, p, q \in X$. Suppose that $F(X \times X)$ is complete and that g and F are w-compatible, then g and F have a unique common fixed point in $x^* \in X$, that is, $x^* = gx^* = F(x^*, x^*)$.

Proof Since $F(X \times X) \subseteq g(X)$, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$gx_{n+1} = F(x_n, y_n) \quad \text{and} \quad gy_{n+1} = F(y_n, x_n), \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.16}$$

From (3.15) and (3.16) we have

$$\begin{aligned} M(gx_n, gx_{n+1}, t) &= M(F(x_{n-1}, y_{n-1}), F(x_n, y_n), t) \\ &\geq M(gx_{n-1}, gx_n, \phi(t)) * M(gy_{n-1}, gy_n, \phi(t)) \end{aligned} \tag{3.17}$$

and

$$
\begin{aligned}
M(gy_n, gy_{n+1}, t) &= M\bigl(F(y_{n-1}, x_{n-1}), F(y_n, x_n), t\bigr) \\
&\geq M\bigl(gy_{n-1}, gy_n, \phi(t)\bigr) * M\bigl(gx_{n-1}, gx_n, \phi(t)\bigr).
\end{aligned} \tag{3.18}
$$

Now, let $E_n(t) = M(gx_{n-1}, gx_n, t) * M(gy_{n-1}, gy_n, t)$. From (3.17) and (3.18) we get $E_{n+1}(t) \geq E_n(\phi(t))$. It follows that

$$
E_{n+1}(t) \geq *^2\bigl(E_n\bigl(\phi(t)\bigr)\bigr) \geq \cdots \geq *^{2n}\bigl(E_1\bigl(\phi^n(t)\bigr)\bigr). \tag{3.19}
$$

Since $\lim_{t\to\infty} E_1(t) = \lim_{t\to\infty} M(gx_0, gx_1, t) * M(gy_0, gy_1, t) = 1$ and $\lim_{n\to\infty} \phi^n(t) = \infty$ for each $t > 0$, we have $\lim_{n\to\infty} E_1(\phi^n(t)) = 1$. By Lemma 3.1 we have

$$
\lim_{n\to\infty} E_n(t) = 1 \quad \text{for all } t > 0. \tag{3.20}
$$

For any fixed $t > 0$, since $\lim_{n\to\infty} \phi^n(t) = \infty$, there exists $n_0 = n_0(t) \in \mathbb{N}$ such that $\phi^{n_0+1}(t) > \phi^{n_0}(t) > t$. Similarly, since $\lim_{n\to\infty} \phi^n(\phi^{n_0+1}(t) - \phi^{n_0}(t)) = \infty$, there exists $m_0 = m_0(t) \in \mathbb{N}$ such that $\phi^{m_0}(\phi^{n_0+1}(t) - \phi^{n_0}(t)) > \phi^{n_0+1}(t) - \phi^{n_0}(t)$. By (3.17) we have

$$
\begin{aligned}
&M\bigl(gx_{n+m_0}, gx_{n+m_0+1}, \phi^{n_0+1}(t) - \phi^{n_0}(t)\bigr) \\
&\quad \geq E_{n+m_0}\bigl(\phi\bigl(\phi^{n_0+1}(t) - \phi^{n_0}(t)\bigr)\bigr) \\
&\quad \geq \cdots \geq *^{2m_0}\bigl(E_n\bigl(\phi^{m_0}\bigl(\phi^{n_0+1}(t) - \phi^{n_0}(t)\bigr)\bigr)\bigr) \\
&\quad \geq *^{2m_0}\bigl(E_n\bigl(\phi^{n_0+1}(t) - \phi^{n_0}(t)\bigr)\bigr).
\end{aligned} \tag{3.21}
$$

Next we show by induction that for any $k \in \mathbb{N} \cup \{0\}$, there exists $b_k \in \mathbb{N}$ such that

$$
\begin{aligned}
&M\bigl(gx_{n+m_0}, gx_{n+m_0+k}, \phi^{n_0+1}(t)\bigr) \geq *^{b_k}\bigl(E_n\bigl(\phi^{n_0+1}(t) - \phi^{n_0}(t)\bigr)\bigr) \quad \text{and} \\
&M\bigl(gy_{n+m_0}, gy_{n+m_0+k}, \phi^{n_0+1}(t)\bigr) \geq *^{b_k}\bigl(E_n\bigl(\phi^{n_0+1}(t) - \phi^{n_0}(t)\bigr)\bigr).
\end{aligned} \tag{3.22}
$$

This is obvious for $k = 0$ since $M(gx_{n+m_0}, gx_{n+m_0}, \phi^{n_0+1}(t)) = 1$ and $M(gy_{n+m_0}, gy_{n+m_0}, \phi^{n_0+1}(t)) = 1$. Assume that (3.22) holds for some $k \in \mathbb{N}$. By (3.15), (3.22), (3.21) and (KM-5), we have

$$
\begin{aligned}
&M\bigl(gx_{n+m_0}, gx_{n+m_0+k+1}, \phi^{n_0+1}(t)\bigr) \\
&\quad = M\bigl(gx_{n+m_0}, gx_{n+m_0+k+1}, \phi^{n_0+1}(t) - \phi^{n_0}(t) + \phi^{n_0}(t)\bigr) \\
&\quad \geq M\bigl(gx_{n+m_0}, gx_{n+m_0+1}, \phi^{n_0+1}(t) - \phi^{n_0}(t)\bigr) * M\bigl(gx_{n+m_0+1}, gx_{n+m_0+k+1}, \phi^{n_0}(t)\bigr) \\
&\quad = M\bigl(gx_{n+m_0}, gx_{n+m_0+1}, \phi^{n_0+1}(t) - \phi^{n_0}(t)\bigr) \\
&\qquad * M\bigl(F(x_{n+m_0}, y_{n+m_0}), F(x_{n+m_0+k}, y_{n+m_0+k}), \phi^{n_0}(t)\bigr) \\
&\quad \geq M\bigl(gx_{n+m_0}, gx_{n+m_0+1}, \phi^{n_0+1}(t) - \phi^{n_0}(t)\bigr) * \bigl(M\bigl(gx_{n+m_0}, gx_{n+m_0+k}, \phi^{n_0+1}(t)\bigr) \\
&\qquad * M\bigl(gy_{n+m_0}, gy_{n+m_0+k}, \phi^{n_0+1}(t)\bigr)\bigr) \\
&\quad \geq M\bigl(gx_{n+m_0}, gx_{n+m_0+1}, \phi^{n_0+1}(t) - \phi^{n_0}(t)\bigr) * (*^{2b_k}\bigl(E_n\bigl(\phi^{n_0+1}(t) - \phi^{n_0}(t)\bigr)\bigr)
\end{aligned}
$$

$$\begin{aligned}&\geq *^{2m_0}\big(E_n\big(\phi^{n_0+1}(t)-\phi^{n_0}(t)\big)*\big(*^{2b_k}\big(E_n\big(\phi^{n_0+1}(t)-\phi^{n_0}(t)\big)\big)\\&= *^{2(m_0+b_k)}\big(E_n\big(\phi^{n_0+1}(t)-\phi^{n_0}(t)\big)\big).\end{aligned}$$

Similarly, we can prove that

$$M\big(gy_{n+m_0},gy_{n+m_0+k+1},\phi^{n_0+1}(t)\big)\geq *^{2(m_0+b_k)}\big(E_n\big(\phi^{n_0+1}(t)-\phi^{n_0}(t)\big)\big).$$

Since $b_{k+1}=2(m_0+b_k)\in\mathbb{N}$, (3.22) holds for $k+1$. Therefore, there exists $b_k\in\mathbb{N}$ such that (3.22) holds for all $k\in\mathbb{N}\cup\{0\}$.

Let $t>0$ and $\epsilon>0$. By hypothesis, $\{*^n : n\in\mathbb{N}\}$ is equicontinuous at 1 and $*(1)=1$, so there is $\delta>0$ such that

$$\text{if } s\in(1-\delta,1],\quad \text{then } *^n(s)>1-\epsilon\quad \text{for all } n\in\mathbb{N}. \tag{3.23}$$

Since by (3.20) $\lim_{n\to\infty}E_n(\phi^{n_0+1}(t)-\phi^{n_0}(t))=1$, there is $N_0\in\mathbb{N}$ such that for all $n>N_0$, $E_n(\phi^{n_0+1}(t)-\phi^{n_0}(t))\in(1-\delta,1]$. Hence, it follows from (3.22) and (3.23) that

$$M\big(gx_{n+m_0},gx_{n+m_0+k},\phi^{n_0+1}(t)\big)*M\big(gy_{n+m_0},gy_{n+m_0+k},\phi^{n_0+1}(t)\big)>1-\epsilon$$

for all $n>N_0$ and any $k\in\mathbb{N}\cup\{0\}$. Noting that (3.17) and (3.18), we have

$$\begin{aligned}&\min\big\{M(gx_{n+m_0+n_0+1},gx_{n+m_0+n_0+1+k},t),M(gy_{n+m_0+n_0+1},gy_{n+m_0+n_0+1+k},t)\big\}\\&\quad\geq *^{2n_0+1}\big(M\big(gx_{n+m_0},gx_{n+m_0+k},\phi^{n_0+1}(t)\big)*M\big(gy_{n+m_0},gy_{n+m_0+k},\phi^{n_0+1}(t)\big)\big)\\&\quad>1-\epsilon.\end{aligned}$$

This implies that for all $k\in\mathbb{N}$,

$$M(gx_m,gx_{m+k},t)>1-\epsilon\quad\text{and}\quad M(gy_m,gy_{m+k},t)>1-\epsilon,$$

where $m>N_0+n_0+m_0+1$. Thus $\{gx_n\}$ and $\{gy_n\}$, *i.e.*, $\{F(x_n,y_n)\}$ and $\{F(y_n,x_n)\}$ are the Cauchy sequences. Since $F(X\times X)$ is complete and $F(X\times X)\subseteq g(X)$, there exists $(\hat{x},\hat{y})\in X\times X$ such that $\{F(x_n,y_n)\}$ converges to $g\hat{x}$ and $\{F(y_n,x_n)\}$ converges to $g\hat{y}$.

Next we prove that $g\hat{x}=F(\hat{x},\hat{y})$ and $g\hat{y}=F(\hat{y},\hat{x})$. By (KM-5) and (3.15), we have, for any $t>0$,

$$M\big(F(\hat{x},\hat{y}),F(x_n,y_n),t\big)\geq M\big(g\hat{x},gx_n,\phi(t)\big)*M\big(g\hat{y},gy_n,\phi(t)\big). \tag{3.24}$$

Since $\lim_{n\to\infty}gx_n=g\hat{x}$ and $\lim_{n\to\infty}gy_n=g\hat{y}$, letting $n\to\infty$ in (3.24), we have $\lim_{n\to\infty}F(x_n,y_n)=F(\hat{x},\hat{y})$. Noting that $\lim_{n\to\infty}F(x_n,y_n)=g\hat{x}$, we have $F(\hat{x},\hat{y})=g\hat{x}$. Similarly, we can prove that $F(\hat{y},\hat{x})=g\hat{y}$.

Let $u=g\hat{x}$ and $v=g\hat{y}$. Since g and F are w-compatible, we have

$$\begin{aligned}&gu=g(g\hat{x})=g\big(F(\hat{x},\hat{y})\big)=F(g\hat{x},g\hat{y})=F(u,v)\quad\text{and}\\&gv=g(g\hat{y})=g\big(F(\hat{y},\hat{x})\big)=F(g\hat{y},g\hat{x})=F(v,u).\end{aligned} \tag{3.25}$$

This shows that (u,v) is a coupled coincidence point of g and F. Now we prove that $gu = g\hat{x}$ and $gv = g\hat{y}$. In fact, from (3.15) we have

$$\begin{aligned} M(gu, gx_n, t) &= M\big(F(u,v), F(x_{n-1}, y_{n-1}), t\big) \\ &\geq M\big(gu, gx_{n-1}, \phi(t)\big) * M\big(gv, gy_{n-t}, \phi(t)\big) \quad \text{and} \\ M(gv, gy_n, t) &= M\big(F(v,u), F(y_{n-1}, x_{n-1}), t\big) \geq M\big(gv, gy_{n-1}, \phi(t)\big) * M\big(gu, gx_{n-t}, \phi(t)\big). \end{aligned}$$

Let $M_n(t) = M(gu, gx_n, t) * M(gv, gy_n, t)$. Then we have

$$M_n(t) \geq *^2\big(M_{n-1}\big(\phi(t)\big)\big) \geq \cdots \geq *^{2n}\big(M_0\big(\phi^n(t)\big)\big).$$

Since $\lim_{n\to\infty} \phi^n(t) = \infty$ and $*$ is continuous, we have

$$*^{2n}\big(M_0\big(\phi^n(t)\big)\big) = *^{2n}\big(M\big(gv, gx_0, \phi^n(t)\big) * M\big(gu, gy_0, \phi^n(t)\big)\big) \to 1 \quad \text{as } n \to \infty.$$

This shows that $M_n(t) \to 1$ as $n \to \infty$, and so we have $gu = g\hat{x}$ and $gv = g\hat{y}$. Therefore, we have $gu = u$ and $gv = v$. Now, from (3.25) it follows that $u = gu = F(u,v)$ and $v = gv = F(v,u)$.

Finally, we prove that $u = v$. In fact, by (3.15) we have, for any $t > 0$,

$$M(u,v,t) = M\big(F(u,v), F(v,u), t\big) \geq M\big(gu, gv, \phi(t)\big) * M\big(gv, gu, \phi(t)\big) = *^2\big(M\big(u,v,\phi(t)\big)\big).$$

By induction we can get $M(u,v,t) \geq *^{2n}(M(u,v,\phi^n(t)))$. Letting $n \to \infty$ and noting that $\phi^n(t) \to \infty$ as $n \to \infty$, we have $M(u,v,t) = 1$ for any $t > 0$, *i.e.*, $u = v$. Therefore, u is a common fixed point of g and F. The uniqueness of u is similar to the final proof line of Theorem 3.1. This completes the proof. □

In Theorem 3.1 and Theorem 3.2, if we let $gx = x$ for all $x \in X$, we get the following result.

Corollary 3.1 *Let $(X, M, *)$ be a fuzzy metric space under a continuous t-norm $*$ of H-type. Let $\phi : (0,\infty) \to (0,\infty)$ be a function satisfying that $\lim_{n\to\infty} \phi^n(t) = 0$ for any $t > 0$. Let $F : X \times X \to X$ be a mapping, and assume that for any $t > 0$,*

$$M\big(F(x,y), F(p,q), \phi(t)\big) \geq M(x,p,t) * M(p,q,t)$$

for all $x, y, p, q \in X$. Suppose that $F(X \times X)$ is complete. Then F has a unique fixed point $x^* \in X$, that is, $x^* = F(x^*, x^*)$.

Corollary 3.2 *Let $(X, M, *)$ be a fuzzy metric space under a continuous t-norm $*$ of H-type. Let $\phi : (0,\infty) \to (0,\infty)$ be a function satisfying that $\lim_{n\to\infty} \phi^n(t) = \infty$ for any $t > 0$. Let $F : X \times X \to X$ be a mapping, and assume that for any $t > 0$,*

$$M_{F(x,y),F(p,q)}(t) \geq M_{x,p}(t) * M_{p,q}\big(\phi(t)\big)$$

for all $x, y, p, q \in X$. Suppose that $F(X \times X)$ is complete. Then F has a unique fixed point $x^* \in X$, that is, $x^* = F(x^*, x^*)$.

Now, we illustrate Theorem 3.1 by the following example.

Example 3.1 Let $X = [0, \frac{1}{4}) \cup \{\frac{1}{2}\}$ and $x * y = \min(x, y)$ for all $x, y \in X$. Define $M(x, y, t) = \frac{t}{t+|x-y|}$ for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a fuzzy metric space, but it is not complete. Define two mappings $g : X \to X$ and $F : X \times X \to X$ by

$$g(x) = \begin{cases} \frac{x}{2} & \text{if } x \in [0, \frac{1}{8}], \\ x & \text{if } x \in (\frac{1}{8}, \frac{1}{4}), \\ \frac{1}{2} & \text{if } x = \frac{1}{2} \end{cases}$$

and

$$F(x, y) = \begin{cases} \frac{x}{8} & \text{if } x \in [0, \frac{1}{4}), \\ \frac{1}{32} & \text{if } x = \frac{1}{2}. \end{cases}$$

It is easy to see that g and F are not commuting since $g(F(\frac{1}{2}, \frac{1}{2})) \neq F(g(\frac{1}{2}), g(\frac{1}{2}))$, $F(X \times X) \subseteq g(X)$, and $F(X \times X)$ is complete.

Let $\phi : (0, \infty) \to (0, \infty)$ by

$$\phi(t) = \begin{cases} \frac{3}{2}, & t = 1, \\ \frac{t}{2}, & t \neq 1. \end{cases}$$

Then $\lim_{n\to\infty} \phi^n(t) = 0$ for any $t > 0$.

Now, we verify (3.1) for $t \neq 1$. We shall consider the following four cases.

Case 1. Let $x \neq \frac{1}{2}$ and $u \neq \frac{1}{2}$. In this case there are four possibilities:

Case 1.1. Let $x \in [0, \frac{1}{8}]$ and $u \in [0, \frac{1}{8}]$. Then we have

$$\begin{aligned} M\bigl(F(x, y), F(u, v), \phi(t)\bigr) &= \frac{\frac{t}{2}}{\frac{t}{2} + |\frac{x}{8} - \frac{u}{8}|} \\ &= \frac{2t}{2t + |\frac{x}{2} - \frac{u}{2}|} \\ &\geq \frac{t}{t + |\frac{x}{2} - \frac{u}{2}|} \\ &\geq \min\left\{\frac{t}{t + |\frac{x}{2} - \frac{u}{2}|}, \frac{t}{t + |\frac{y}{2} - \frac{v}{2}|}\right\} \\ &\geq \min\bigl\{M\bigl(g(x), g(u), t\bigr), M\bigl(g(y), g(v), t\bigr)\bigr\} \quad \text{for all } y, v \in X. \end{aligned}$$

Case 1.2. Let $x \in [0, \frac{1}{8}]$ and $u \in (\frac{1}{8}, \frac{1}{4})$. Then

$$\begin{aligned} M\bigl(F(x, y), F(u, v), \phi(t)\bigr) &= \frac{\frac{t}{2}}{\frac{t}{2} + |\frac{x}{8} - \frac{u}{8}|} \\ &= \frac{2t}{2t + (\frac{u}{2} - \frac{x}{2})} \end{aligned}$$

$$\begin{aligned}
&\geq \frac{2t}{2t + (u - \frac{x}{2})} \\
&\geq \min\left\{\frac{t}{t + |u - \frac{x}{2}|}, \frac{t}{t + |\frac{y}{2} - \frac{v}{2}|}\right\} \\
&\geq \min\bigl\{M\bigl(g(x), g(u), t\bigr), M\bigl(g(y), g(v), t\bigr)\bigr\} \quad \text{for all } y, v \in X.
\end{aligned}$$

Case 1.3. Let $x \in (\frac{1}{8}, \frac{1}{4})$ and $u \in [0, \frac{1}{8}]$. This case is similar to Case 1.2.
Case 1.4. Let $x \in (\frac{1}{8}, \frac{1}{4})$ and $u \in (\frac{1}{8}, \frac{1}{4})$. Then

$$\begin{aligned}
M\bigl(F(x,y), F(u,v), \phi(t)\bigr) &= \frac{\frac{t}{2}}{\frac{t}{2} + |\frac{x}{8} - \frac{u}{8}|} \\
&= \frac{2t}{2t + |\frac{x}{2} - \frac{u}{2}|} \\
&\geq \frac{t}{t + |\frac{x}{2} - \frac{u}{2}|} \\
&\geq \min\left\{\frac{t}{t + |x - u|}, \frac{t}{t + |\frac{y}{2} - \frac{v}{2}|}\right\} \\
&\geq \min\bigl\{M\bigl(g(x), g(u), t\bigr), M\bigl(g(y), g(v), t\bigr)\bigr\} \quad \text{for all } y, v \in X.
\end{aligned}$$

Case 2. Let $x = \frac{1}{2}$ and $u = \frac{1}{2}$. Then we have

$$\begin{aligned}
&M\bigl(F(x,y), F(u,v), \phi(t)\bigr) \\
&\quad = M\left(F\left(\frac{1}{2}, y\right), F\left(\frac{1}{2}, v\right), \phi(t)\right) = \frac{\frac{t}{2}}{\frac{t}{2} + |\frac{1}{32} - \frac{1}{32}|} \\
&\quad = 1 \geq \min\left\{M\left(g\left(\frac{1}{2}\right), g\left(\frac{1}{2}\right), t\right), M\bigl(g(y), g(v), t\bigr)\right\} \quad \text{for all } y, v \in X.
\end{aligned}$$

Case 3. Let $x = \frac{1}{2}$ and $u \neq \frac{1}{2}$. Then we have:
Case 3.1. If $u \in [0, \frac{1}{8}]$, then

$$\begin{aligned}
M\bigl(F(x,y), F(u,v), \phi(t)\bigr) &= M\left(F\left(\frac{1}{2}, y\right), F(u,v), \phi(t)\right) \\
&= \frac{\frac{t}{2}}{\frac{t}{2} + |\frac{1}{32} - \frac{u}{8}|} = \frac{t}{t + |\frac{1}{16} - \frac{u}{4}|} \geq \frac{t}{t + |\frac{1}{4} - u|} \geq \frac{t}{t + |\frac{1}{2} - \frac{u}{2}|} \\
&\geq \min\left\{M\left(g\left(\frac{1}{2}\right), g(u), t\right), M\bigl(g(y), g(v), t\bigr)\right\} \quad \text{for all } y, v \in X.
\end{aligned}$$

Case 3.2. If $u \in (\frac{1}{8}, \frac{1}{4})$, then

$$\begin{aligned}
M\bigl(F(x,y), F(u,v), \phi(t)\bigr) &= M\left(F\left(\frac{1}{2}, y\right), F(u,v), \phi(t)\right) \\
&= \frac{\frac{t}{2}}{\frac{t}{2} + |\frac{1}{32} - \frac{u}{8}|} = \frac{t}{t + |\frac{1}{16} - \frac{u}{4}|} \geq \frac{t}{t + |\frac{1}{2} - u|} \\
&\geq \min\left\{M\left(g\left(\frac{1}{2}\right), g(u), t\right), M\bigl(g(y), g(v), t\bigr)\right\} \quad \text{for all } y, v \in X.
\end{aligned}$$

Case 4. $x \neq \frac{1}{2}$ and $u = \frac{1}{2}$. This case is similar to Case 3.

For $t = 1$, since $M(F(x,y),F(u,v),\phi(1)) = M(F(x,y),F(u,v),\frac{3}{2})$, by Cases 1-4 above, we can see that $M(F(x,y),F(u,v),\phi(1)) \geq \min\{M(gx,gu,1),M(gy,gv,1)\}$ for all $x,y,u,v \in X$. It is easy to see that $(0,0)$ is a coupled coincidence point of g and F. Also, g and F are w-compatible at $(0,0)$. By Theorem 3.1, we conclude that g and F have a unique common fixed point in X. Obviously, in this example, 0 is the unique common fixed point of g and F.

Since $g(x)$ is not continuous at $x = \frac{1}{8}$ and $(X,M,*)$ is not complete, Hu's Theorem 3.1 [17, Theorem 1] cannot be applied to Example 3.1.

Remark 3.1 Our results improve the ones of Sedghi *et al.* [18] as follows:

(i) from kt to $\phi(t)$;

(ii) the functions F and g are not required to be commutable.

Our results also improve the corresponding ones of Hu [17] as follows:

(a) in our Theorem 3.1, the function $\phi(t)$ is only required to satisfy the condition $\lim_{n\to\infty}\phi^n(t) = 0$ for any $t > 0$. However, the function $\phi(t)$ in Hu's result is required to satisfy the conditions (ϕ-1)-(ϕ-3);

(b) in our results, the mappings F and g are required to be weakly compatible, but in Hu's result the mappings F and g are required to be compatible.

Also, in our results the mapping g is not required to be continuous, but the condition is imposed on the mapping g in the results of Sedghi *et al.* and Hu.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

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