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Coupled coincidence point and common coupled fixed point results in cone b -metric spaces

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Abstract

In this paper, we establish common coupled fixed point and coupled fixed point theorems in cone b -metric spaces. The presented theorems extend and generalize several well-known comparable results in literature. We supply some examples to elucidate our obtained results.

Keywords: cone b -metric spaces; coupled fixed points; coupled coincidence points; common coupled fixed points

1 Introduction and preliminaries

In [1], Bakhtin introduced b -metric spaces (or metric-type spaces) as a generalization of metric spaces. He evidenced the contraction mapping principle in b -metric spaces that generalized the famous Banach contraction principle in metric spaces. From that time on, manifold papers have treated fixed point theory or the variational principle for single-valued and multi-valued operators in b -metric spaces (see [2–7] and the references therein).

Ordered normed spaces, cones and topical functions have applications in applied mathematics, for instance, in using Newton's approximation method [8–12] and optimization theory [13, 14]. In the mid-twentieth century [9], k -metric and k -normed spaces were established (see also [11, 12]) by replacing an ordered Banach space instead of the set of real numbers, as the codomain for a metric. Due to defining convergent and Cauchy sequences in terms of interior points of the underlying cone, Huang and Zhang [15] re-introduced such spaces under the name of cone metric space. Even though they used only normal cones, nonnormal cones can be used as well in such a way but by taking into the consideration that the Sandwich theorem and continuity of the metric may not hold. Some fixed point theorems for contractive-type mappings in cone metric spaces have been substantiated; for more details, see [16–25].

As a generalization of b -metric spaces and cone metric spaces, Hussain and Shah [26] announced cone b -metric spaces, which was in 2011. They built up some topological properties in such spaces and upgraded some latest results about KKM mappings in the setting of a cone b -metric space. Hussain and Shah [26] have done initial work that stimulated many authors to prove fixed point theorems, as well as common fixed point theorems for two or more mappings on cone b -metric spaces (see [27–30] and the references therein).

The following will be needed in the sequel.

Let E be a real Banach space and let θ denote the zero element in E . A cone P is a subset of E such that:

1. P is nonempty set closed and $P \neq \{\theta\}$.
2. If a, b are nonnegative real numbers and $x, y \in P$, then $ax + by \in P$.
3. $x \in P$ and $-x \in P$ implies $x = \theta$.

For any cone $P \subset E$, the partial ordering \preceq with respect to P is defined by $x \preceq y$ if and only if $y - x \in P$. The notation of $<$ stands for $x \preceq y$ but $x \neq y$. Also, we used $x \ll y$ to indicate that $y - x \in \text{int}P$, where $\text{int}P$ denotes the interior of P . A cone P is called normal if there exists a number K such that

$$\theta \preceq x \preceq y \implies \|x\| \leq K\|y\|$$

for all $x, y \in E$. The least positive number K satisfying the above condition is called the normal constant of P . Throughout this paper, we do not impose the normality condition for the cones, but the only assumption is that the cone P is solid, that is, $\text{int}P \neq \emptyset$.

Definition 1.1 [26] Let X be a nonempty set and E be a real Banach space equipped with the partial ordering \preceq with respect to the cone P . A vector-valued function $d : X \times X \rightarrow E$ is said to be a cone b -metric function on X with the constant $s \geq 1$ if the following conditions are satisfied:

1. $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$ for all $x, y \in X$,
3. $d(x, y) \preceq s(d(x, y) + d(y, z))$ for all $x, y, z \in X$.

Then the pair (X, d) is called a cone b -metric space (or a cone metric-type space); we will use the first mentioned term.

Observe that if $s = 1$ then the ordinary triangle inequality in a cone metric space is satisfied; however, it does not hold true when $s > 1$. Thus the class of tvs-cone b -metric spaces is effectively larger than that of the ordinary cone metric spaces. That is, every cone metric space is a cone b -metric space, but the converse need not be true. The following examples illustrate the above remarks.

Example 1.2 [29] Let $X = \{-1, 0, 1\}$, $E = \mathbb{R}^2$, $P = \{(x, y) : x \geq 0, y \geq 0\}$. Define $d : X \times X \rightarrow P$ by $d(x, y) = d(y, x)$ for all $x, y \in X$, $d(x, x) = \theta$, $x \in X$ and $d(-1, 0) = (3, 3)$, $d(-1, 1) = d(0, 1) = (1, 1)$. Then (X, d) is a complete cone b -metric space, but the triangle inequality is not satisfied. Indeed, we have that $d(-1, 1) + d(1, 0) = (1, 1) + (1, 1) = (2, 2) < (3, 3) = d(-1, 0)$. It is not hard to verify that $s = \frac{3}{2}$.

The following example is a modification of Example 3 from [31].

Example 1.3 Let $X = \mathbb{N} \cup \{\infty\}$, $E = \mathbb{R}^2$ and $P = \{(x, y) \in E : x \geq 0, y \geq 0\}$. Define $d : X \times X \rightarrow E$ by

$$d(x, y) = \begin{cases} (0, 0) & \text{if } x = y, \\ (|\frac{1}{x} - \frac{1}{y}|, |\frac{1}{x} - \frac{1}{y}|) & \text{if } x \text{ and } y \text{ are even or } xy = \infty, \\ (5, 5) & \text{if } x \text{ and } y \text{ are odd and } x \neq y, \\ (2, 2) & \text{otherwise.} \end{cases}$$

Then (X, d) is a cone b -metric space with the coefficient $s = 3$. But it is not a cone metric space since the triangle inequality is not satisfied. Indeed, $(5, 5) = d(1, 3) \succ d(1, 2) + d(2, 3) = (2, 2) + (2, 2) = (4, 4)$.

Definition 1.4 [26] Let (X, d) be a cone b -metric space, and let $\{x_n\}$ be a sequence in X and $x \in X$.

1. For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x) \ll c$ for all $n > N$, then x_n is said to be convergent and x is the limit of $\{x_n\}$. We denote this by $x_n \rightarrow x$.
2. For all $c \in E$ with $\theta \ll c$, if there exists a positive integer N such that $d(x_n, x_m) \ll c$ for all $n, m > N$, then $\{x_n\}$ is called a Cauchy sequence in X .
3. A cone metric space (X, d) is called complete if every Cauchy sequence in X is convergent.

The following lemma is helpful in proving our results.

Lemma 1.5 [24]

1. If E is a real Banach space with a cone P and $a \leq \lambda a$, where $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.
2. If $c \in \text{int } P$, $\theta \leq a_n$ and $a_n \rightarrow \theta$, then there exists a positive integer N such that $a_n \ll c$ for all $n \geq N$.
3. If $a \leq b$ and $b \ll c$, then $a \ll c$.
4. If $\theta \leq u \ll c$ for each $\theta \ll c$, then $u = \theta$.

Recall the following definitions.

Definition 1.6 [32] An element $(x, y) \in X^2$ is said to be a coupled fixed point of the mapping $F : X^2 \rightarrow X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 1.7 [33] An element $(x, y) \in X^2$ is called

1. a coupled coincidence point of mappings $F : X^2 \rightarrow X$ and $g : X \rightarrow X$ if $gx = F(x, y)$ and $gy = F(y, x)$, and (gx, gy) is called a coupled point of coincidence;
2. a common coupled fixed point of mappings $F : X^2 \rightarrow X$ and $g : X \rightarrow X$ if $x = gx = F(x, y)$ and $y = gy = F(y, x)$.

Definition 1.8 [25] The mappings $F : X^2 \rightarrow X$ and $g : X \rightarrow X$ are called w -compatible if $g(F(x, y)) = F(gx, gy)$ whenever $gx = F(x, y)$ and $gy = F(y, x)$.

2 Coupled coincidence point results

In this section, we prove some coupled coincidence point results in cone b -metric spaces.

Theorem 2.1 Let (X, d) be a cone b -metric space with the coefficient $s \geq 1$ relative to a solid cone P . Let $F : X^2 \rightarrow X$ and $g : X \rightarrow X$ be two mappings and suppose that there exist nonnegative constants $a_i \in [0, 1]$, $i = 1, 2, \dots, 10$, with $(s + 1)(a_1 + a_2 + a_3 + a_4) + s(s + 1)(a_5 + a_6 + a_7 + a_8) + 2s(a_9 + a_{10}) < 2$ and $\sum_{i=1}^{10} a_i < 1$ such that the following contractive condition

holds for all $x, y, u, v \in X$:

$$\begin{aligned} d(F(x, y), F(u, v)) \leq & [a_1d(gx, F(x, y)) + a_2d(gy, F(y, x))] \\ & + [a_3d(gu, F(u, v)) + a_4d(gv, F(v, u))] \\ & + [a_5d(gx, F(u, v)) + a_6d(gy, F(v, u))] \\ & + [a_7d(gu, F(x, y)) + a_8d(gv, F(y, x))] \\ & + [a_9d(gx, gu) + a_{10}d(gy, gv)]. \end{aligned}$$

If $F(X^2) \subseteq g(X)$ and $g(X)$ is a complete subspace of X , then F and g have a coupled coincidence point $(x^*, y^*) \in X^2$.

Proof Choose $x_0, y_0 \in X$. Set $gx_1 = F(x_0, y_0)$, $gy_1 = F(y_0, x_0)$, this can be done because $F(X^2) \subseteq g(X)$. Continuing this process, we obtain two sequences $\{x_n\}$, $\{y_n\}$ such that $gx_{n+1} = F(x_n, y_n)$ and $gy_{n+1} = F(y_n, x_n)$. Then we have

$$\begin{aligned} d(gx_n, gx_{n+1}) = d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ \leq [a_1d(gx_{n-1}, F(x_{n-1}, y_{n-1})) + a_2d(gy_{n-1}, F(y_{n-1}, x_{n-1}))] \\ + [a_3d(gx_n, F(x_n, y_n)) + a_4d(gy_n, F(y_n, x_n))] \\ + [a_5d(gx_{n-1}, F(x_n, y_n)) + a_6d(gy_{n-1}, F(y_n, x_n))] \\ + [a_7d(gx_n, F(x_{n-1}, y_{n-1})) + a_8d(gy_n, F(y_{n-1}, x_{n-1}))] \\ + [a_9d(gx_{n-1}, gx_n) + a_{10}d(gy_{n-1}, gy_n)]. \end{aligned}$$

So that

$$\begin{aligned} d(gx_n, gx_{n+1}) = d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ \leq [a_1d(gx_{n-1}, gx_n) + a_2d(gy_{n-1}, gy_n)] + [a_3d(gx_n, gx_{n+1}) + a_4d(gy_n, gy_{n+1})] \\ + [a_5d(gx_{n-1}, gx_{n+1}) + a_6d(gy_{n-1}, gy_{n+1})] \\ + [a_7d(gx_n, gx_n) + a_8d(gy_n, gy_n)] \\ + [a_9d(gx_{n-1}, gx_n) + a_{10}d(gy_{n-1}, gy_n)] \\ \leq [a_1d(gx_{n-1}, gx_n) + a_2d(gy_{n-1}, gy_n)] + [a_3d(gx_n, gx_{n+1}) + a_4d(gy_n, gy_{n+1})] \\ + [sa_5(d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1}))] \\ + [sa_6(d(gy_{n-1}, gy_n) + d(gy_n, gy_{n+1}))] \\ + [a_9d(gx_{n-1}, gx_n) + a_{10}d(gy_{n-1}, gy_n)]. \end{aligned}$$

Hence,

$$\begin{aligned} d(gx_n, gx_{n+1}) \leq & [(a_1 + a_5s + a_9)d(gx_{n-1}, gx_n) + (a_2 + a_6s + a_{10})d(gy_{n-1}, gy_n)] \\ & + [(a_3 + a_5s)d(gx_n, gx_{n+1}) + (a_4 + a_6s)d(gy_n, gy_{n+1})]. \end{aligned} \tag{2.1}$$

Similarly, we can prove that

$$d(gy_n, gy_{n+1}) \leq [(a_1 + sa_5 + a_9)d(gy_{n-1}, gy_n) + (a_2 + sa_6 + a_{10})d(gx_{n-1}, gx_n)] + [(a_3 + sa_5)d(gy_n, gy_{n+1}) + (a_4 + sa_6)d(gx_n, gx_{n+1})]. \tag{2.2}$$

Put

$$d_n = d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}).$$

Adding inequalities (2.1) and (2.2), one can assert that

$$d_n \leq (a_1 + a_2 + sa_5 + sa_6 + a_9 + a_{10})d_{n-1} + (a_3 + a_4 + sa_5 + sa_6)d_n. \tag{2.3}$$

On the other hand, we have

$$\begin{aligned} d(gx_{n+1}, gx_n) &= d(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\ &\leq [a_1d(gx_n, F(x_n, y_n)) + a_2d(gy_n, F(y_n, x_n))] \\ &\quad + [a_3d(gx_{n-1}, F(x_{n-1}, y_{n-1})) + a_4d(gy_{n-1}, F(y_{n-1}, x_{n-1}))] \\ &\quad + [a_5d(gx_n, F(x_{n-1}, y_{n-1})) + a_6d(gy_n, F(y_{n-1}, x_{n-1}))] \\ &\quad + [a_7d(gx_{n-1}, F(x_n, y_n)) + a_8d(gy_{n-1}, F(y_n, x_n))] \\ &\quad + [a_9d(gx_n, gx_{n-1}) + a_{10}d(gy_n, gy_{n-1})]. \end{aligned}$$

So that

$$\begin{aligned} d(gx_{n+1}, gx_n) &= d(F(x_n, y_n), F(x_{n-1}, y_{n-1})) + [a_1d(gx_n, gx_{n+1}) + a_2d(gy_n, gy_{n+1})] \\ &\leq [a_3d(gx_{n-1}, gx_n) + a_4d(gy_{n-1}, gy_n)] \\ &\quad + [a_5d(gx_n, gx_n) + a_6d(gy_n, gy_n)] \\ &\quad + [a_7d(gx_{n-1}, gx_{n+1}) + a_8d(gy_{n-1}, gy_{n+1})] \\ &\quad + [a_9d(gx_n, gx_{n-1}) + a_{10}d(gy_n, gy_{n-1})] \\ &\leq [a_1d(gx_n, gx_{n+1}) + a_2d(gy_n, gy_{n+1})] \\ &\quad + [a_3d(gx_{n-1}, gx_n) + a_4d(gy_{n-1}, gy_n)] \\ &\quad + [sa_7(d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1}))] \\ &\quad + [sa_8(d(gy_{n-1}, gy_n) + d(gy_n, gy_{n+1}))] \\ &\quad + [a_9d(gx_{n-1}, gx_n) + a_{10}d(gy_{n-1}, gy_n)]. \end{aligned}$$

Hence,

$$d(gx_{n+1}, gx_n) \leq [(a_3 + sa_7 + a_9)d(gx_{n-1}, gx_n) + (a_4 + sa_8 + a_{10})d(gy_{n-1}, gy_n)] + [(a_1 + sa_7)d(gx_n, gx_{n+1}) + (a_2 + sa_8)d(gy_n, gy_{n+1})]. \tag{2.4}$$

Similarly,

$$d(gy_{n+1}, gy_n) \leq [(a_3 + sa_7 + a_9)d(gy_{n-1}, gy_n) + (a_4 + sa_8 + a_{10})d(gx_{n-1}, gx_n)] + [(a_1 + sa_7)d(gy_n, gy_{n+1}) + (a_2 + sa_8)d(gx_n, gx_{n+1})]. \tag{2.5}$$

Adding inequalities (2.4) and (2.5), one can assert that

$$d_n \leq (a_3 + a_4 + sa_7 + sa_8 + a_9 + a_{10})d_{n-1} + (a_1 + a_2 + sa_7 + sa_8)d_n. \tag{2.6}$$

Finally, from (2.3) and (2.6), we have

$$2d_n \leq (a_1 + a_2 + a_3 + a_4 + sa_5 + sa_6 + sa_7 + sa_8 + 2(a_9 + a_{10}))d_{n-1} + (a_1 + a_2 + a_3 + a_4 + sa_5 + sa_6 + sa_7 + sa_8)d_n,$$

that is,

$$d_n \leq hd_{n-1},$$

where $h = \frac{(a_1+a_2+a_3+a_4+sa_5+sa_6+sa_7+sa_8+2(a_9+a_{10}))}{2-(a_1+a_2+a_3+a_4+sa_5+sa_6+sa_7+sa_8)} < \frac{1}{s}$.

Consequently, we have

$$\begin{aligned} d_n &\leq hd_{n-1} \\ &\leq h^2 d_{n-2} \\ &\leq h^3 d_{n-3} \\ &\vdots \\ &\leq h^n d_0. \end{aligned} \tag{2.7}$$

Let $m > n \geq 1$. It follows that

$$d(gx_n, gx_m) \leq sd(gx_n, gx_{n+1}) + s^2 d(gx_{n+1}, gx_{n+2}) + \dots + s^{m-n} d(gx_{m-1}, gx_m)$$

and

$$d(gy_n, gy_m) \leq sd(gy_n, gy_{n+1}) + s^2 d(gy_{n+1}, gy_{n+2}) + \dots + s^{m-n} d(gy_{m-1}, gy_m).$$

Now, (2.7) and $sh < 1$ imply that

$$\begin{aligned} d(gx_n, gx_m) + d(gy_n, gy_m) &\leq sd_n + s^2 d_{n+1} + \dots + s^{m-n} d_{m-1} \\ &\leq sh^n d_0 + s^2 h^{n+1} d_0 + \dots + s^{m-n} h^{m-1} d_0 \\ &= (sh^n + s^2 h^{n+1} + \dots + s^{m-n} h^{m-1}) d_0 \\ &= sh^n (1 + sh + (sh)^2 + \dots + (sh)^{m-n-1}) d_0 \\ &\leq \frac{sh^n}{1 - sh} d_0 \rightarrow \theta \quad \text{as } n \rightarrow \infty. \end{aligned} \tag{2.8}$$

According to Lemma 1.5(2), and for any $c \in E$ with $c \gg \theta$, there exists $N_0 \in \mathbb{N}$ such that for any $n > N_0$, $\frac{h^n}{1-h}d_0 \ll c$. Furthermore, from (2.8) and for any $m > n > N_0$, Lemma 1.5(3) shows that

$$d(gx_n, gx_m) + d(gy_n, gy_m) \ll c,$$

which implies that

$$d(gx_n, gx_m) \ll c$$

and

$$d(gy_n, gy_m) \ll c.$$

Hence, by Definition 1.4(2), $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$. Since $g(X)$ is complete, there exist x^* and $y^* \in X$ such that $gx_n \rightarrow gx^*$ and $gy_n \rightarrow gy^*$ as $n \rightarrow \infty$.

On the other hand,

$$\begin{aligned} d(F(x^*, y^*), gx^*) &\leq s(d(F(x^*, y^*), gx_{n+1}) + d(gx_{n+1}, gx^*)) \\ &= s(d(F(x^*, y^*), F(x_n, y_n)) + d(gx_{n+1}, gx^*)) \\ &\leq s[a_1d(gx^*, F(x^*, y^*)) + a_2d(gy^*, F(y^*, x^*))] \\ &\quad + s[a_3d(gx_n, F(x_n, y_n)) + a_4d(gy_n, F(y_n, x_n))] \\ &\quad + s[a_5d(gx^*, F(x_n, y_n)) + a_6d(gy^*, F(y_n, x_n))] \\ &\quad + s[a_7d(gx_n, F(x^*, y^*)) + a_8d(gy_n, F(y^*, x^*))] \\ &\quad + s[a_9d(gx^*, gx_n) + a_{10}d(gy^*, gy_n)] + sd(gx_{n+1}, gx^*) \\ &\leq s[a_1d(F(x^*, y^*), gx^*) + a_2d(F(y^*, x^*), gy^*)] \\ &\quad + s[sa_3d(gx_n, gx^*) + sa_3d(gx^*, gx_{n+1}) \\ &\quad + sa_4d(gy_n, gy^*) + sa_4d(gy^*, gy_{n+1})] \\ &\quad + s[a_5d(gx^*, gx_{n+1}) + a_6d(gy^*, gy_{n+1})] \\ &\quad + s[sa_7d(gx_n, gx^*) + sa_7d(gx^*, F(x^*, y^*)) \\ &\quad + sa_8d(gy_n, gy^*) + sa_8d(gy^*, F(y^*, x^*))] \\ &\quad + s[a_9d(gx^*, gx_n) + a_{10}d(gy^*, gy_n)] + sd(gx_{n+1}, gx^*) \\ &= s[a_1d(F(x^*, y^*), gx^*) + a_2d(F(y^*, x^*), gy^*)] \\ &\quad + s[sa_3d(gx_n, gx^*) + sa_3d(gx_{n+1}, gx^*) \\ &\quad + sa_4d(gy_n, gy^*) + sa_4d(gy_{n+1}, gy^*)] \\ &\quad + s[a_5d(gx_{n+1}, gx^*) + a_6d(gy_{n+1}, gy^*)] \\ &\quad + s[sa_7d(gx_n, gx^*) + sa_7d(F(x^*, y^*), gx^*) \\ &\quad + sa_8d(gy_n, gy^*) + sa_8d(F(y^*, x^*), gy^*)] \\ &\quad + s[a_9d(gx_n, gx^*) + a_{10}d(gy_n, gy^*)] + sd(gx_{n+1}, gx^*). \end{aligned}$$

Hence,

$$\begin{aligned} d(F(x^*, y^*), gx^*) &\leq (sa_1 + s^2a_7)d(F(x^*, y^*), gx^*) + (a_2 + s^2a_8)d(F(y^*, x^*), gy^*) \\ &\quad + (s^2a_3 + s^2a_7 + sa_9)d(gx_n, gx^*) + (s^2a_3 + sa_5 + s)d(gx_{n+1}, gx^*) \\ &\quad + (s^2a_4 + s^2a_8 + sa_{10})d(gy_n, gy^*) + (s^2a_4 + sa_6)d(gy_{n+1}, gy^*). \end{aligned}$$

Similarly,

$$\begin{aligned} d(F(y^*, x^*), gy^*) &\leq (sa_1 + s^2a_7)d(F(y^*, x^*), gy^*) + (a_2 + s^2a_8)d(F(x^*, y^*), gx^*) \\ &\quad + (s^2a_3 + s^2a_7 + sa_9)d(gy_n, gy^*) + (s^2a_3 + sa_5 + s)d(gy_{n+1}, gy^*) \\ &\quad + (s^2a_4 + s^2a_8 + sa_{10})d(gx_n, gx^*) + (s^2a_4 + sa_6)d(gx_{n+1}, gx^*). \end{aligned}$$

Put

$$\delta = d(F(x^*, y^*), gx^*) + d(F(y^*, x^*), gy^*).$$

Adding the above inequalities, we get

$$\begin{aligned} \delta &\leq (sa_1 + s^2a_7 + a_2 + s^2a_8)\delta \\ &\quad + (s^2a_3 + s^2a_7 + sa_9 + s^2a_4 + s^2a_8 + sa_{10})d(gx_n, gx^*) \\ &\quad + (s^2a_3 + s^2a_7 + sa_9 + s^2a_4 + s^2a_8 + sa_{10})d(gy_n, gy^*) \\ &\quad + (s^2a_3 + sa_5 + s + s^2a_4 + sa_6)d(gx_{n+1}, gx^*) \\ &\quad + (s^2a_3 + sa_5 + s + s^2a_4 + sa_6)d(gy_{n+1}, gy^*). \end{aligned}$$

Then

$$\delta \leq \frac{B}{1-A}d(gx_n, gx^*) + \frac{B}{1-A}d(gy_n, gy^*) + \frac{C}{1-A}d(gx_{n+1}, gx^*) + \frac{C}{1-A}d(gy_{n+1}, gy^*),$$

where $A = sa_1 + s^2a_7 + a_2 + s^2a_8$, $B = s^2a_3 + s^2a_7 + sa_9 + s^2a_4 + s^2a_8 + sa_{10}$ and $C = s^2a_3 + sa_5 + s + s^2a_4 + sa_6$. Since $gx_n \rightarrow gx^*$ and $gy_n \rightarrow gy^*$ as $n \rightarrow \infty$, then by Definition 1.4(1) and for $c \gg \theta$, there exists $N_0 \in \mathbb{N}$ such that for all $n > N_0$, $d(gx_n, gx^*) \ll c \frac{1-A}{4B}$, $d(gy_n, gy^*) \ll c \frac{1-A}{4B}$, $d(gx_{n+1}, gx^*) \ll c \frac{1-A}{4C}$ and $d(gy_{n+1}, gy^*) \ll c \frac{1-A}{4C}$. Hence,

$$\begin{aligned} \delta &\leq \frac{B}{A}d(gx_n, gx^*) + \frac{B}{A}d(gy_n, gy^*) + \frac{B}{C}d(gx_{n+1}, gx^*) + \frac{B}{C}d(gy_{n+1}, gy^*) \\ &\ll c \frac{1-A}{4B} \frac{B}{1-A} + c \frac{1-A}{4B} \frac{B}{1-A} + c \frac{1-A}{4C} \frac{C}{1-A} + c \frac{1-A}{4C} \frac{C}{1-A} \\ &= \frac{c}{4} + \frac{c}{4} + \frac{c}{4} + \frac{c}{4} \\ &= c. \end{aligned}$$

Now, according to Lemma 1.5(4), it follows that $\delta = \theta$, that is, $(F(x^*, y^*), gx^*) + d(F(y^*, x^*), gy^*) = \theta$, which implies that $(F(x^*, y^*), gx^*) = \theta$ and $d(F(y^*, x^*), gy^*) = \theta$. Hence, $gx^* = F(x^*, y^*)$ and $gy^* = F(y^*, x^*)$. Therefore (x^*, y^*) is a coupled coincidence point of F and g . \square

Remark 2.2 Theorem 2.1 extends and generalizes Theorem 2.4 of Abbas *et al.* [25] to cone b -metric spaces.

Corollary 2.3 Let (X, d) be a cone b -metric space with the coefficient $s \geq 1$ relative to a solid cone P . Let $F : X^2 \rightarrow X$ and $g : X \rightarrow X$ be two mappings and suppose that there exist nonnegative constants k, l with $k + l < \frac{1}{s}$ such that the following contractive condition holds for all $x, y, u, v \in X$:

$$d(F(x, y), F(u, v)) \leq kd(gx, gu) + ld(gy, gv).$$

If $F(X^2) \subseteq g(X)$ and $g(X)$ is a complete subspace of X , then F and g have a coupled coincidence point $(x^*, y^*) \in X^2$.

Corollary 2.4 Let (X, d) be a cone b -metric space with the coefficient $s \geq 1$ relative to a solid cone P . Let $F : X^2 \rightarrow X$ and $g : X \rightarrow X$ be two mappings and suppose that there exist nonnegative constants k, l with $k + l < \frac{2}{s+1}$ such that the following contractive condition holds for all $x, y, u, v \in X$:

$$d(F(x, y), F(u, v)) \leq kd(gx, F(x, y)) + ld(gu, F(u, v)).$$

If $F(X^2) \subseteq g(X)$ and $g(X)$ is a complete subspace of X , then F and g have a coupled coincidence point $(x^*, y^*) \in X^2$.

Corollary 2.5 Let (X, d) be a cone b -metric space with the coefficient $s \geq 1$ relative to a solid cone P . Let $F : X^2 \rightarrow X$ and $g : X \rightarrow X$ be two mappings and suppose that there exist nonnegative constants k, l with $k + l < \frac{2}{s(s+1)}$ such that the following contractive condition holds for all $x, y, u, v \in X$:

$$d(F(x, y), F(u, v)) \leq kd(gx, F(u, v)) + ld(gu, F(x, y)).$$

If $F(X^2) \subseteq g(X)$ and $g(X)$ is a complete subspace of X , then F and g have a coupled coincidence point $(x^*, y^*) \in X^2$.

Remark 2.6 All of the coupled coincidence point results may be proved for a partially ordered cone b -metric space (X, \sqsubseteq, d) by inserting well-known conditions like

- (1) F has the mixed g -monotone property and g is b -continuous;
- and either
- (a) F is b -continuous, or
 - (b) (X, d) is regular.

3 Common coupled fixed point results

The conditions of Theorem 2.1 are not enough to prove the existence of a common coupled fixed point for the mappings F and g . By restricting to w -compatibility for F and g , we obtain the following theorem.

Theorem 3.1 In addition to the hypotheses of Theorem 2.1, if F and g are w -compatible, then F and g have a unique common coupled fixed point. Moreover, a common coupled fixed point of F and g is of the form (u, u) for some $u \in X$.

Proof From Theorem 2.1, F and g have a coupled coincidence point (x^*, y^*) . Then (gx^*, gy^*) is a coupled point of coincidence of F and g such that $gx^* = F(x^*, y^*)$ and $gy^* = F(y^*, x^*)$. First, we will show that the coupled point of coincidence is unique. Suppose that F and g have another coupled point of coincidence (gx', gy') such that $gx' = F(x', y')$ and $gy' = F(y', x')$, where $(x', y') \in X^2$. Then we have

$$\begin{aligned} d(gx^*, gx') &= d(F(x^*, y^*), F(x', y')) \\ &\leq [a_1 d(gx^*, F(x^*, y^*)) + a_2 d(gy^*, F(y^*, x^*))] \\ &\quad + [a_3 d(gx', F(x', y')) + a_4 d(gy', F(y', x'))] \\ &\quad + [a_5 d(gx^*, F(x', y')) + a_6 d(gy^*, F(y', x'))] \\ &\quad + [a_7 d(gx', F(x^*, y^*)) + a_8 d(gy', F(y^*, x^*))] \\ &\quad + [a_9 d(gx^*, gx') + a_{10} d(gy^*, gy')] \\ &= [a_1 d(gx^*, gx^*) + a_2 d(gy^*, gy^*)] \\ &\quad + [a_3 d(gx', gx') + a_4 d(gy', gy')] \\ &\quad + [a_5 d(gx^*, gx') + a_6 d(gy^*, gy')] \\ &\quad + [a_7 d(gx', gx^*) + a_8 d(gy', gy^*)] \\ &\quad + [a_9 d(gx^*, gx') + a_{10} d(gy^*, gy')] \\ &= [a_5 d(gx^*, gx') + a_6 d(gy^*, gy')] \\ &\quad + [a_7 d(gx^*, gx') + a_8 d(gy^*, gy')] \\ &\quad + [a_9 d(gx^*, gx') + a_{10} d(gy^*, gy')]. \end{aligned}$$

Hence,

$$d(gx^*, gx') \leq (a_5 + a_7 + a_9)d(gx^*, gx') + (a_6 + a_8 + a_{10})d(gy^*, gy'). \tag{3.1}$$

By a similar way, we can show that

$$d(gy^*, gy') \leq (a_5 + a_7 + a_9)d(gy^*, gy') + (a_6 + a_8 + a_{10})d(gx^*, gx'). \tag{3.2}$$

By adding inequalities (3.1) and (3.2), we get

$$d(gx^*, gx') + d(gy^*, gy') \leq (a_5 + a_6 + a_7 + a_8 + a_9 + a_{10})(d(gx^*, gx') + d(gy^*, gy')).$$

Since $(a_5 + a_6 + a_7 + a_8 + a_9 + a_{10}) < 1$, Lemma 1.5(1) shows that $d(gx^*, gx') + d(gy^*, gy') = \theta$. But $d(gx^*, gx') \geq \theta$ and $d(gy^*, gy') \geq \theta$. Hence, $d(gx^*, gx') = \theta$ and $d(gy^*, gy') = \theta$, that is,

$$gx^* = gx' \quad \text{and} \quad gy^* = gy', \tag{3.3}$$

which implies the uniqueness of the coupled point of coincidence of F and g , that is, (gx^*, gy^*) . By a similar way, we can prove that

$$gx^* = gy' \quad \text{and} \quad gy^* = gx'. \tag{3.4}$$

In view of (3.3) and (3.4), one can assert that

$$gx^* = gy^*. \tag{3.5}$$

That is, the unique coupled point of coincidence of F and g is (gx^*, gx^*) .

Now, let $u = gx^* = F(x^*, y^*)$. Since F and g are w -compatible, then we have

$$gu = g(gx^*) = gF(x^*, y^*) = F(gx^*, gy^*) = F(gx^*, gx^*) = F(u, u).$$

Then (gu, gu) is a coupled point of coincidence, and also we have (u, u) is a coupled point of coincidence. The uniqueness of the coupled point of coincidence implies that $gu = u$. Therefore $u = gu = F(u, u)$. Hence, (u, u) is the unique common coupled fixed point of F and g . This completes the proof. \square

Now, we present one example to illustrate our results.

Example 3.2 Let $X = [0, 1]$ and $E = C_{\mathbb{R}}^1[0, 1]$ with $\|u\| = \|u\|_{\infty} + \|u'\|_{\infty}$, $u \in E$ and let $P = \{u \in E : u(t) \geq 0 \text{ on } [0, 1]\}$. It is well known that this cone is solid but it is not normal. Define a cone b -metric $d : X \times X \rightarrow E$ by $d(x, y)(t) = |x - y|^2 e^t$. Then (X, d) is a complete cone b -metric space with the coefficient $s = 2$. Let us define $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ as $F(x, y) = \frac{1}{9}x + \frac{1}{7}y$ and $g(x) = \frac{1}{2}x$ for all $x \in X$. Now we obtain that

$$\begin{aligned} d(F(x, y), F(u, v))(t) &= \left| \frac{1}{9}x + \frac{1}{7}y - \frac{1}{9}u - \frac{1}{7}v \right|^2 e^t \\ &= \left| \left(\frac{1}{9}x - \frac{1}{9}u \right) + \left(\frac{1}{7}y - \frac{1}{7}v \right) \right|^2 e^t \\ &= \left| \frac{2}{9} \left(\frac{1}{2}x - \frac{1}{2}u \right) + \frac{2}{7} \left(\frac{1}{2}y - \frac{1}{2}v \right) \right|^2 e^t \\ &\leq 2 \left(\left| \frac{2}{9} \left(\frac{1}{2}x - \frac{1}{2}u \right) \right|^2 e^t + \left| \frac{2}{7} \left(\frac{1}{2}y - \frac{1}{2}v \right) \right|^2 e^t \right) \\ &= \frac{8}{81} \left| \frac{1}{2}x - \frac{1}{2}u \right|^2 e^t + \frac{8}{49} \left| \frac{1}{2}y - \frac{1}{2}v \right|^2 e^t \\ &= \frac{8}{81} d(gx, gu)(t) + \frac{8}{49} d(gy, gv)(t), \end{aligned}$$

where $a_9 = \frac{8}{81}$, $a_{10} = \frac{8}{49}$, $a_i = 0$, $i = 1, 2, \dots, 8$. Note that $2s(a_9 + a_{10}) = 4(\frac{8}{81} + \frac{8}{49}) < 2$, $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X . Hence, the conditions of Theorem 2.1 are satisfied, that is, F and g have a coupled coincidence point $(0, 0)$. Also, F and g are w -compatible at $(0, 0)$. Hence, Theorem 3.1 shows that $(0, 0)$ is the unique common coupled fixed point of F and g .

Finally, we have the following result (immediate consequence of Theorems 2.1 and 3.1).

Theorem 3.3 Let (X, d) be a complete cone b -metric space with the coefficient $s \geq 1$ relative to a solid cone P . Let $F : X^2 \rightarrow X$ be a mapping and suppose that there exist nonnegative constants $a_i \in [0, 1]$, $i = 1, 2, \dots, 10$, with $(s + 1)(a_1 + a_2 + a_3 + a_4) + s(s + 1)(a_5 + a_6 + a_7 +$

$a_8) + 2s(a_9 + a_{10}) < 2$ and $\sum_{i=1}^{10} a_i < 1$ such that the following contractive condition holds for all $x, y, u, v \in X$:

$$\begin{aligned} d(F(x, y), F(u, v)) \leq & [a_1 d(x, F(x, y)) + a_2 d(y, F(y, x))] \\ & + [a_3 d(u, F(u, v)) + a_4 d(v, F(v, u))] \\ & + [a_5 d(x, F(u, v)) + a_6 d(y, F(v, u))] \\ & + [a_7 d(u, F(x, y)) + a_8 d(v, F(y, x))] \\ & + [a_9 d(x, u) + a_{10} d(y, v)]. \end{aligned}$$

Then F has a coupled fixed point $(x^*, y^*) \in X^2$. Moreover, the coupled fixed point is unique and of the form (x^*, x^*) for some $x^* \in X$.

As consequences of Theorem 3.3, we have the following results which are the extension of main results of Sabetghadam *et al.* [34] to cone b -metric spaces.

Corollary 3.4 *Let (X, d) be a complete cone b -metric space with the coefficient $s \geq 1$ relative to a solid cone P . Let $F : X^2 \rightarrow X$ be a mapping and suppose that there exist nonnegative constants k, l with $k + l < \frac{1}{s}$ such that the following contractive condition holds for all $x, y, u, v \in X$:*

$$d(F(x, y), F(u, v)) \leq kd(x, u) + ld(y, v).$$

Then F has a coupled fixed point $(x^*, y^*) \in X^2$. Moreover, the coupled fixed point is unique and of the form (x^*, x^*) for some $x^* \in X$.

Corollary 3.5 *Let (X, d) be a complete cone b -metric space with the coefficient $s \geq 1$ relative to a solid cone P . Let $F : X^2 \rightarrow X$ be a mapping and suppose that there exist nonnegative constants k, l with $k + l < \frac{2}{s+1}$ such that the following contractive condition holds for all $x, y, u, v \in X$:*

$$d(F(x, y), F(u, v)) \leq kd(x, F(x, y)) + ld(u, F(u, v)).$$

Then F has a coupled fixed point $(x^*, y^*) \in X^2$. Moreover, the coupled fixed point is unique and of the form (x^*, x^*) for some $x^* \in X$.

Corollary 3.6 *Let (X, d) be a complete cone b -metric space with the coefficient $s \geq 1$ relative to a solid cone P . Let $F : X^2 \rightarrow X$ be a mapping and suppose that there exist nonnegative constants k, l with $k + l < \frac{2}{s(s+1)}$ such that the following contractive condition holds for all $x, y, u, v \in X$:*

$$d(F(x, y), F(u, v)) \leq kd(x, F(u, v)) + ld(u, F(x, y)).$$

Then F has a coupled fixed point $(x^*, y^*) \in X^2$. Moreover, the coupled fixed point is unique and of the form (x^*, x^*) for some $x^* \in X$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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