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On solutions of a system of variational inequalities and fixed point problems in Banach spaces

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Abstract
In this paper, considering the problem of solving a system of variational inequalities and a common fixed point problem of an infinite family of nonexpansive mappings in Banach spaces, we propose a two-step relaxed extragradient method which is based on Korpelevich's extragradient method and viscosity approximation method. Strong convergence results are established.
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1 Introduction

In the last three decades, the theory of variational inequalities has been used as a tool to study the Nash equilibrium problem for a finite or infinite number of players; see, for example, [1–6] and the references therein. There are two ways to study the Nash equilibrium problem by using variational inequality technique: (1) system of variational inequalities; (2) variational inequalities defined over the product of sets. If the number of players is finite, then the system of variational inequalities is equivalent to the variational inequality defined over the product of sets; see, for example, [7, 8] and the references therein.

Very recently, Cai and Bu [9] considered the following system of two variational inequalities in the setting of Banach spaces.

Let C be a nonempty, closed and convex subset of a real Banach space X, let $B_1, B_2 : C \to X$ be two nonlinear mappings and μ_1 and μ_2 be two positive constants. The problem of system of variational inequalities (SVI) [9] is to find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B_1 y^* + x^* - y^*, J(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle \mu_2 B_2 x^* + y^* - x^*, J(x - y^*)\rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.1}$$

where J is the normalized duality mapping. The set of solutions of GSVI (1.1) is denoted by GSVI(C, B_1, B_2). This system could be useful to study the Nash equilibrium problem for two players. They proposed an iterative scheme to compute the approximate solutions of such a system.

In particular, if $X = H$, a real Hilbert space, then GSVI (1.1) reduces to the following problem of a system of variational inequalities of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \mu_2 B_2 x^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.2}$$

where μ_1 and μ_2 are two positive constants. The set of solutions of problem (1.2) is still denoted by $\mathrm{GSVI}(C, B_1, B_2)$.

In this paper, we introduce two-step relaxed extragradient method for solving SVI (1.1) and the common fixed point problem of an infinite family $\{S_n\}$ of nonexpansive mappings of C into itself. Here, the two-step relaxed extragradient method is based on Korpelevich's extragradient method [10] and viscosity approximation method. We first suggest and analyze an implicit iterative algorithm by the two-step relaxed extragradient method in a uniformly convex and 2-uniformly smooth Banach space X, and then another explicit iterative algorithm in a uniformly convex Banach space X with a uniformly Gâteaux differentiable norm. On the other hand, we also propose and analyze a composite explicit iterative algorithm by the two-step relaxed extragradient method for solving SVI (1.1) and the common fixed point problem of $\{S_n\}$ in a uniformly convex and 2-uniformly smooth Banach space. The results presented in this paper improve, extend, supplement and develop the corresponding results that have appeared very recently in the literature.

2 Preliminaries

Let X^* be the dual of X. The normalized duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in X,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

Let C be a nonempty closed convex subset of a real Banach space X. A mapping $A : C \to X$ is said to be accretive if for each $x, y \in C$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq 0,$$

where J is the normalized duality mapping. A is said to be α-strongly accretive if for each $x, y \in C$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq \alpha \|x - y\|^2$$

for some $\alpha \in (0, 1)$. It is said to be β-inverse-strongly-accretive if for each $x, y \in C$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \geq \beta \|Ax - Ay\|^2$$

for some $\beta > 0$; and finally A is said to be λ-strictly pseudocontractive if for each $x, y \in C$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y) \rangle \leq \|x - y\|^2 - \lambda \|x - y - (Ax - Ay)\|^2$$

for some $\lambda \in (0, 1)$.

Let D be a subset of C and let Π be a mapping of C into D. Then Π is said to be sunny if

$$\Pi\big[\Pi(x) + t\big(x - \Pi(x)\big)\big] = \Pi(x),$$

whenever $\Pi(x) + t(x - \Pi(x)) \in C$ for $x \in C$ and $t \geq 0$. A mapping Π of C into itself is called a retraction if $\Pi^2 = \Pi$. If a mapping Π of C into itself is a retraction, then $\Pi(z) = z$ for every $z \in R(\Pi)$, where $R(\Pi)$ is the range of Π. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D.

It is well known that if $X = H$, a Hilbert space, then a sunny nonexpansive retraction Π_C is coincident with the metric projection from X onto C; that is, $\Pi_C = P_C$. If C is a nonempty closed convex subset of a strictly convex and uniformly smooth Banach space X and if $T : C \to C$ is a nonexpansive mapping with the fixed point set $\mathrm{Fix}(T) \neq \emptyset$, then the set $\mathrm{Fix}(T)$ is a sunny nonexpansive retract of C.

The following lemma concerns the sunny nonexpansive retraction.

Lemma 2.1 (see [11]) *Let C be a nonempty closed convex subset of a real smooth Banach space X. Let D be a nonempty subset of C. Let Π be a retraction of C onto D and let J be a normalized duality mapping on X. Then the following are equivalent*:

(i) *Π is sunny and nonexpansive*;
(ii) $\|\Pi(x) - \Pi(y)\|^2 \leq \langle x - y, J(\Pi(x) - \Pi(y))\rangle$, $\forall x, y \in C$;
(iii) $\langle x - \Pi(x), J(y - \Pi(x))\rangle \leq 0$, $\forall x \in C$, $y \in D$.

Next, we present some more lemmas which are crucial for the proofs of our results.

Lemma 2.2 (see [12]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n\beta_n + \gamma_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the conditions:

(i) $\{\alpha_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\limsup_{n\to\infty} \beta_n \leq 0$;
(iii) $\gamma_n \geq 0$, $\forall n \geq 0$, *and* $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then $\limsup_{n\to\infty} s_n = 0$.

Lemma 2.3 (see [12]) *In a smooth Banach space X, the following inequality holds*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\big\langle y, J(x + y)\big\rangle, \quad \forall x, y \in X.$$

Lemma 2.4 (see [13]) *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\alpha_n\}$ be a sequence in* $[0,1]$ *which satisfies the following condition*:

$$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1.$$

Suppose $x_{n+1} = \alpha_n x_n + (1 - \alpha_n)z_n$, $\forall n \geq 0$ *and* $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. *Then* $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Lemma 2.5 (see [14]) *Given a number $r > 0$. A real Banach space X is uniformly convex if and only if there exists a continuous strictly increasing function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$, such that*

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|)$$

for all $\lambda \in [0,1]$ and $x, y \in X$ such that $\|x\| \leq r$ and $\|y\| \leq r$.

Lemma 2.6 (see [15]) *Let C be a nonempty closed convex subset of a Banach space X. Let $S_0, S_1, \ldots$ be a sequence of mappings of C into itself. Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_n x - S_{n-1}x\| : x \in C\} < \infty$. Then, for each $y \in C$, $\{S_n y\}$ converges strongly to some point of C. Moreover, let S be a mapping of C into itself defined by $Sy = \lim_{n\to\infty} S_n y$ for all $y \in C$. Then $\lim_{n\to\infty} \sup\{\|Sx - S_n x\| : x \in C\} = 0$.*

Let C be a nonempty closed convex subset of a Banach space X and let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. As previously, let Ξ_C be a set of all contractions on C. For $t \in (0,1)$ and $f \in \Xi_C$, let $x_t \in C$ be a unique fixed point of the contraction $x \mapsto tf(x) + (1-t)Tx$ on C; that is,

$$x_t = tf(x_t) + (1-t)Tx_t.$$

Lemma 2.7 (see [16]) *Let X be a uniformly smooth Banach space, or a reflexive and strictly convex Banach space with a uniformly Gateaux differentiable norm. Let C be a nonempty closed convex subset of X, let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$, and $f \in \Xi_C$. Then the net $\{x_t\}$ defined by $x_t = tf(x_t) + (1-t)Tx_t$ converges strongly to a point in $\mathrm{Fix}(T)$. If we define a mapping $Q : \Xi_C \to \mathrm{Fix}(T)$ by $Q(f) := s-\lim_{t\to 0} x_t$, $\forall f \in \Xi_C$, then $Q(f)$ solves the VIP:*

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \leq 0, \quad \forall f \in \Xi_C, p \in \mathrm{Fix}(T).$$

Lemma 2.8 (see [17]) *Let C be a nonempty closed convex subset of a strictly convex Banach space X. Let $\{T_n\}_{n=0}^{\infty}$ be a sequence of nonexpansive mappings on C. Suppose $\bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=0}^{\infty} \lambda_n = 1$. Then a mapping S on C defined by $Sx = \sum_{n=0}^{\infty} \lambda_n T_n x$ for $x \in C$ is well defined, nonexpansive and $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n)$ holds.*

Lemma 2.9 (see [12]) *Let C be a nonempty closed convex subset of a smooth Banach space X and let the mapping $B_i : C \to X$ be λ_i-strictly pseudocontractive and α_i-strongly accretive with $\alpha_i + \lambda_i \geq 1$ for $i = 1, 2$. Then, for $\mu_i \in (0,1]$, we have*

$$\|(I-\mu_i B_i)x - (I-\mu_i B_i)y\| \leq \left\{\sqrt{\frac{1-\alpha_i}{\lambda_i}} + (1-\mu_i)\left(1+\frac{1}{\lambda_i}\right)\right\}\|x-y\|, \quad \forall x, y \in C,$$

for $i = 1, 2$. In particular, if $1 - \frac{\lambda_i}{1+\lambda_i}(1 - \sqrt{\frac{1-\alpha_i}{\lambda_i}}) \leq \mu_i \leq 1$, then $I - \mu_i B_i$ is nonexpansive for $i = 1, 2$.

Lemma 2.10 (see [9]) *Let C be a nonempty closed convex subset of a smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C and let the mapping*

$B_i : C \to X$ be λ_i-strictly pseudocontractive and α_i-strongly accretive with $\alpha_i + \lambda_i \geq 1$ for $i = 1, 2$. Let $G : C \to C$ be a mapping defined by

$$G(x) = \Pi_C\bigl[\Pi_C(x - \mu_2 B_2 x) - \mu_1 B_1 \Pi_C(x - \mu_2 B_2 x)\bigr], \quad \forall x \in C.$$

If $1 - \frac{\lambda_i}{1+\lambda_i}(1 - \sqrt{\frac{1-\alpha_i}{\lambda_i}}) \leq \mu_i \leq 1$, then $G : C \to C$ is nonexpansive.

Lemma 2.11 (see [9]) *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X. Let the mapping $B_i : C \to X$ be α_i-inverse-strongly accretive. Then we have*

$$\bigl\|(I - \mu_i B_i)x - (I - \mu_i B_i)y\bigr\| \leq \|x - y\|^2 + 2\mu_i\bigl(\mu_i \kappa^2 - \alpha_i\bigr)\|x - y\|, \quad \forall x, y \in C,$$

for $i = 1, 2$, where $\mu_i > 0$. In particular, if $0 < \mu_i \leq \frac{\alpha_i}{\kappa^2}$, then $I - \mu_i B_i$ is nonexpansive for $i = 1, 2$.

Lemma 2.12 (see [9]) *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mapping $B_i : C \to X$ be α_i-inverse-strongly accretive for $i = 1, 2$. Let $\psi : C \to C$ be the mapping defined by*

$$\psi(x) = \Pi_C\bigl[\Pi_C(x - \mu_2 B_2 x) - \mu_1 B_1 \Pi_C(x - \mu_2 B_2 x)\bigr], \quad \forall x \in C.$$

If $0 < \mu_i \leq \frac{\alpha_i}{\kappa^2}$ for $i = 1, 2$, then $\psi : C \to C$ is nonexpansive.

Lemma 2.13 (see [12]) *Let C be a nonempty closed convex subset of a smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C and let the mapping $B_i : C \to X$ be λ_i-strictly pseudocontractive and α_i-strongly accretive for $i = 1, 2$. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of GSVI (1.1) if and only if $x^* = \Pi_C(y^* - \mu_1 B_1 y^*)$, where $y^* = \Pi_C(x^* - \mu_2 B_2 x^*)$.*

By Lemma 2.12, we observe that

$$x^* = \Pi_C\bigl[\Pi_C\bigl(x^* - \mu_2 B_2 x^*\bigr) - \mu_1 B_1 \Pi_C\bigl(x^* - \mu_2 B_2 x^*\bigr)\bigr],$$

which implies that x^* is a fixed point of the mapping $G = \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$.

Proposition 2.1 (see [18]) *Let X be a real smooth and uniform convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to$ $\mathbf{R}$, $g(0) = 0$ such that*

$$g\bigl(\|x - y\|\bigr) \leq \|x\|^2 - 2\langle x, J(y)\rangle + \|y\|^2, \quad \forall x, y \in B_r,$$

where $B_r = \{x \in X : \|x\| \leq r\}$.

3 Two-step relaxed extragradient algorithms

In this section, we first suggest and analyze an implicit iterative algorithm by the two-step relaxed extragradient method in the setting of uniformly convex and 2-uniformly smooth Banach spaces, and then another explicit iterative algorithm in the setting of uniformly convex Banach spaces with a uniformly Gateaux differentiable norm.

Theorem 3.1 *Let C be a nonempty closed convex subset of a uniformly convex and* 2-uniformly *smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mapping $B_i : C \to X$ be α_i-inverse-strongly accretive for $i = 1, 2$. Let $f : C \to C$ be a contraction with coefficient $\rho \in (0, 1)$. Let $\{S_n\}_{n=0}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself such that $F = \bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i) \cap \Omega \neq \emptyset$, where Ω is a fixed point set of the mapping G. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = \alpha_n f(y_n) + (1 - \alpha_n)\Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)x_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)S_n y_n, \quad \forall n \geq 0, \end{cases} \tag{3.1}$$

where $0 < \mu_i < \frac{\alpha_i}{\kappa^2}$ for $i = 1, 2$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Assume that $\sum_{n=1}^{\infty} \sup_{x\in D} \|S_n x - S_{n-1}x\| < \infty$ for any bounded subset D of C and let S be a mapping of C into itself defined by $Sx = \lim_{n\to\infty} S_n x$ for all $x \in C$ and suppose that $\mathrm{Fix}(S) = \bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i)$. *Then $\{x_n\}$ converges strongly to $q \in F$, which solves the following VIP*:

$$\langle q - f(q), J(q - p)\rangle \leq 0, \quad \forall p \in F.$$

Proof It is easy to see that scheme (3.1) can be rewritten as

$$\begin{cases} y_n = \alpha_n f(y_n) + (1 - \alpha_n)G(x_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)S_n y_n, \quad \forall n \geq 0. \end{cases} \tag{3.2}$$

Take a fixed $p \in F$ arbitrarily. Then by Lemma 2.13 we know that $p = G(p)$. Moreover, by Lemma 2.12 we have

$$\begin{aligned} \|y_n - p\| &= \left\|\alpha_n\big(f(y_n) - p\big) + (1 - \alpha_n)\big(G(x_n) - p\big)\right\| \\ &\leq \alpha_n\left\|f(y_n) - f(p)\right\| + \alpha_n\left\|f(p) - p\right\| + (1 - \alpha_n)\left\|G(x_n) - p\right\| \\ &\leq \alpha_n\rho\|y_n - p\| + \alpha_n\left\|f(p) - p\right\| + (1 - \alpha_n)\|x_n - p\|, \end{aligned}$$

which hence implies that

$$\|y_n - p\| \leq \left(1 - \frac{1 - \rho}{1 - \alpha_n\rho}\alpha_n\right)\|x_n - p\| + \frac{1}{1 - \alpha_n\rho}\alpha_n\left\|f(p) - p\right\|. \tag{3.3}$$

Thus, from (3.2) we have

$$\begin{aligned}
&\|x_{n+1}-p\| \\
&\quad= \big\|\beta_n(x_n-p)+(1-\beta_n)(S_ny_n-p)\big\| \\
&\quad\le \beta_n\|x_n-p\|+(1-\beta_n)\|S_ny_n-p\| \\
&\quad\le \beta_n\|x_n-p\|+(1-\beta_n)\|y_n-p\| \\
&\quad\le \beta_n\|x_n-p\|+(1-\beta_n)\left\{\left(1-\frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right)\|x_n-p\|+\frac{1}{1-\alpha_n\rho}\alpha_n\big\|f(p)-p\big\|\right\} \\
&\quad= \left[1-\frac{(1-\beta_n)(1-\rho)}{1-\alpha_n\rho}\alpha_n\right]\|x_n-p\|+\frac{(1-\beta_n)(1-\rho)}{1-\alpha_n\rho}\alpha_n\frac{\|f(p)-p\|}{1-\rho} \\
&\quad\le \max\left\{\|x_0-p\|,\frac{\|f(p)-p\|}{1-\rho}\right\}.
\end{aligned}$$

It immediately follows that $\{x_n\}$ is bounded, and so are the sequences $\{y_n\}$, $\{G(x_n)\}$ due to (3.3) and the nonexpansivity of G.

Let us show that $\|x_{n+1}-x_n\|\to 0$ as $n\to\infty$. As a matter of fact, from (3.2) we have

$$\begin{cases}
y_n=\alpha_nf(y_n)+(1-\alpha_n)G(x_n), \\
y_{n-1}=\alpha_{n-1}f(y_{n-1})+(1-\alpha_{n-1})G(x_{n-1}), \quad \forall n\ge 1.
\end{cases}$$

Simple calculations show that

$$y_n-y_{n-1}=\alpha_n\big(f(y_n)-f(y_{n-1})\big)+(\alpha_n-\alpha_{n-1})\big(f(y_{n-1})-G(x_{n-1})\big)+(1-\alpha_n)\big(G(x_n)-G(x_{n-1})\big).$$

It follows that

$$\begin{aligned}
\|y_n-y_{n-1}\| &\le \alpha_n\big\|f(y_n)-f(y_{n-1})\big\|+|\alpha_n-\alpha_{n-1}|\big\|f(y_{n-1})-G(x_{n-1})\big\| \\
&\quad+(1-\alpha_n)\big\|G(x_n)-G(x_{n-1})\big\| \\
&\le \alpha_n\rho\|y_n-y_{n-1}\|+|\alpha_n-\alpha_{n-1}|\big\|f(y_{n-1})-G(x_{n-1})\big\| \\
&\quad+(1-\alpha_n)\|x_n-x_{n-1}\|,
\end{aligned}$$

which hence yields

$$\|y_n-y_{n-1}\|\le\left(1-\frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right)\|x_n-x_{n-1}\|+\frac{|\alpha_n-\alpha_{n-1}|}{1-\alpha_n\rho}\big\|f(y_{n-1})-G(x_{n-1})\big\|. \tag{3.4}$$

Thus we have from (3.4)

$$\begin{aligned}
&\|S_ny_n-S_{n-1}y_{n-1}\| \\
&\quad\le \|S_ny_n-S_ny_{n-1}\|+\|S_ny_{n-1}-S_{n-1}y_{n-1}\| \\
&\quad\le \|y_n-y_{n-1}\|+\|S_ny_{n-1}-S_{n-1}y_{n-1}\| \\
&\quad\le \left(1-\frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right)\|x_n-x_{n-1}\|+\frac{|\alpha_n-\alpha_{n-1}|}{1-\alpha_n\rho}\big\|f(y_{n-1})-G(x_{n-1})\big\|
\end{aligned}$$

$$
\begin{aligned}
&+ \|S_n y_{n-1} - S_{n-1} y_{n-1}\| \\
&\le \|x_n - x_{n-1}\| + \frac{|\alpha_n - \alpha_{n-1}|}{1-\alpha_n \rho} \big\|f(y_{n-1}) - G(x_{n-1})\big\| + \|S_n y_{n-1} - S_{n-1} y_{n-1}\|,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\|S_n y_n - S_{n-1} y_{n-1}\| - \|x_n - x_{n-1}\| \\
&\quad \le \frac{|\alpha_n - \alpha_{n-1}|}{1-\alpha_n \rho} \big\|f(y_{n-1}) - G(x_{n-1})\big\| + \|S_n y_{n-1} - S_{n-1} y_{n-1}\|.
\end{aligned}
$$

From condition (i) and the assumption on $\{S_n\}$, we have

$$
\limsup_{n\to\infty} \big(\|S_n y_n - S_{n-1} y_{n-1}\| - \|x_n - x_{n-1}\|\big) \le 0.
$$

It follows from Lemma 2.4 that

$$
\lim_{n\to\infty} \|S_n y_n - x_n\| = 0. \tag{3.5}
$$

Hence we obtain

$$
\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1-\beta_n)\|S_n y_n - x_n\| = 0. \tag{3.6}
$$

Next we show that $\|x_n - G(x_n)\| \to 0$ as $n \to \infty$.

For simplicity, put $q = \Pi_C(p - \mu_2 B_2 p)$, $u_n = \Pi_C(x_n - \mu_2 B_2 x_n)$ and $v_n = \Pi_C(u_n - \mu_1 B_1 u_n)$. Then $v_n = G(x_n)$. From Lemma 2.11 we have

$$
\begin{aligned}
\|u_n - q\|^2 &= \big\|\Pi_C(x_n - \mu_2 B_2 x_n) - \Pi_C(p - \mu_2 B_2 p)\big\|^2 \\
&\le \big\|x_n - p - \mu_2(B_2 x_n - B_2 p)\big\|^2 \\
&\le \|x_n - p\|^2 - 2\mu_2\big(\alpha_2 - \kappa^2 \mu_2\big)\|B_2 x_n - B_2 p\|^2
\end{aligned} \tag{3.7}
$$

and

$$
\begin{aligned}
\|v_n - p\|^2 &= \big\|\Pi_C(u_n - \mu_1 B_1 u_n) - \Pi_C(q - \mu_1 B_1 q)\big\|^2 \\
&\le \big\|u_n - q - \mu_1(B_1 u_n - B_1 q)\big\|^2 \\
&\le \|u_n - q\|^2 - 2\mu_1\big(\alpha_1 - \kappa^2 \mu_1\big)\|B_1 u_n - B_1 q\|^2.
\end{aligned} \tag{3.8}
$$

Substituting (3.7) into (3.8), we obtain

$$
\begin{aligned}
\|v_n - p\|^2 \le{}& \|x_n - p\|^2 - 2\mu_2\big(\alpha_2 - \kappa^2 \mu_2\big)\|B_2 x_n - B_2 p\|^2 \\
&- 2\mu_1\big(\alpha_1 - \kappa^2 \mu_1\big)\|B_1 u_n - B_1 q\|^2.
\end{aligned} \tag{3.9}
$$

According to Lemma 2.3, we have from (3.2)

$$
\begin{aligned}
\|y_n - p\|^2 &= \big\|\alpha_n\big(f(y_n) - f(p)\big) + (1-\alpha_n)(v_n - p) + \alpha_n\big(f(p) - p\big)\big\|^2 \\
&\le \big\|\alpha_n\big(f(y_n) - f(p)\big) + (1-\alpha_n)(v_n - p)\big\|^2 + 2\alpha_n\big\langle f(p) - p, J(y_n - p)\big\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\le \alpha_n \big\| f(y_n) - f(p) \big\|^2 + (1-\alpha_n)\|v_n - p\|^2 + 2\alpha_n \big\langle f(p) - p, J(y_n - p) \big\rangle \\
&\le \alpha_n \rho \|y_n - p\|^2 + (1-\alpha_n)\|v_n - p\|^2 + 2\alpha_n \big\| f(p) - p \big\| \|y_n - p\|,
\end{aligned}
$$

which hence yields

$$
\|y_n - p\|^2 \le \left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right)\|v_n - p\|^2 + \frac{2\alpha_n}{1-\alpha_n\rho}\big\| f(p) - p \big\| \|y_n - p\|.
$$

From this together with (3.9) and the convexity of $\|\cdot\|^2$ we have

$$
\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad = \big\| \beta_n(x_n - p) + (1-\beta_n)(S_n y_n - p) \big\|^2 \\
&\quad \le \beta_n \|x_n - p\|^2 + (1-\beta_n)\|S_n y_n - p\|^2 \\
&\quad \le \beta_n \|x_n - p\|^2 + (1-\beta_n)\|y_n - p\|^2 \\
&\quad \le \beta_n \|x_n - p\|^2 + (1-\beta_n)\left\{\left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right)\|v_n - p\|^2 + \frac{2\alpha_n}{1-\alpha_n\rho}\big\| f(p) - p \big\| \|y_n - p\|\right\} \\
&\quad \le \beta_n \|x_n - p\|^2 + (1-\beta_n)\left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right)\|v_n - p\|^2 + \alpha_n M_1 \\
&\quad \le \beta_n \|x_n - p\|^2 + (1-\beta_n)\left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right)\big[\|x_n - p\|^2 \\
&\qquad - 2\mu_2\big(\alpha_2 - \kappa^2\mu_2\big)\|B_2 x_n - B_2 p\|^2 - 2\mu_1\big(\alpha_1 - \kappa^2\mu_1\big)\|B_1 u_n - B_1 q\|^2\big] + \alpha_n M_1 \\
&\quad = \left(1 - \frac{(1-\beta_n)(1-\rho)}{1-\alpha_n\rho}\alpha_n\right)\|x_n - p\|^2 \\
&\qquad - 2(1-\beta_n)\left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right)\big[\mu_2\big(\alpha_2 - \kappa^2\mu_2\big)\|B_2 x_n - B_2 p\|^2 \\
&\qquad + \mu_1\big(\alpha_1 - \kappa^2\mu_1\big)\|B_1 u_n - B_1 q\|^2\big] + \alpha_n M_1 \\
&\quad \le \|x_n - p\|^2 - 2(1-\beta_n)\left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right)\big[\mu_2\big(\alpha_2 - \kappa^2\mu_2\big)\|B_2 x_n - B_2 p\|^2 \\
&\qquad + \mu_1\big(\alpha_1 - \kappa^2\mu_1\big)\|B_1 u_n - B_1 q\|^2\big] + \alpha_n M_1,
\end{aligned}
\tag{3.10}
$$

where $\sup_{n\ge 0}\{\frac{2(1-\beta_n)}{1-\alpha_n\rho}\|f(p) - p\|\|y_n - p\|\} \le M_1$ for some $M_1 > 0$. So, it follows that

$$
\begin{aligned}
&2(1-\beta_n)\left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right) \\
&\qquad \times \big[\mu_2\big(\alpha_2 - \kappa^2\mu_2\big)\|B_2 x_n - B_2 p\|^2 + \mu_1\big(\alpha_1 - \kappa^2\mu_1\big)\|B_1 u_n - B_1 q\|^2\big] \\
&\quad \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n M_1 \\
&\quad \le \big(\|x_n - p\| + \|x_{n+1} - p\|\big)\|x_n - x_{n+1}\| + \alpha_n M_1.
\end{aligned}
$$

Since $0 < \mu_i < \frac{\alpha_i}{\kappa^2}$ for $i = 1, 2$, from conditions (i), (ii) and (3.6) we obtain

$$
\lim_{n\to\infty}\|B_2 x_n - B_2 p\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|B_1 u_n - B_1 q\| = 0.
\tag{3.11}
$$

Utilizing Proposition 2.1 and Lemma 2.1, we have

$$\begin{aligned}
&\|u_n - q\|^2 \\
&\quad = \left\| \Pi_C(x_n - \mu_2 B_2 x_n) - \Pi_C(p - \mu_2 B_2 p) \right\|^2 \\
&\quad \le \left\langle x_n - \mu_2 B_2 x_n - (p - \mu_2 B_2 p), J(u_n - q) \right\rangle \\
&\quad = \left\langle x_n - p, J(u_n - q) \right\rangle + \mu_2 \left\langle B_2 p - B_2 x_n, J(u_n - q) \right\rangle \\
&\quad \le \frac{1}{2}\left[\|x_n - p\|^2 + \|u_n - q\|^2 - g_1\left(\left\| x_n - u_n - (p - q) \right\| \right) \right] + \mu_2 \|B_2 p - B_2 x_n\| \|u_n - q\|,
\end{aligned}$$

which implies that

$$\|u_n - q\|^2 \le \|x_n - p\|^2 - g_1\left(\left\| x_n - u_n - (p - q) \right\| \right) + 2\mu_2 \|B_2 p - B_2 x_n\| \|u_n - q\|. \tag{3.12}$$

In the same way, we derive

$$\begin{aligned}
&\|v_n - p\|^2 \\
&\quad = \left\| \Pi_C(u_n - \mu_1 B_1 u_n) - \Pi_C(q - \mu_1 B_1 q) \right\|^2 \\
&\quad \le \left\langle u_n - \mu_1 B_1 u_n - (q - \mu_1 B_1 q), J(v_n - p) \right\rangle \\
&\quad = \left\langle u_n - q, J(v_n - p) \right\rangle + \mu_1 \left\langle B_1 q - B_1 u_n, J(v_n - p) \right\rangle \\
&\quad \le \frac{1}{2}\left[\|u_n - q\|^2 + \|v_n - p\|^2 - g_2\left(\left\| u_n - v_n + (p - q) \right\| \right) \right] + \mu_1 \|B_1 q - B_1 u_n\| \|v_n - p\|,
\end{aligned}$$

which implies that

$$\|v_n - p\|^2 \le \|u_n - q\|^2 - g_2\left(\left\| u_n - v_n + (p - q) \right\| \right) + 2\mu_1 \|B_1 q - B_1 u_n\| \|v_n - p\|. \tag{3.13}$$

Substituting (3.12) into (3.13), we get

$$\begin{aligned}
\|v_n - p\|^2 \le{}& \|x_n - p\|^2 - g_1\left(\left\| x_n - u_n - (p - q) \right\| \right) - g_2\left(\left\| u_n - v_n + (p - q) \right\| \right) \\
&+ 2\mu_2 \|B_2 p - B_2 x_n\| \|u_n - q\| + 2\mu_1 \|B_1 q - B_1 u_n\| \|v_n - p\|.
\end{aligned} \tag{3.14}$$

From (3.10) and (3.14), we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad \le \alpha_n M_1 + \beta_n \|x_n - p\|^2 + (1 - \beta_n)\left(1 - \frac{1 - \rho}{1 - \alpha_n \rho}\alpha_n \right)\left[\|x_n - p\|^2 \right. \\
&\qquad - g_1\left(\left\| x_n - u_n - (p - q) \right\| \right) - g_2\left(\left\| u_n - v_n + (p - q) \right\| \right) \\
&\qquad \left. + 2\mu_2 \|B_2 p - B_2 x_n\| \|u_n - q\| + 2\mu_1 \|B_1 q - B_1 u_n\| \|v_n - p\| \right] \\
&\quad \le \alpha_n M_1 + \left(1 - \frac{(1 - \beta_n)(1 - \rho)}{1 - \alpha_n \rho}\alpha_n \right) \|x_n - p\|^2 \\
&\qquad - (1 - \beta_n)\left(1 - \frac{1 - \rho}{1 - \alpha_n \rho}\alpha_n \right)\left[g_1\left(\left\| x_n - u_n - (p - q) \right\| \right) + g_2\left(\left\| u_n - v_n + (p - q) \right\| \right) \right]
\end{aligned}$$

$$
\begin{aligned}
&\quad + 2\mu_2\|B_2p - B_2x_n\|\|u_n - q\| + 2\mu_1\|B_1q - B_1u_n\|\|v_n - p\| \\
&\le \alpha_n M_1 + \|x_n - p\|^2 - (1-\beta_n)\left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right)\big[g_1\big(\big\|x_n - u_n - (p-q)\big\|\big) \\
&\quad + g_2\big(\big\|u_n - v_n + (p-q)\big\|\big)\big] + 2\mu_2\|B_2p - B_2x_n\|\|u_n - q\| \\
&\quad + 2\mu_1\|B_1q - B_1u_n\|\|v_n - p\|,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&(1-\beta_n)\left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right)\big[g_1\big(\big\|x_n - u_n - (p-q)\big\|\big) + g_2\big(\big\|u_n - v_n + (p-q)\big\|\big)\big] \\
&\le \alpha_n M_1 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\mu_2\|B_2p - B_2x_n\|\|u_n - q\| \\
&\quad + 2\mu_1\|B_1q - B_1u_n\|\|v_n - p\| \\
&\le \alpha_n M_1 + \big(\|x_n - p\| + \|x_{n+1} - p\|\big)\|x_n - x_{n+1}\| + 2\mu_2\|B_2p - B_2x_n\|\|u_n - q\| \\
&\quad + 2\mu_1\|B_1q - B_1u_n\|\|v_n - p\|.
\end{aligned}
$$

Utilizing conditions (i), (ii), from (3.6) and (3.11) we have

$$
\lim_{n\to\infty} g_1\big(\big\|x_n - u_n - (p-q)\big\|\big) = 0, \qquad \lim_{n\to\infty} g_2\big(\big\|u_n - v_n + (p-q)\big\|\big) = 0. \tag{3.15}
$$

Utilizing the properties of g_1 and g_2, we deduce that

$$
\lim_{n\to\infty}\big\|x_n - u_n - (p-q)\big\| = 0, \qquad \lim_{n\to\infty}\big\|u_n - v_n + (p-q)\big\| = 0. \tag{3.16}
$$

From (3.16) we obtain

$$
\|x_n - v_n\| \le \big\|x_n - u_n - (p-q)\big\| + \big\|u_n - v_n + (p-q)\big\| \to 0 \quad \text{as } n \to \infty.
$$

That is,

$$
\lim_{n\to\infty}\big\|x_n - G(x_n)\big\| = 0. \tag{3.17}
$$

On the other hand, we observe that

$$
y_n - G(x_n) = \alpha_n\big(f(y_n) - G(x_n)\big).
$$

Since $\alpha_n \to 0$ as $n \to \infty$, we have

$$
\lim_{n\to\infty}\big\|y_n - G(x_n)\big\| = 0. \tag{3.18}
$$

We note that

$$
\begin{aligned}
\big\|S_nG(x_n) - G(x_n)\big\| &\le \big\|S_nG(x_n) - S_ny_n\big\| + \|S_ny_n - x_n\| + \big\|x_n - G(x_n)\big\| \\
&\le \big\|G(x_n) - y_n\big\| + \|S_ny_n - x_n\| + \big\|x_n - G(x_n)\big\|.
\end{aligned}
$$

From (3.5), (3.17) and (3.18), we obtain that

$$\lim_{n\to\infty} \|S_n G(x_n) - G(x_n)\| = 0. \tag{3.19}$$

By (3.19) and Lemma 2.6, we have

$$\begin{aligned} \|SG(x_n) - G(x_n)\| &\le \|SG(x_n) - S_n G(x_n)\| + \|S_n G(x_n) - G(x_n)\| \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.20}$$

In terms of (3.17) and (3.20), we have

$$\begin{aligned} \|x_n - Sx_n\| &\le \|x_n - G(x_n)\| + \|G(x_n) - SG(x_n)\| + \|SG(x_n) - Sx_n\| \\ &\le 2\|x_n - G(x_n)\| + \|G(x_n) - SG(x_n)\| \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.21}$$

Define a mapping $Wx = (1-\theta)Sx + \theta G(x)$, $\theta \in (0,1)$ is a constant. Then by Lemma 2.8 we have that $\mathrm{Fix}(W) = \mathrm{Fix}(S) \cap \mathrm{Fix}(G) = F$. We observe that

$$\begin{aligned} \|x_n - Wx_n\| &= \|(1-\theta)(x_n - Sx_n) + \theta(x_n - G(x_n))\| \\ &\le (1-\theta)\|x_n - Sx_n\| + \theta\|x_n - G(x_n)\|. \end{aligned}$$

From (3.17) and (3.21), we obtain

$$\lim_{n\to\infty} \|x_n - Wx_n\| = 0. \tag{3.22}$$

Now, we claim that

$$\limsup_{n\to\infty} \langle f(q) - q, J(x_n - q)\rangle \le 0, \tag{3.23}$$

where $q = s - \lim_{t\to 0} x_t$ with x_t being a fixed point of the contraction

$$x \mapsto tf(x) + (1-t)Wx.$$

Then x_t solves the fixed point equation $x_t = tf(x_t) + (1-t)Wx_t$. Thus we have

$$\|x_t - x_n\| = \|(1-t)(Wx_t - x_n) + t(f(x_t) - x_n)\|.$$

By Lemma 2.3 we conclude that

$$\begin{aligned} &\|x_t - x_n\|^2 \\ &\quad = \|(1-t)(Wx_t - x_n) + t(f(x_t) - x_n)\|^2 \\ &\quad \le (1-t)^2\|Wx_t - x_n\|^2 + 2t\langle f(x_t) - x_n, J(x_t - x_n)\rangle \\ &\quad \le (1-t)^2(\|Wx_t - Wx_n\| + \|Wx_n - x_n\|)^2 + 2t\langle f(x_t) - x_n, J(x_t - x_n)\rangle \end{aligned}$$

$$\begin{aligned}
&\leq (1-t)^2\big(\|x_t - x_n\| + \|Wx_n - x_n\|\big)^2 + 2t\big\langle f(x_t) - x_n, J(x_t - x_n)\big\rangle \\
&= (1-t)^2\big[\|x_t - x_n\|^2 + 2\|x_t - x_n\|\|Wx_n - x_n\| + \|Wx_n - x_n\|^2\big] \\
&\quad + 2t\big\langle f(x_t) - x_t, J(x_t - x_n)\big\rangle + 2t\big\langle x_t - x_n, J(x_t - x_n)\big\rangle \\
&= \big(1 - 2t + t^2\big)\|x_t - x_n\|^2 + f_n(t) + 2t\big\langle f(x_t) - x_t, J(x_t - x_n)\big\rangle + 2t\|x_t - x_n\|^2,
\end{aligned} \tag{3.24}$$

where

$$f_n(t) = (1-t)^2\big(2\|x_t - x_n\| + \|x_n - Wx_n\|\big)\|x_n - Wx_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.25}$$

It follows from (3.24) that

$$\big\langle x_t - f(x_t), J(x_t - x_n)\big\rangle \leq \frac{t}{2}\|x_t - x_n\|^2 + \frac{1}{2t}f_n(t). \tag{3.26}$$

Letting $n \to \infty$ in (3.26) and noticing (3.25), we derive

$$\limsup_{n\to\infty}\big\langle x_t - f(x_t), J(x_t - x_n)\big\rangle \leq \frac{t}{2}M_2, \tag{3.27}$$

where $M_2 > 0$ is a constant such that $\|x_t - x_n\|^2 \leq M_2$ for all $t \in (0,1)$ and $n \geq 0$. Taking $t \to 0$ in (3.27), we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\big\langle x_t - f(x_t), J(x_t - x_n)\big\rangle \leq 0.$$

On the other hand, we have

$$\begin{aligned}
&\big\langle f(q) - q, J(x_n - q)\big\rangle \\
&\quad = \big\langle f(q) - q, J(x_n - q)\big\rangle - \big\langle f(q) - q, J(x_n - x_t)\big\rangle + \big\langle f(q) - q, J(x_n - x_t)\big\rangle \\
&\qquad - \big\langle f(q) - x_t, J(x_n - x_t)\big\rangle + \big\langle f(q) - x_t, J(x_n - x_t)\big\rangle - \big\langle f(x_t) - x_t, J(x_n - x_t)\big\rangle \\
&\qquad + \big\langle f(x_t) - x_t, J(x_n - x_t)\big\rangle \\
&\quad = \big\langle f(q) - q, J(x_n - q) - J(x_n - x_t)\big\rangle + \big\langle x_t - q, J(x_n - x_t)\big\rangle \\
&\qquad + \big\langle f(q) - f(x_t), J(x_n - x_t)\big\rangle + \big\langle f(x_t) - x_t, J(x_n - x_t)\big\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
\limsup_{n\to\infty}\big\langle f(q) - q, J(x_n - q)\big\rangle &\leq \limsup_{n\to\infty}\big\langle f(q) - q, J(x_n - q) - J(x_n - x_t)\big\rangle \\
&\quad + \|x_t - q\|\limsup_{n\to\infty}\|x_n - x_t\| + \rho\|q - x_t\|\limsup_{n\to\infty}\|x_n - x_t\| \\
&\quad + \limsup_{n\to\infty}\big\langle f(x_t) - x_t, J(x_n - x_t)\big\rangle.
\end{aligned}$$

Taking into account that $x_t \to q$ as $t \to 0$, we have

$$\begin{aligned}
\limsup_{n\to\infty}\big\langle f(q) - q, J(x_n - q)\big\rangle &= \limsup_{t\to 0}\limsup_{n\to\infty}\big\langle f(q) - q, J(x_n - q)\big\rangle \\
&\leq \limsup_{t\to 0}\limsup_{n\to\infty}\big\langle f(q) - q, J(x_n - q) - J(x_n - x_t)\big\rangle.
\end{aligned} \tag{3.28}$$

Since X has a uniformly Fréchet differentiable norm, the duality mapping J is norm-to-norm uniformly continuous on bounded subsets of X. Consequently, the two limits are interchangeable and hence (3.23) holds. From (3.17) and (3.18) we get $(y_n - q) - (x_n - q) \to 0$. Noticing that J is norm-to-norm uniformly continuous on bounded subsets of X, we deduce from (3.23) that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle f(q) - q, J(y_n - q)\rangle \\
&\quad= \limsup_{n\to\infty}\bigl(\langle f(q) - q, J(x_n - q)\rangle + \langle f(q) - q, J(y_n - q) - J(x_n - q)\rangle\bigr) \\
&\quad= \limsup_{n\to\infty}\langle f(q) - q, J(x_n - q)\rangle \le 0.
\end{aligned} \tag{3.29}$$

Finally, let us show that $x_n \to q$ as $n \to \infty$. We observe that

$$\begin{aligned}
&\|y_n - q\|^2 \\
&\quad= \bigl\|\alpha_n\bigl(f(y_n) - f(q)\bigr) + (1 - \alpha_n)\bigl(G(x_n) - q\bigr) + \alpha_n\bigl(f(q) - q\bigr)\bigr\|^2 \\
&\quad\le \bigl\|\alpha_n\bigl(f(y_n) - f(q)\bigr) + (1 - \alpha_n)\bigl(G(x_n) - q\bigr)\bigr\|^2 + 2\alpha_n\langle f(q) - q, J(y_n - q)\rangle \\
&\quad\le \alpha_n\bigl\|f(y_n) - f(q)\bigr\|^2 + (1 - \alpha_n)\bigl\|G(x_n) - q\bigr\|^2 + 2\alpha_n\langle f(q) - q, J(y_n - q)\rangle \\
&\quad\le \alpha_n\rho\|y_n - q\|^2 + (1 - \alpha_n)\|x_n - q\|^2 + 2\alpha_n\langle f(q) - q, J(y_n - q)\rangle,
\end{aligned}$$

which implies that

$$\|y_n - q\|^2 \le \left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right)\|x_n - q\|^2 + \frac{\alpha_n(1-\rho)}{1-\alpha_n\rho} \cdot \frac{2\langle f(q) - q, J(y_n - q)\rangle}{1-\rho}. \tag{3.30}$$

By the convexity of $\|\cdot\|^2$ and (3.2), we get

$$\|x_{n+1} - q\|^2 \le \beta_n\|x_n - q\|^2 + (1 - \beta_n)\|y_n - q\|^2,$$

which together with (3.30) leads to

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\le \beta_n\|x_n - q\|^2 + (1 - \beta_n)\left\{\left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right)\|x_n - q\|^2 \right.\\
&\qquad\left. + \frac{\alpha_n(1-\rho)}{1-\alpha_n\rho} \cdot \frac{2\langle f(q) - q, J(y_n - q)\rangle}{1-\rho}\right\} \\
&= \left[1 - \frac{(1-\beta_n)(1-\rho)}{1-\alpha_n\rho}\alpha_n\right]\|x_n - q\|^2 \\
&\qquad + \frac{(1-\beta_n)(1-\rho)}{1-\alpha_n\rho}\alpha_n \cdot \frac{2\langle f(q) - q, J(y_n - q)\rangle}{1-\rho}.
\end{aligned} \tag{3.31}$$

Applying Lemma 2.2 to (3.31), we obtain that $x_n \to q$ as $n \to \infty$. This completes the proof. □

Corollary 3.1 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X*

onto C. Let the mapping $B_i : C \to X$ be α_i-inverse-strongly accretive for $i = 1,2$. Let $f : C \to C$ be a contraction with coefficient $\rho \in (0,1)$. Let S be a nonexpansive mapping of C into itself such that $F = \mathrm{Fix}(S) \cap \Omega \neq \emptyset$, where Ω is the fixed point set of the mapping G. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} y_n = \alpha_n f(y_n) + (1 - \alpha_n)\Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)x_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)Sy_n, \quad \forall n \geq 0, \end{cases}$$

where $0 < \mu_i < \frac{\alpha_i}{\kappa^2}$ for $i = 1,2$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ converges strongly to $q \in F$, which solves the following VIP:

$$\langle q - f(q), J(q - p)\rangle \leq 0, \quad \forall p \in F.$$

Theorem 3.2 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X which has a uniformly Gâteaux differentiable norm. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mapping $B_i : C \to X$ be λ_i-strictly pseudocontractive and α_i-strongly accretive with $\alpha_i + \lambda_i \geq 1$ for $i = 1,2$. Let $f : C \to C$ be a contraction with coefficient $\rho \in (0,1)$. Let $\{S_n\}_{n=0}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself such that $F = \bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i) \cap \Omega \neq \emptyset$, where Ω is a fixed point set of the mapping G. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)\Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)x_n, \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)S_n y_n, \quad \forall n \geq 0, \end{cases} \tag{3.32}$$

where $1 - \frac{\lambda_i}{1+\lambda_i}(1 - \sqrt{\frac{1-\alpha_i}{\lambda_i}}) \leq \mu_i \leq 1$ for $i = 1,2$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ satisfying the following conditions:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$;
(iii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *or* $\lim_{n\to\infty} |\alpha_n - \alpha_{n-1}|/\beta_n = 0$;
(iv) $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \beta_{n-1}/\beta_n = 1$.

Assume that $\sum_{n=1}^{\infty} \sup_{x\in D} \|S_n x - S_{n-1}x\| < \infty$ for any bounded subset D of C and let S be a mapping of C into itself defined by $Sx = \lim_{n\to\infty} S_n x$ for all $x \in C$ and suppose that $\mathrm{Fix}(S) = \bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i)$. Then $\{x_n\}$ converges strongly to $q \in F$, which solves the following VIP:

$$\langle q - f(q), J(q - p)\rangle \leq 0, \quad \forall p \in F.$$

Proof It is easy to see that scheme (3.32) can be rewritten as

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)G(x_n), \\ x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)S_n y_n, \quad \forall n \geq 0. \end{cases} \tag{3.33}$$

Take a fixed $p \in F$ arbitrarily. Then by Lemma 2.13 we know that $p = G(p)$. Moreover, by Lemma 2.10 we have

$$\begin{aligned} \|y_n - p\| &= \bigl\|\alpha_n(x_n - p) + (1-\alpha_n)\bigl(G(x_n) - p\bigr)\bigr\| \\ &\le \alpha_n\|x_n - p\| + (1-\alpha_n)\bigl\|G(x_n) - p\bigr\| \\ &\le \alpha_n\|x_n - p\| + (1-\alpha_n)\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned} \tag{3.34}$$

From (3.34) we obtain

$$\begin{aligned} \|x_{n+1} - p\| &= \bigl\|\beta_n\bigl(f(x_n) - p\bigr) + (1-\beta_n)(S_n y_n - p)\bigr\| \\ &\le \beta_n\bigl(\bigl\|f(x_n) - f(p)\bigr\| + \bigl\|f(p) - p\bigr\|\bigr) + (1-\beta_n)\|S_n y_n - p\| \\ &\le \beta_n\rho\|x_n - p\| + \beta_n\bigl\|f(p) - p\bigr\| + (1-\beta_n)\|y_n - p\| \\ &\le \beta_n\rho\|x_n - p\| + \beta_n\bigl\|f(p) - p\bigr\| + (1-\beta_n)\|x_n - p\| \\ &= \bigl(1 - \beta_n(1-\rho)\bigr)\|x_n - p\| + \beta_n\bigl\|f(p) - p\bigr\| \\ &= \bigl(1 - \beta_n(1-\rho)\bigr)\|x_n - p\| + \beta_n(1-\rho)\cdot\frac{\|f(p) - p\|}{1-\rho} \\ &\le \max\biggl\{\|x_0 - p\|, \frac{\|f(p) - p\|}{1-\rho}\biggr\}, \end{aligned}$$

which implies that $\{x_n\}$ is bounded. By Lemma 2.10 we know from (3.34) that $\{G(x_n)\}$ and $\{y_n\}$ both are bounded.

Let us show that $\|x_{n+1} - x_n\| \to 0$ and $\|x_n - y_n\| \to 0$ as $n \to \infty$. As a matter of fact, from (3.3) we have

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)G(x_n), \\ y_{n-1} = \alpha_{n-1}x_{n-1} + (1-\alpha_{n-1})G(x_{n-1}), \quad \forall n \ge 1. \end{cases}$$

Simple calculations show that

$$y_n - y_{n-1} = \alpha_n(x_n - x_{n-1}) + (\alpha_n - \alpha_{n-1})\bigl(x_{n-1} - G(x_{n-1})\bigr) + (1-\alpha_n)\bigl(G(x_n) - G(x_{n-1})\bigr).$$

It follows that

$$\begin{aligned} \|y_n - y_{n-1}\| &\le \alpha_n\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\bigl\|x_{n-1} - G(x_{n-1})\bigr\| \\ &\quad + (1-\alpha_n)\bigl\|G(x_n) - G(x_{n-1})\bigr\| \\ &\le \alpha_n\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\bigl\|x_{n-1} - G(x_{n-1})\bigr\| \\ &\quad + (1-\alpha_n)\|x_n - x_{n-1}\| \\ &= \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\bigl\|x_{n-1} - G(x_{n-1})\bigr\|. \end{aligned} \tag{3.35}$$

Furthermore, from (3.33) we have

$$\begin{cases} x_{n+1} = \beta_n f(x_n) + (1-\beta_n)S_n y_n, \\ x_n = \beta_{n-1} f(x_{n-1}) + (1-\beta_{n-1})S_{n-1}y_{n-1}, \quad \forall n \ge 1. \end{cases}$$

Also, simple calculations show that

$$\begin{aligned} x_{n+1} - x_n &= \beta_n \big(f(x_n) - f(x_{n-1})\big) + (\beta_n - \beta_{n-1})\big(f(x_{n-1}) - S_{n-1}y_{n-1}\big) \\ &\quad + (1-\beta_n)(S_n y_n - S_{n-1}y_{n-1}). \end{aligned} \tag{3.36}$$

It follows from (3.35) and (3.36) that

$$\begin{aligned} \|x_{n+1} - x_n\| &\le \beta_n \big\|f(x_n) - f(x_{n-1})\big\| + |\beta_n - \beta_{n-1}| \big\|f(x_{n-1}) - S_{n-1}y_{n-1}\big\| \\ &\quad + (1-\beta_n)\|S_n y_n - S_{n-1}y_{n-1}\| \\ &\le \beta_n \rho \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \big\|f(x_{n-1}) - S_{n-1}y_{n-1}\big\| \\ &\quad + (1-\beta_n)\big(\|S_n y_n - S_n y_{n-1}\| + \|S_n y_{n-1} - S_{n-1}y_{n-1}\|\big) \\ &\le \beta_n \rho \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \big\|f(x_{n-1}) - S_{n-1}y_{n-1}\big\| \\ &\quad + (1-\beta_n)\big(\|y_n - y_{n-1}\| + \|S_n y_{n-1} - S_{n-1}y_{n-1}\|\big) \\ &\le \beta_n \rho \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \big\|f(x_{n-1}) - S_{n-1}y_{n-1}\big\| \\ &\quad + (1-\beta_n)\big[\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \big\|x_{n-1} - G(x_{n-1})\big\|\big] \\ &\quad + \|S_n y_{n-1} - S_{n-1}y_{n-1}\| \\ &= \big(1 - \beta_n(1-\rho)\big)\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \big\|f(x_{n-1}) - S_{n-1}y_{n-1}\big\| \\ &\quad + |\alpha_n - \alpha_{n-1}| \big\|x_{n-1} - G(x_{n-1})\big\| + \|S_n y_{n-1} - S_{n-1}y_{n-1}\| \\ &\le \big(1 - \beta_n(1-\rho)\big)\|x_n - x_{n-1}\| + M\big(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|\big) \\ &\quad + \|S_n y_{n-1} - S_{n-1}y_{n-1}\|, \end{aligned} \tag{3.37}$$

where $\sup_{n\ge 0}\{\|f(x_n) - S_n y_n\| + \|x_n - G(x_n)\|\} \le M$ for some $M > 0$. Utilizing Lemma 2.2, from conditions (ii)-(iv) and the assumption on $\{S_n\}$, we deduce that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.38}$$

Since $\{x_n\}$ and $\{G(x_n)\}$ both are bounded, by Lemma 2.5 there exists a continuous strictly increasing function $g : [0,\infty) \to [0,\infty)$, $g(0) = 0$ such that for $p \in F$

$$\begin{aligned} &\|y_n - p\|^2 \\ &\quad \le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\big\|G(x_n) - p\big\|^2 - \alpha_n(1-\alpha_n)g\big(\big\|x_n - G(x_n)\big\|\big) \\ &\quad \le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - \alpha_n(1-\alpha_n)g\big(\big\|x_n - G(x_n)\big\|\big) \\ &\quad = \|x_n - p\|^2 - \alpha_n(1-\alpha_n)g\big(\big\|x_n - G(x_n)\big\|\big), \end{aligned}$$

which together with (3.33) implies that

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad= \big\|\beta_n\big(f(x_n)-f(p)\big)+(1-\beta_n)(S_ny_n-p)+\beta_n\big(f(p)-p\big)\big\|^2 \\
&\quad\le \big\|\beta_n\big(f(x_n)-f(p)\big)+(1-\beta_n)(S_ny_n-p)\big\|^2+2\beta_n\big\langle f(p)-p,J(x_{n+1}-p)\big\rangle \\
&\quad\le \beta_n\big\|f(x_n)-f(p)\big\|^2+(1-\beta_n)\|S_ny_n-p\|^2+2\beta_n\big\|f(p)-p\big\|\|x_{n+1}-p\| \\
&\quad\le \beta_n\rho^2\|x_n-p\|^2+(1-\beta_n)\|y_n-p\|^2+2\beta_n\big\|f(p)-p\big\|\|x_{n+1}-p\| \\
&\quad\le \beta_n\rho\|x_n-p\|^2+(1-\beta_n)\big[\|x_n-p\|^2-\alpha_n(1-\alpha_n)g\big(\big\|x_n-G(x_n)\big\|\big)\big] \\
&\qquad+2\beta_n\big\|f(p)-p\big\|\|x_{n+1}-p\| \\
&\quad= \big(1-\beta_n(1-\rho)\big)\|x_n-p\|^2-(1-\beta_n)\alpha_n(1-\alpha_n)g\big(\big\|x_n-G(x_n)\big\|\big) \\
&\qquad+2\beta_n\big\|f(p)-p\big\|\|x_{n+1}-p\| \\
&\quad\le \|x_n-p\|^2-(1-\beta_n)\alpha_n(1-\alpha_n)g\big(\big\|x_n-G(x_n)\big\|\big)+2\beta_n\big\|f(p)-p\big\|\|x_{n+1}-p\|.
\end{aligned} \tag{3.39}$$

It immediately follows that

$$\begin{aligned}
&(1-\beta_n)\alpha_n(1-\alpha_n)g\big(\big\|x_n-G(x_n)\big\|\big) \\
&\quad\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+2\beta_n\big\|f(p)-p\big\|\|x_{n+1}-p\| \\
&\quad\le \big(\|x_n-p\|+\|x_{n+1}-p\|\big)\|x_n-x_{n+1}\|+2\beta_n\big\|f(p)-p\big\|\|x_{n+1}-p\|.
\end{aligned}$$

Since $\beta_n\to 0$, $\|x_{n+1}-x_n\|\to 0$ and $0<\liminf_{n\to\infty}\alpha_n\le\limsup_{n\to\infty}\alpha_n<1$, we get $\lim_{n\to\infty}g(\|x_n-G(x_n)\|)=0$ and hence

$$\lim_{n\to\infty}\big\|x_n-G(x_n)\big\|=0. \tag{3.40}$$

Thus, from (3.33) and (3.40) it follows that

$$\lim_{n\to\infty}\|y_n-x_n\|=\lim_{n\to\infty}(1-\alpha_n)\big\|G(x_n)-x_n\big\|=0. \tag{3.41}$$

On the other hand, we observe that

$$\begin{aligned}
x_{n+1}-x_n &= \beta_n\big(f(x_n)-x_n\big)+(1-\beta_n)(S_ny_n-x_n) \\
&= \beta_n\big(f(x_n)-x_n\big)+(1-\beta_n)(S_ny_n-y_n)+(1-\beta_n)(y_n-x_n).
\end{aligned}$$

Then we have

$$\begin{aligned}
&(1-\beta_n)\|S_ny_n-y_n\| \\
&\quad= \big\|x_{n+1}-x_n-\beta_n\big(f(x_n)-x_n\big)-(1-\beta_n)(y_n-x_n)\big\| \\
&\quad\le \|x_{n+1}-x_n\|+\beta_n\big\|f(x_n)-x_n\big\|+(1-\beta_n)\|y_n-x_n\| \\
&\quad\le \|x_{n+1}-x_n\|+\beta_n\big\|f(x_n)-x_n\big\|+\|y_n-x_n\|.
\end{aligned}$$

Since $\beta_n \to 0$, $\|x_{n+1} - x_n\| \to 0$ and $\|x_n - y_n\| \to 0$ as $n \to \infty$, we get

$$\lim_{n\to\infty} \|S_n y_n - y_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|S_n y_n - x_n\| = 0. \tag{3.42}$$

In the meantime, since $\|x_n - G(x_n)\| \to 0$ and $\|x_n - y_n\| \to 0$ as $n \to \infty$, we also get

$$\lim_{n\to\infty} \|y_n - G(x_n)\| = 0. \tag{3.43}$$

We note that

$$\begin{aligned} \|S_n G(x_n) - G(x_n)\| &\le \|S_n G(x_n) - S_n y_n\| + \|S_n y_n - x_n\| + \|x_n - G(x_n)\| \\ &\le \|G(x_n) - y_n\| + \|S_n y_n - x_n\| + \|x_n - G(x_n)\|. \end{aligned}$$

From (3.40), (3.42) and (3.43), we obtain

$$\lim_{n\to\infty} \|S_n G(x_n) - G(x_n)\| = 0. \tag{3.44}$$

By (3.44) and Lemma 2.6, we have

$$\begin{aligned} \|SG(x_n) - G(x_n)\| &\le \|SG(x_n) - S_n G(x_n)\| + \|S_n G(x_n) - G(x_n)\| \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.45}$$

In terms of (3.40) and (3.45), we have

$$\begin{aligned} \|x_n - Sx_n\| &\le \|x_n - G(x_n)\| + \|G(x_n) - SG(x_n)\| + \|SG(x_n) - Sx_n\| \\ &\le 2\|x_n - G(x_n)\| + \|G(x_n) - SG(x_n)\| \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.46}$$

Define a mapping $Wx = (1-\theta)Sx + \theta G(x)$, $\theta \in (0,1)$ is a constant. Then by Lemma 2.8 we have that $\mathrm{Fix}(W) = \mathrm{Fix}(S) \cap \mathrm{Fix}(G) = F$. We observe that

$$\begin{aligned} \|x_n - Wx_n\| &= \|(1-\theta)(x_n - Sx_n) + \theta(x_n - G(x_n))\| \\ &\le (1-\theta)\|x_n - Sx_n\| + \theta\|x_n - G(x_n)\|. \end{aligned}$$

From (3.40) and (3.46), we obtain

$$\lim_{n\to\infty} \|x_n - Wx_n\| = 0. \tag{3.47}$$

Now, we claim that

$$\limsup_{n\to\infty} \langle f(q) - q, J(x_n - q)\rangle \le 0, \tag{3.48}$$

where $q = s - \lim_{t\to 0} x_t$ with x_t being the fixed point of the contraction $x \mapsto tf(x) + (1-t)Wx$. Then x_t solves the fixed point equation $x_t = tf(x_t) + (1-t)Wx_t$. Utilizing the arguments similar to those of (3.28) in the proof of Theorem 3.1, we can deduce that

$$\begin{aligned}\limsup_{n\to\infty}\langle f(q)-q, J(x_n-q)\rangle &= \limsup_{t\to 0}\limsup_{n\to\infty}\langle f(q)-q, J(x_n-q)\rangle \\ &\le \limsup_{t\to 0}\limsup_{n\to\infty}\langle f(q)-q, J(x_n-q)-J(x_n-x_t)\rangle.\end{aligned}$$

Since X has a uniformly Gâteaux differentiable norm, the duality mapping J is norm-to-weak* uniformly continuous on bounded subsets of X. Consequently, the two limits are interchangeable and hence the following holds:

$$\limsup_{n\to\infty}\langle f(q)-q, J(x_n-q)\rangle \le 0. \tag{3.49}$$

From (3.38) we get $(x_{n+1}-q)-(x_n-q)\to 0$. Noticing the norm-to-weak* uniform continuity of J on bounded subsets of X, we deduce from (3.48) that

$$\begin{aligned}&\limsup_{n\to\infty}\langle f(q)-q, J(x_{n+1}-q)\rangle \\ &\quad= \limsup_{n\to\infty}\bigl(\langle f(q)-q, J(x_{n+1}-q)-J(x_n-q)\rangle + \langle f(q)-q, J(x_{n+1}-q)\rangle\bigr) \\ &\quad= \limsup_{n\to\infty}\langle f(q)-q, J(x_n-q)\rangle \le 0.\end{aligned} \tag{3.50}$$

Finally, let us show that $x_n\to q$ as $n\to\infty$. We observe that

$$\begin{aligned}\|y_n-q\| &= \bigl\|\alpha_n(x_n-q)+(1-\alpha_n)\bigl(G(x_n)-q\bigr)\bigr\| \\ &\le \alpha_n\|x_n-q\|+(1-\alpha_n)\|x_n-q\| = \|x_n-q\|,\end{aligned}$$

and

$$\begin{aligned}&\|x_{n+1}-q\|^2 \\ &\quad= \bigl\|\beta_n\bigl(f(x_n)-f(q)\bigr)+(1-\beta_n)(S_ny_n-q)+\beta_n\bigl(f(q)-q\bigr)\bigr\|^2 \\ &\quad\le \bigl\|\beta_n\bigl(f(x_n)-f(q)\bigr)+(1-\beta_n)(S_ny_n-q)\bigr\|^2+2\beta_n\langle f(q)-q, J(x_{n+1}-q)\rangle \\ &\quad\le \beta_n\bigl\|f(x_n)-f(q)\bigr\|^2+(1-\beta_n)\|S_ny_n-q\|^2+2\beta_n\langle f(q)-q, J(x_{n+1}-q)\rangle \\ &\quad\le \beta_n\rho\|x_n-q\|^2+(1-\beta_n)\|y_n-q\|^2+2\beta_n\langle f(q)-q, J(x_{n+1}-q)\rangle \\ &\quad\le \beta_n\rho\|x_n-q\|^2+(1-\beta_n)\|x_n-q\|^2+2\beta_n\langle f(q)-q, J(x_{n+1}-q)\rangle \\ &\quad= \bigl(1-\beta_n(1-\rho)\bigr)\|x_n-q\|^2+2\beta_n\langle f(q)-q, J(x_{n+1}-q)\rangle.\end{aligned} \tag{3.51}$$

Since $\sum_{n=0}^{\infty}\beta_n=\infty$ and $\limsup_{n\to\infty}\langle f(q)-q, J(x_{n+1}-q)\rangle\le 0$, by Lemma 2.2 we conclude from (3.51) that $x_n\to q$ as $n\to\infty$. This completes the proof. □

Corollary 3.2 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X which has a uniformly Gâteaux differentiable norm. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mapping $B_i: C\to X$ be λ_i-strictly pseudocontractive and α_i-strongly accretive with $\alpha_i+\lambda_i\ge 1$ for $i=1,2$. Let $f: C\to C$ be a contraction with coefficient $\rho\in(0,1)$. Let S be a nonexpansive mapping of C into itself such that*

$F = \mathrm{Fix}(S) \cap \Omega \neq \emptyset$, where Ω is a fixed point set of the mapping G. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)\Pi_C(I-\mu_1 B_1)\Pi_C(I-\mu_2 B_2)x_n, \\ x_{n+1} = \beta_n f(x_n) + (1-\beta_n)Sy_n, \quad \forall n \geq 0, \end{cases}$$

where $1 - \frac{\lambda_i}{1+\lambda_i}(1-\sqrt{\frac{1-\alpha_i}{\lambda_i}}) \leq \mu_i \leq 1$ for $i = 1,2$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ satisfying the following conditions:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$;
(iii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *or* $\lim_{n\to\infty} |\alpha_n - \alpha_{n-1}|/\beta_n = 0$;
(iv) $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \beta_{n-1}/\beta_n = 1$.

Then $\{x_n\}$ converges strongly to $q \in F$, which solves the following VIP:

$$\langle q - f(q), J(q-p)\rangle \leq 0, \quad \forall p \in F.$$

Remark 3.1 Theorems 3.1 and 3.2 improve, extend, supplement and develop [14, Theorem 3.1] in the following aspects. Although the iterative algorithm in Theorem 3.1 is an implicit algorithm, we can derive the strong convergence of the proposed algorithm under the same conditions on the parameter sequences $\{\alpha_n\}$, $\{\beta_n\}$ as in [14, Theorem 3.1]. The assumption of the uniformly convex and 2-uniformly smooth Banach space X in [14, Theorem 3.1] is weakened to the one of the uniformly convex Banach space X having a uniformly Gâteaux differentiable norm in Theorem 3.2.

4 Relaxed extragradient composite algorithms

In this section, we propose and analyze a composite explicit iterative algorithm by the two-step relaxed extragradient method for solving GSVI (1.1) and the common fixed point problem of an infinite family of nonexpansive self-mappings $\{S_n\}$ in a 2-uniformly smooth and uniformly convex Banach space.

Theorem 4.1 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mapping $B_i : C \to X$ be α_i-inverse-strongly accretive for $i = 1,2$. Let $f : C \to C$ be a contraction with coefficient $\rho \in (0,1)$. Let $\{S_n\}_{n=0}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself such that $F = \bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i) \cap \Omega \neq \emptyset$, where Ω is a fixed point set of the mapping G. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)S_n\Pi_C(I-\mu_1 B_1)\Pi_C(I-\mu_2 B_2)x_n, \\ x_{n+1} = \beta_n y_n + (1-\beta_n)S_n\Pi_C(I-\mu_1 B_1)\Pi_C(I-\mu_2 B_2)y_n, \quad \forall n \geq 0, \end{cases} \tag{4.1}$$

where $0 < \mu_i < \frac{\alpha_i}{\kappa^2}$ for $i = 1,2$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1]$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\{\beta_n\} \subset [a,1]$ *for some* $a \in (0,1)$;

(iii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \alpha_{n-1}/\alpha_n = 1$;
(iv) $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$ *or* $\lim_{n\to\infty} |\beta_n - \beta_{n-1}|/\alpha_n = 0$.

Assume that $\sum_{n=1}^{\infty} \sup_{x\in D} \|S_n x - S_{n-1}x\| < \infty$ *for any bounded subset D of C and let S be a mapping of C into itself defined by* $Sx = \lim_{n\to\infty} S_n x$ *for all* $x \in C$ *and suppose that* $\operatorname{Fix}(S) = \bigcap_{i=0}^{\infty} \operatorname{Fix}(S_i)$. *Then* $\{x_n\}$ *converges strongly to* $q \in F$, *which solves the following VIP*:

$$\langle q - f(q), J(q-p)\rangle \le 0, \quad \forall p \in F.$$

Proof It is easy to see that scheme (4.1) can be rewritten as

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)S_n G(x_n), \\ x_{n+1} = \beta_n y_n + (1-\beta_n)S_n G(y_n), \quad \forall n \ge 0. \end{cases} \tag{4.2}$$

Take a fixed $p \in F$ arbitrarily. Then by Lemma 2.13 we know that $p = G(p)$. Moreover, by Lemma 2.12 we have

$$\begin{aligned} \|y_n - p\| &= \big\|\alpha_n\big(f(x_n) - p\big) + (1-\alpha_n)\big(S_n G(x_n) - p\big)\big\| \\ &\le \alpha_n \big\|f(x_n) - f(p)\big\| + \alpha_n \big\|f(p) - p\big\| + (1-\alpha_n)\big\|S_n G(x_n) - p\big\| \\ &\le \alpha_n \rho \|x_n - p\| + \alpha_n \big\|f(p) - p\big\| + (1-\alpha_n)\big\|G(x_n) - p\big\| \\ &\le \alpha_n \rho \|x_n - p\| + \alpha_n \big\|f(p) - p\big\| + (1-\alpha_n)\|x_n - p\| \\ &= \big(1 - \alpha_n(1-\rho)\big)\|x_n - p\| + \alpha_n \big\|f(p) - p\big\|. \end{aligned} \tag{4.3}$$

From (4.3) we have

$$\begin{aligned} \|x_{n+1} - p\| &= \big\|\beta_n(y_n - p) + (1-\beta_n)\big(S_n G(y_n) - p\big)\big\| \\ &\le \beta_n \|y_n - p\| + (1-\beta_n)\big\|S_n G(y_n) - p\big\| \\ &\le \beta_n \|y_n - p\| + (1-\beta_n)\big\|G(y_n) - p\big\| \\ &\le \beta_n \|y_n - p\| + (1-\beta_n)\|y_n - p\| \\ &= \|y_n - p\| \\ &\le \big(1 - \alpha_n(1-\rho)\big)\|x_n - p\| + \alpha_n \big\|f(p) - p\big\| \\ &= \big(1 - \alpha_n(1-\rho)\big)\|x_n - p\| + \alpha_n(1-\rho)\frac{\|f(p) - p\|}{1-\rho} \\ &\le \max\left\{\|x_0 - p\|, \frac{\|f(p) - p\|}{1-\rho}\right\}. \end{aligned}$$

It immediately follows that $\{x_n\}$ is bounded, and so are the sequences $\{y_n\}$, $\{G(x_n)\}$, $\{G(y_n)\}$ due to (4.3) and the nonexpansivity of G.

Let us show that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. As a matter of fact, from (4.2) we have

$$\begin{cases} y_n = \alpha_n f(x_n) + (1-\alpha_n)S_n G(x_n), \\ y_{n-1} = \alpha_{n-1} f(x_{n-1}) + (1-\alpha_{n-1})S_{n-1} G(x_{n-1}), \quad \forall n \ge 1. \end{cases}$$

Simple calculations show that

$$\begin{aligned} y_n - y_{n-1} &= \alpha_n\big(f(x_n) - f(x_{n-1})\big) + (\alpha_n - \alpha_{n-1})\big(f(x_{n-1}) - S_{n-1}G(x_{n-1})\big) \\ &\quad + (1-\alpha_n)\big(S_nG(x_n) - S_{n-1}G(x_{n-1})\big). \end{aligned}$$

It follows that

$$\begin{aligned} \|y_n - y_{n-1}\| &\le \alpha_n\big\|f(x_n) - f(x_{n-1})\big\| + |\alpha_n - \alpha_{n-1}|\big\|f(x_{n-1}) - S_{n-1}G(x_{n-1})\big\| \\ &\quad + (1-\alpha_n)\big\|S_nG(x_n) - S_{n-1}G(x_{n-1})\big\| \\ &\le \alpha_n\rho\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\big\|f(x_{n-1}) - S_{n-1}G(x_{n-1})\big\| \\ &\quad + (1-\alpha_n)\big(\big\|S_nG(x_n) - S_nG(x_{n-1})\big\| + \big\|S_nG(x_{n-1}) - S_{n-1}G(x_{n-1})\big\|\big) \\ &\le \alpha_n\rho\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\big\|f(x_{n-1}) - S_{n-1}G(x_{n-1})\big\| \\ &\quad + (1-\alpha_n)\big(\big\|G(x_n) - G(x_{n-1})\big\| + \big\|S_nG(x_{n-1}) - S_{n-1}G(x_{n-1})\big\|\big) \\ &\le \alpha_n\rho\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\big\|f(x_{n-1}) - S_{n-1}G(x_{n-1})\big\| \\ &\quad + (1-\alpha_n)\big(\|x_n - x_{n-1}\| + \big\|S_nG(x_{n-1}) - S_{n-1}G(x_{n-1})\big\|\big) \\ &\le \big(1 - \alpha_n(1-\rho)\big)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\big\|f(x_{n-1}) - S_{n-1}G(x_{n-1})\big\| \\ &\quad + \big\|S_nG(x_{n-1}) - S_{n-1}G(x_{n-1})\big\|. \end{aligned} \tag{4.4}$$

So, we have from (4.4)

$$\begin{aligned} &\big\|S_nG(y_n) - S_{n-1}G(y_{n-1})\big\| \\ &\quad\le \big\|S_nG(y_n) - S_nG(y_{n-1})\big\| + \big\|S_nG(y_{n-1}) - S_{n-1}G(y_{n-1})\big\| \\ &\quad\le \big\|G(y_n) - G(y_{n-1})\big\| + \big\|S_nG(y_{n-1}) - S_{n-1}G(y_{n-1})\big\| \\ &\quad\le \|y_n - y_{n-1}\| + \big\|S_nG(y_{n-1}) - S_{n-1}G(y_{n-1})\big\|. \end{aligned} \tag{4.5}$$

On the other hand, from (4.2) we have

$$\begin{cases} x_{n+1} = \beta_n y_n + (1-\beta_n)S_nG(y_n), \\ x_n = \beta_{n-1}y_{n-1} + (1-\beta_{n-1})S_{n-1}G(y_{n-1}). \end{cases}$$

Also, simple calculations show that

$$\begin{aligned} x_{n+1} - x_n &= \beta_n(y_n - y_{n-1}) + (\beta_n - \beta_{n-1})\big(y_{n-1} - S_{n-1}G(y_{n-1})\big) \\ &\quad + (1-\beta_n)\big(S_nG(y_n) - S_{n-1}G(y_{n-1})\big). \end{aligned} \tag{4.6}$$

Thus, it follows from (4.4)-(4.6) that for all $n \ge 1$

$$\begin{aligned} &\|x_{n+1} - x_n\| \\ &\quad\le \beta_n\|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}|\big\|y_{n-1} - S_{n-1}G(y_{n-1})\big\| \end{aligned}$$

$$
\begin{aligned}
&\quad + (1-\beta_n)\|S_nG(y_n) - S_{n-1}G(y_{n-1})\| \\
&\le \beta_n\|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}|\|y_{n-1} - S_{n-1}G(y_{n-1})\| \\
&\quad + (1-\beta_n)\bigl(\|y_n - y_{n-1}\| + \|S_nG(y_{n-1}) - S_{n-1}G(y_{n-1})\|\bigr) \\
&\le \|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}|\|y_{n-1} - S_{n-1}G(y_{n-1})\| \\
&\quad + \|S_nG(y_{n-1}) - S_{n-1}G(y_{n-1})\| \\
&\le \bigl(1-\alpha_n(1-\rho)\bigr)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\quad + \|S_nG(x_{n-1}) - S_{n-1}G(x_{n-1})\| + |\beta_n - \beta_{n-1}|\|y_{n-1} - S_{n-1}G(y_{n-1})\| \\
&\quad + \|S_nG(y_{n-1}) - S_{n-1}G(y_{n-1})\| \\
&\le \bigl(1-\alpha_n(1-\rho)\bigr)\|x_n - x_{n-1}\| + M\bigl(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|\bigr) \\
&\quad + \|S_nG(x_{n-1}) - S_{n-1}G(x_{n-1})\| + \|S_nG(y_{n-1}) - S_{n-1}G(y_{n-1})\|,
\end{aligned}
$$

where $\sup_{n\ge 0}\{\|f(x_n) - S_nG(x_n)\| + \|y_n - S_nG(y_n)\|\} \le M$ for some $M > 0$. Utilizing Lemma 2.2, we deduce from conditions (i), (iii), (iv) and the assumption on $\{S_n\}$ that

$$
\limsup_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{4.7}
$$

In terms of (4.4), we also have that $\|y_n - y_{n-1}\| \to 0$ as $n \to \infty$.

Let us show that $\|x_n - y_n\|$ and $\|x_n - S_nG(x_n)\| \to 0$ as $n \to \infty$. Indeed, since $y_n = \alpha_n f(x_n) + (1-\alpha_n)S_nG(x_n)$, we get

$$
\lim_{n\to\infty}(1-\alpha_n)\|S_nG(x_n) - y_n\| = \lim_{n\to\infty}\alpha_n\|f(x_n) - y_n\| = 0,
$$

which together with $\alpha_n \to 0$ implies that

$$
\lim_{n\to\infty}\|S_nG(x_n) - y_n\| = 0. \tag{4.8}
$$

Observe that

$$
\begin{aligned}
\|x_{n+1} - y_n\| &= (1-\beta_n)\|S_nG(y_n) - y_n\| \\
&\le (1-\beta_n)\bigl(\|S_nG(y_n) - S_nG(x_n)\| + \|S_nG(x_n) - y_n\|\bigr) \\
&\le (1-\beta_n)\bigl(\|G(y_n) - G(x_n)\| + \|S_nG(x_n) - y_n\|\bigr) \\
&\le (1-\beta_n)\bigl(\|y_n - x_n\| + \|S_nG(x_n) - y_n\|\bigr) \\
&\le (1-\beta_n)\bigl(\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| + \|S_nG(x_n) - y_n\|\bigr),
\end{aligned}
$$

which together with condition (ii) implies that

$$
\begin{aligned}
\|x_{n+1} - y_n\| &\le \frac{1-\beta_n}{\beta_n}\bigl(\|x_{n+1} - x_n\| + \|S_nG(x_n) - y_n\|\bigr) \\
&\le \frac{1-a}{a}\bigl(\|x_{n+1} - x_n\| + \|S_nG(x_n) - y_n\|\bigr).
\end{aligned}
$$

Obviously, from (4.7) and (4.8) we know that $\|x_{n+1}-y_n\| \to 0$ as $n \to \infty$. This implies that

$$\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \to 0 \quad \text{as } n \to \infty. \tag{4.9}$$

Also, from (4.8) and (4.9) we have

$$\|x_n - S_nG(x_n)\| \le \|x_n - y_n\| + \|y_n - S_nG(x_n)\| \to 0 \quad \text{as } n \to \infty. \tag{4.10}$$

Let us show that $\|x_n - G(x_n)\| \to 0$ as $n \to \infty$. Indeed, for simplicity, put $q = \Pi_C(p - \mu_2B_2p)$, $u_n = \Pi_C(x_n - \mu_2B_2x_n)$ and $v_n = \Pi_C(u_n - \mu_1B_1u_n)$. Then $v_n = G(x_n)$. From Lemma 2.11 we have

$$\begin{aligned}
\|u_n - q\|^2 &= \|\Pi_C(x_n - \mu_2B_2x_n) - \Pi_C(p - \mu_2B_2p)\|^2 \\
&\le \|x_n - p - \mu_2(B_2x_n - B_2p)\|^2 \\
&\le \|x_n - p\|^2 - 2\mu_2(\alpha_2 - \kappa^2\mu_2)\|B_2x_n - B_2p\|^2,
\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}
\|v_n - p\|^2 &= \|\Pi_C(u_n - \mu_1B_1u_n) - \Pi_C(q - \mu_1B_1q)\|^2 \\
&\le \|u_n - q - \mu_1(B_1u_n - B_1q)\|^2 \\
&\le \|u_n - q\|^2 - 2\mu_1(\alpha_1 - \kappa^2\mu_1)\|B_1u_n - B_1q\|^2.
\end{aligned} \tag{4.12}$$

Substituting (4.11) into (4.12), we obtain

$$\begin{aligned}
\|v_n - p\|^2 \le{}& \|x_n - p\|^2 - 2\mu_2(\alpha_2 - \kappa^2\mu_2)\|B_2x_n - B_2p\|^2 \\
&- 2\mu_1(\alpha_1 - \kappa^2\mu_1)\|B_1u_n - B_1q\|^2.
\end{aligned} \tag{4.13}$$

According to Lemma 2.2, we have from (4.2)

$$\begin{aligned}
\|y_n - p\|^2 &= \|\alpha_n(f(x_n) - f(p)) + (1-\alpha_n)(S_nv_n - p) + \alpha_n(f(p) - p)\|^2 \\
&\le \|\alpha_n(f(x_n) - f(p)) + (1-\alpha_n)(S_nv_n - p)\|^2 + 2\alpha_n\langle f(p) - p, J(y_n - p)\rangle \\
&\le \alpha_n\|f(x_n) - f(p)\|^2 + (1-\alpha_n)\|S_nv_n - p\|^2 + 2\alpha_n\langle f(p) - p, J(y_n - p)\rangle \\
&\le \alpha_n\rho^2\|x_n - p\|^2 + (1-\alpha_n)\|v_n - p\|^2 + 2\alpha_n\langle f(p) - p, J(y_n - p)\rangle \\
&\le \alpha_n\rho\|x_n - p\|^2 + (1-\alpha_n)\|v_n - p\|^2 + 2\alpha_n\|f(p) - p\|\|y_n - p\|,
\end{aligned}$$

which together with (4.13) and the convexity of $\|\cdot\|^2$ implies that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad = \|\beta_n(y_n - p) + (1-\beta_n)(S_nG(y_n) - p)\|^2 \\
&\quad \le \beta_n\|y_n - p\|^2 + (1-\beta_n)\|S_nG(y_n) - p\|^2 \\
&\quad \le \beta_n\|y_n - p\|^2 + (1-\beta_n)\|G(y_n) - p\|^2
\end{aligned}$$

$$
\begin{aligned}
&\le \beta_n\|y_n - p\|^2 + (1-\beta_n)\|y_n - p\|^2 \\
&= \|y_n - p\|^2 \\
&\le \alpha_n\rho\|x_n - p\|^2 + (1-\alpha_n)\|v_n - p\|^2 + 2\alpha_n\big\|f(p) - p\big\|\|y_n - p\| \\
&\le \alpha_n\rho\|x_n - p\|^2 + (1-\alpha_n)\|v_n - p\|^2 + M_1\alpha_n \\
&\le \alpha_n\rho\|x_n - p\|^2 + (1-\alpha_n)\big[\|x_n - p\|^2 - 2\mu_2\big(\alpha_2 - \kappa^2\mu_2\big)\|B_2x_n - B_2p\|^2 \\
&\quad - 2\mu_1\big(\alpha_1 - \kappa^2\mu_1\big)\|B_1u_n - B_1q\|^2\big] + M_1\alpha_n \\
&= \big(1 - \alpha_n(1-\rho)\big)\|x_n - p\|^2 - 2(1-\alpha_n)\big[\mu_2\big(\alpha_2 - \kappa^2\mu_2\big)\|B_2x_n - B_2p\|^2 \\
&\quad + \mu_1\big(\alpha_1 - \kappa^2\mu_1\big)\|B_1u_n - B_1q\|^2\big] + M_1\alpha_n \\
&\le \|x_n - p\|^2 - 2(1-\alpha_n)\big[\mu_2\big(\alpha_2 - \kappa^2\mu_2\big)\|B_2x_n - B_2p\|^2 \\
&\quad + \mu_1\big(\alpha_1 - \kappa^2\mu_1\big)\|B_1u_n - B_1q\|^2\big] + M_1\alpha_n,
\end{aligned} \tag{4.14}
$$

where $\sup_{n\ge 0}\{2\|f(p) - p\|\|y_n - p\|\} \le M_1$ for some $M_1 > 0$. So, it follows that

$$
\begin{aligned}
&2(1-\alpha_n)\big[\mu_2\big(\alpha_2 - \kappa^2\mu_2\big)\|B_2x_n - B_2p\|^2 + \mu_1\big(\alpha_1 - \kappa^2\mu_1\big)\|B_1u_n - B_1q\|^2\big] \\
&\quad \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + M_1\alpha_n \\
&\quad \le \big(\|x_n - p\| + \|x_{n+1} - p\|\big)\|x_n - x_{n+1}\| + M_1\alpha_n.
\end{aligned}
$$

Since $0 < \mu_i < \frac{\alpha_i}{\kappa^2}$ for $i = 1, 2$, from (4.7) and $\alpha_n \to 0$ we obtain

$$
\lim_{n\to\infty}\|B_2x_n - B_2p\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|B_1u_n - B_1q\| = 0. \tag{4.15}
$$

Utilizing Proposition 2.1 and Lemma 2.1, we have

$$
\begin{aligned}
\|u_n - q\|^2 &= \big\|\Pi_C(x_n - \mu_2B_2x_n) - \Pi_C(p - \mu_2B_2p)\big\|^2 \\
&\le \big\langle x_n - \mu_2B_2x_n - (p - \mu_2B_2p), J(u_n - q)\big\rangle \\
&= \big\langle x_n - p, J(u_n - q)\big\rangle + \mu_2\big\langle B_2p - B_2x_n, J(u_n - q)\big\rangle \\
&\le \frac{1}{2}\big[\|x_n - p\|^2 + \|u_n - q\|^2 - g_1\big(\big\|x_n - u_n - (p - q)\big\|\big)\big] \\
&\quad + \mu_2\|B_2p - B_2x_n\|\|u_n - q\|,
\end{aligned}
$$

which implies that

$$
\|u_n - q\|^2 \le \|x_n - p\|^2 - g_1\big(\big\|x_n - u_n - (p - q)\big\|\big) + 2\mu_2\|B_2p - B_2x_n\|\|u_n - q\|. \tag{4.16}
$$

In the same way, we derive

$$
\begin{aligned}
\|v_n - p\|^2 &= \big\|\Pi_C(u_n - \mu_1B_1u_n) - \Pi_C(q - \mu_1B_1q)\big\|^2 \\
&\le \big\langle u_n - \mu_1B_1u_n - (q - \mu_1B_1q), J(v_n - p)\big\rangle \\
&= \big\langle u_n - q, J(v_n - p)\big\rangle + \mu_1\big\langle B_1q - B_1u_n, J(v_n - p)\big\rangle
\end{aligned}
$$

$$\begin{aligned}&\le \frac{1}{2}\big[\|u_n-q\|^2+\|v_n-p\|^2-g_2\big(\big\|u_n-v_n+(p-q)\big\|\big)\big]\\&\quad+\mu_1\|B_1q-B_1u_n\|\|v_n-p\|,\end{aligned}$$

which implies that

$$\|v_n-p\|^2\le\|u_n-q\|^2-g_2\big(\big\|u_n-v_n+(p-q)\big\|\big)+2\mu_1\|B_1q-B_1u_n\|\|v_n-p\|. \tag{4.17}$$

Substituting (4.16) into (4.17), we get

$$\begin{aligned}\|v_n-p\|^2&\le\|x_n-p\|^2-g_1\big(\big\|x_n-u_n-(p-q)\big\|\big)-g_2\big(\big\|u_n-v_n+(p-q)\big\|\big)\\&\quad+2\mu_2\|B_2p-B_2x_n\|\|u_n-q\|+2\mu_1\|B_1q-B_1u_n\|\|v_n-p\|.\end{aligned} \tag{4.18}$$

From (4.14) and (4.18), we have

$$\begin{aligned}&\|x_{n+1}-p\|^2\\&\quad\le\alpha_n\rho\|x_n-p\|^2+(1-\alpha_n)\|v_n-p\|^2+M_1\alpha_n\\&\quad\le\alpha_n\rho\|x_n-p\|^2+(1-\alpha_n)\big[\|x_n-p\|^2-g_1\big(\big\|x_n-u_n-(p-q)\big\|\big)\\&\qquad-g_2\big(\big\|u_n-v_n+(p-q)\big\|\big)+2\mu_2\|B_2p-B_2x_n\|\|u_n-q\|\\&\qquad+2\mu_1\|B_1q-B_1u_n\|\|v_n-p\|\big]+M_1\alpha_n\\&\quad\le\big(1-\alpha_n(1-\rho)\big)\|x_n-p\|^2-(1-\alpha_n)\big[g_1\big(\big\|x_n-u_n-(p-q)\big\|\big)\\&\qquad+g_2\big(\big\|u_n-v_n+(p-q)\big\|\big)\big]+2\mu_2\|B_2p-B_2x_n\|\|u_n-q\|\\&\qquad+2\mu_1\|B_1q-B_1u_n\|\|v_n-p\|+M_1\alpha_n\\&\quad\le\|x_n-p\|^2-(1-\alpha_n)\big[g_1\big(\big\|x_n-u_n-(p-q)\big\|\big)+g_2\big(\big\|u_n-v_n+(p-q)\big\|\big)\big]\\&\qquad+2\mu_2\|B_2p-B_2x_n\|\|u_n-q\|+2\mu_1\|B_1q-B_1u_n\|\|v_n-p\|+M_1\alpha_n,\end{aligned}$$

which hence implies that

$$\begin{aligned}&(1-\alpha_n)\big[g_1\big(\big\|x_n-u_n-(p-q)\big\|\big)+g_2\big(\big\|u_n-v_n+(p-q)\big\|\big)\big]\\&\quad\le\|x_n-p\|^2-\|x_{n+1}-p\|^2+2\mu_2\|B_2p-B_2x_n\|\|u_n-q\|\\&\qquad+2\mu_1\|B_1q-B_1u_n\|\|v_n-p\|+M_1\alpha_n\\&\quad\le\alpha_nM_1+\big(\|x_n-p\|+\|x_{n+1}-p\|\big)\|x_n-x_{n+1}\|+2\mu_2\|B_2p-B_2x_n\|\|u_n-q\|\\&\qquad+2\mu_1\|B_1q-B_1u_n\|\|v_n-p\|.\end{aligned}$$

Thus, from (4.7), (4.15) and $\alpha_n\to 0$ we have

$$\lim_{n\to\infty}g_1\big(\big\|x_n-u_n-(p-q)\big\|\big)=0\quad\text{and}\quad\lim_{n\to\infty}g_2\big(\big\|u_n-v_n+(p-q)\big\|\big)=0. \tag{4.19}$$

Utilizing the properties of g_1 and g_2, we deduce that

$$\lim_{n\to\infty}\big\|x_n-u_n-(p-q)\big\|=0\quad\text{and}\quad\lim_{n\to\infty}\big\|u_n-v_n+(p-q)\big\|=0. \tag{4.20}$$

From (4.20), we obtain

$$\|x_n - v_n\| \le \big\|x_n - u_n - (p-q)\big\| + \big\|u_n - v_n + (p-q)\big\| \to 0 \quad \text{as } n \to \infty.$$

That is,

$$\lim_{n\to\infty} \big\|x_n - G(x_n)\big\| = 0. \tag{4.21}$$

On the other hand, we observe that

$$\big\|S_n G(x_n) - G(x_n)\big\| \le \big\|S_n G(x_n) - y_n\big\| + \|y_n - x_n\| + \big\|x_n - G(x_n)\big\|. \tag{4.22}$$

So, it follows from (4.8), (4.9) and (4.21) that

$$\lim_{n\to\infty} \big\|S_n G(x_n) - G(x_n)\big\| = 0. \tag{4.23}$$

By (4.23) and Lemma 2.6, we have

$$\begin{aligned} \big\|SG(x_n) - G(x_n)\big\| &\le \big\|SG(x_n) - S_n G(x_n)\big\| + \big\|S_n G(x_n) - G(x_n)\big\| \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{4.24}$$

In terms of (4.21) and (4.24), we have

$$\begin{aligned} \|x_n - Sx_n\| &\le \big\|x_n - G(x_n)\big\| + \big\|G(x_n) - SG(x_n)\big\| + \big\|SG(x_n) - Sx_n\big\| \\ &\le 2\big\|x_n - G(x_n)\big\| + \big\|G(x_n) - SG(x_n)\big\| \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{4.25}$$

Define a mapping $Wx = (1-\theta)Sx + \theta G(x)$ and $\theta \in (0,1)$ is a constant. Then by Lemma 2.8 we have that $\mathrm{Fix}(W) = \mathrm{Fix}(S) \cap \mathrm{Fix}(G) = F$. We observe that

$$\|x_n - Wx_n\| = \big\|(1-\theta)(x_n - Sx_n) + \theta\big(x_n - G(x_n)\big)\big\| \le (1-\theta)\|x_n - Sx_n\| + \theta\big\|x_n - G(x_n)\big\|.$$

From (4.21) and (4.25), we obtain

$$\lim_{n\to\infty} \|x_n - Wx_n\| = 0. \tag{4.26}$$

Utilizing the arguments similar to those of (3.29) in the proof of Theorem 3.1, we can deduce that

$$\limsup_{n\to\infty} \big\langle f(q) - q, J(y_n - q)\big\rangle \le 0. \tag{4.27}$$

Finally, let us show that $x_n \to q$ as $n \to \infty$. We observe that

$$\begin{aligned} &\|y_n - q\|^2 \\ &\quad = \big\|\alpha_n\big(f(x_n) - f(q)\big) + (1-\alpha_n)\big(S_n G(x_n) - q\big) + \alpha_n\big(f(q) - q\big)\big\|^2 \end{aligned}$$

$$
\begin{aligned}
&\le \big\|\alpha_n\big(f(x_n)-f(q)\big)+(1-\alpha_n)\big(S_nG(x_n)-q\big)\big\|^2+2\alpha_n\big\langle f(q)-q,J(y_n-q)\big\rangle\\
&\le \alpha_n\big\|f(x_n)-f(q)\big\|^2+(1-\alpha_n)\big\|S_nG(x_n)-q\big\|^2+2\alpha_n\big\langle f(q)-q,J(y_n-q)\big\rangle\\
&\le \alpha_n\rho\|x_n-q\|^2+(1-\alpha_n)\|x_n-q\|^2+2\alpha_n\big\langle f(q)-q,J(y_n-q)\big\rangle\\
&= \big(1-\alpha_n(1-\rho)\big)\|x_n-q\|^2+2\alpha_n\big\langle f(q)-q,J(y_n-q)\big\rangle.
\end{aligned}
\tag{4.28}
$$

By the convexity of $\|\cdot\|^2$ and (4.2), we get

$$
\|x_{n+1}-q\|^2\le\beta_n\|y_n-q\|^2+(1-\beta_n)\big\|S_nG(y_n)-q\big\|^2\le\|y_n-q\|^2,
$$

which together with (4.28) leads to

$$
\begin{aligned}
\|x_{n+1}-q\|^2&\le\big(1-\alpha_n(1-\rho)\big)\|x_n-q\|^2+2\alpha_n\big\langle f(q)-q,J(y_n-q)\big\rangle\\
&=\big(1-\alpha_n(1-\rho)\big)\|x_n-q\|^2+\alpha_n(1-\rho)\cdot\frac{2\langle f(q)-q,J(y_n-q)\rangle}{1-\rho}.
\end{aligned}
\tag{4.29}
$$

Applying Lemma 2.2 to (4.29), we obtain that $x_n\to q$ as $n\to\infty$. This completes the proof. □

Corollary 4.1 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mapping $B_i: C\to X$ be α_i-inverse-strongly accretive for $i=1,2$. Let $f: C\to C$ be a contraction with coefficient $\rho\in(0,1)$. Let S be a nonexpansive mapping of C into itself such that $F=\mathrm{Fix}(S)\cap\Omega\ne\emptyset$, where Ω is a fixed point set of the mapping G. For arbitrarily given $x_0\in C$, let $\{x_n\}$ be a sequence generated by*

$$
\begin{cases}
y_n=\alpha_nf(x_n)+(1-\alpha_n)S\Pi_C(I-\mu_1B_1)\Pi_C(I-\mu_2B_2)x_n,\\
x_{n+1}=\beta_ny_n+(1-\beta_n)S\Pi_C(I-\mu_1B_1)\Pi_C(I-\mu_2B_2)y_n,\quad\forall n\ge 0,
\end{cases}
$$

where $0<\mu_i<\frac{\alpha_i}{\kappa^2}$ for $i=1,2$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1]$ satisfying the following conditions:

(i) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=0}^\infty\alpha_n=\infty$;
(ii) $\{\beta_n\}\subset[a,1]$ *for some* $a\in(0,1)$;
(iii) $\sum_{n=1}^\infty|\alpha_n-\alpha_{n-1}|<\infty$ *or* $\lim_{n\to\infty}\alpha_{n-1}/\alpha_n=1$;
(iv) $\sum_{n=1}^\infty|\beta_n-\beta_{n-1}|<\infty$ *or* $\lim_{n\to\infty}|\beta_n-\beta_{n-1}|/\alpha_n=0$.

Then $\{x_n\}$ converges strongly to $q\in F$, which solves the following VIP:

$$
\big\langle q-f(q),J(q-p)\big\rangle\le 0,\quad\forall p\in F.
$$

Remark 4.1 Theorem 4.1 improves, extends, supplements and develops [14, Theorem 3.1] in the following aspects. The composite iterative algorithm in [14, Theorem 3.1] is extended to develop the composite iterative algorithm in Theorem 4.1. Compared with the iterative algorithm in [14, Theorem 3.1], each iteration step in the iterative algorithm of Theorem 4.1 is very different from the corresponding step in the iterative algorithm of [14, Theorem 3.1] because each iteration step in the iterative algorithm of Theorem 4.1 involves the composite operator $S_n\Pi_C(I-\mu_1B_1)\Pi_C(I-\mu_2B_2)$. In the proof of [14, Theorem 3.1], Lemma 2.4 was used to derive $\|x_{n+1} - x_n\| \to 0$. However, in the proof of Theorem 4.1, we only use Lemma 2.2 to derive $\|x_{n+1} - x_n\| \to 0$. Thus, Theorem 4.1 drops the restriction $\limsup_{n\to\infty} \beta_n < 1$.

Corollary 4.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let the mapping $B_i : C \to H$ be α_i-inverse-strongly monotone for $i = 1, 2$. Let $f : C \to C$ be a contraction with coefficient $\rho \in (0, 1)$. Let $\{S_n\}_{n=0}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself such that $F = \bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i) \cap \Omega \neq \emptyset$, where Ω is a fixed point set of the mapping G. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = \alpha_n f(x_n) + (1 - \alpha_n)S_n P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)x_n, \\ x_{n+1} = \beta_n y_n + (1 - \beta_n)S_n P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)y_n, \quad \forall n \geq 0, \end{cases}$$

where $0 < \mu_i < 2\alpha_i$ for $i = 1, 2$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1]$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\{\beta_n\} \subset [a, 1]$ *for some* $a \in (0, 1)$;

(iii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \alpha_{n-1}/\alpha_n = 1$;

(iv) $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$ *or* $\lim_{n\to\infty} |\beta_n - \beta_{n-1}|/\alpha_n = 0$.

Assume that $\sum_{n=1}^{\infty} \sup_{x\in D} \|S_n x - S_{n-1}x\| < \infty$ for any bounded subset D of C and let S be a mapping of C into itself defined by $Sx = \lim_{n\to\infty} S_n x$ for all $x \in C$ and suppose that $\mathrm{Fix}(S) = \bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i)$. Then $\{x_n\}$ converges strongly to $q \in F$, which solves the following VIP:

$$\langle q - f(q), J(q - p) \rangle \leq 0, \quad \forall p \in F.$$

Corollary 4.3 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let the mapping $B_i : C \to H$ be α_i-inverse-strongly monotone for $i = 1, 2$. Let $f : C \to C$ be a contraction with coefficient $\rho \in (0, 1)$. Let S be a nonexpansive mapping of C into itself such that $F = \mathrm{Fix}(S) \cap \Omega \neq \emptyset$, where Ω is a fixed point set of the mapping G. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = \alpha_n f(x_n) + (1 - \alpha_n)SP_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)x_n, \\ x_{n+1} = \beta_n y_n + (1 - \beta_n)SP_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)y_n, \quad \forall n \geq 0, \end{cases}$$

where $0 < \mu_i < \frac{\alpha_i}{\kappa^2}$ for $i = 1, 2$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1]$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\{\beta_n\} \subset [a, 1]$ *for some* $a \in (0, 1)$;

(iii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \alpha_{n-1}/\alpha_n = 1$;

(iv) $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$ *or* $\lim_{n\to\infty} |\beta_n - \beta_{n-1}|/\alpha_n = 0$.

Then $\{x_n\}$ converges strongly to $q \in F$, which solves the following VIP:

$$\langle q - f(q), q - p \rangle \leq 0, \quad \forall p \in F.$$

Now, we say that a mapping $T : C \to C$ has property $(*)$ if there exists a constant $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\big\|(I - T)x - (I - T)y\big\|^2, \quad \forall x, y \in C.$$

Whenever $k = 0$, then T is nonexpansive. Put $A = I - T$, where $T : C \to C$ is a mapping having property $(*)$. Then A is $(1 - k)/2$-inverse-strongly monotone. Indeed, we have

$$\big\|(I - A)x - (I - A)y\big\|^2 \le \|x - y\|^2 + k\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Since H is a real Hilbert space, we have

$$\big\|(I - A)x - (I - A)y\big\|^2 = \|x - y\|^2 + \|Ax - Ay\|^2 - 2\langle x - y, Ax - Ay\rangle,$$

and hence

$$\begin{aligned}
&\langle x - y, Ax - Ay\rangle \ge \frac{1-k}{2}\|Ax - Ay\|^2 \\
&\quad\Rightarrow\quad \big\|(I - T)x - (I - T)y\big\| \le \frac{2}{1-k}\|x - y\| \\
&\quad\Rightarrow\quad \|Tx - Ty\|^2 \le \|x - y\|^2 + k\big\|(I - T)x - (I - T)y\big\|^2 \le \left(\frac{1+k}{1-k}\right)^2\|x - y\|^2.
\end{aligned}$$

Thus, if T is a mapping having property $(*)$, then T is Lipschitz continuous with constant $\frac{1+k}{1-k}$, *i.e.*, $\|Tx - Ty\| \le \frac{1+k}{1-k}\|x - y\|$ for all $x, y \in C$. We denote by $\mathrm{Fix}(T)$ a fixed point set of T. It is obvious that the class of mappings having property $(*)$ strictly includes the class of nonexpansive mappings.

Further, utilizing Corollary 4.3 we first derive a strong convergence result for finding a common fixed point of a nonexpansive mapping and a mapping having property $(*)$.

Corollary 4.4 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a mapping having property $(*)$ and let $S : C \to C$ be a nonexpansive mapping such that $\mathrm{Fix}(S) \cap \mathrm{Fix}(T) \ne \emptyset$. Let $f : C \to C$ be a contraction with coefficient $\rho \in (0,1)$. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = \alpha_n f(x_n) + (1 - \alpha_n)S((1 - \lambda)x_n + \lambda Tx_n), \\ x_{n+1} = \beta_n y_n + (1 - \beta_n)S((1 - \lambda)y_n + \lambda Ty_n), \quad \forall n \ge 0, \end{cases} \tag{4.30}$$

where $0 < \lambda < 1 - k$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1]$ satisfying the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(ii) $\{\beta_n\} \subset [a,1]$ *for some* $a \in (0,1)$;

(iii) $\sum_{n=1}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$ *or* $\lim_{n\to\infty}\alpha_{n-1}/\alpha_n = 1$;

(iv) $\sum_{n=1}^{\infty}|\beta_n - \beta_{n-1}| < \infty$ *or* $\lim_{n\to\infty}|\beta_n - \beta_{n-1}|/\alpha_n = 0$.

Then $\{x_n\}$ converges strongly to $q \in \mathrm{Fix}(S) \cap \mathrm{Fix}(T)$, which solves the following VIP:

$$\big\langle q - f(q), q - p\big\rangle \le 0, \quad \forall p \in \mathrm{Fix}(S) \cap \mathrm{Fix}(T).$$

Proof In Corollary 4.3, we put $B_1 = I - T$, $B_2 = 0$ and $\mu_1 = \lambda$. Then GSVI (1.1) is equivalent to the VIP of finding $x^* \in C$ such that

$$\langle B_1 x^*, x - x^* \rangle \geq 0, \quad \forall x \in C.$$

In this case, B_1 is $(1-k)/2$-inverse-strongly monotone. It is not hard to see that $\mathrm{Fix}(T) = \mathrm{VI}(C, B_1)$. As a matter of fact, we have, for $\lambda > 0$,

$$\begin{aligned}
u \in \mathrm{VI}(C, B_1) \quad &\Leftrightarrow \quad \langle B_1 u, y - u \rangle \geq 0 \quad \forall y \in C \\
&\Leftrightarrow \quad \langle u - \lambda B_1 u - u, u - y \rangle \geq 0 \quad \forall y \in C \\
&\Leftrightarrow \quad u = P_C(u - \lambda B_1 u) \\
&\Leftrightarrow \quad u = P_C(u - \lambda u + \lambda Tu) \\
&\Leftrightarrow \quad \langle u - \lambda u + \lambda Tu - u, u - y \rangle \geq 0 \quad \forall y \in C \\
&\Leftrightarrow \quad \langle u - Tu, u - y \rangle \leq 0 \quad \forall y \in C \\
&\Leftrightarrow \quad u = Tu \\
&\Leftrightarrow \quad u \in \mathrm{Fix}(T).
\end{aligned}$$

Accordingly, we know that $F = \mathrm{Fix}(S) \cap \varOmega = \mathrm{Fix}(S) \cap \mathrm{Fix}(T)$,

$$\begin{aligned}
P_C(I - \mu_1 B_1) P_C(I - \mu_2 B_2) x_n &= P_C(I - \mu_1 B_1) x_n \\
&= P_C\big((1-\lambda)x_n + \lambda T x_n\big) = (1-\lambda)x_n + \lambda T x_n,
\end{aligned}$$

and

$$\begin{aligned}
P_C(I - \mu_1 B_1) P_C(I - \mu_2 B_2) y_n &= P_C(I - \mu_1 B_1) y_n \\
&= P_C\big((1-\lambda)y_n + \lambda T y_n\big) = (1-\lambda)y_n + \lambda T y_n.
\end{aligned}$$

So, scheme (4.2) reduces to (4.30). Therefore, the desired result follows from Corollary 4.3. □

Utilizing Corollary 4.3, we also have the following result.

Corollary 4.5 *Let H be a real Hilbert space. Let A be an α-inverse-strongly monotone mapping of H into itself and let S be a nonexpansive mapping of H into itself such that* $\mathrm{Fix}(S) \cap \cap A^{-1}0 \neq \emptyset$. *Let $f : H \to H$ be a contraction with coefficient $\rho \in (0,1)$. For arbitrarily given $x_0 \in H$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases}
y_n = \alpha_n f(x_n) + (1-\alpha_n) S(x_n - \lambda A x_n), \\
x_{n+1} = \beta_n y_n + (1-\beta_n) S(y_n - \lambda A y_n), \quad \forall n \geq 0,
\end{cases} \tag{4.31}$$

where $0 < \lambda < 2\alpha$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1]$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\{\beta_n\} \subset [a,1]$ *for some* $a \in (0,1)$;
(iii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \alpha_{n-1}/\alpha_n = 1$;
(iv) $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$ *or* $\lim_{n\to\infty} |\beta_n - \beta_{n-1}|/\alpha_n = 0$.

Then $\{x_n\}$ *converges strongly to* $q \in \mathrm{Fix}(S) \cap A^{-1}0$, *which solves the following VIP*:

$$\langle q - f(q), q - p\rangle \le 0, \quad \forall p \in \mathrm{Fix}(S) \cap A^{-1}0.$$

Proof In Corollary 4.3, we put $C = H$, $B_1 = A$, $B_2 = 0$ and $\mu_1 = \lambda$. Then we know that $P_H = I$ and $A^{-1}0 = \mathrm{VI}(H, A) = \Omega$. Moreover, we know that $F = \mathrm{Fix}(S) \cap \Omega = \mathrm{Fix}(S) \cap A^{-1}0$,

$$P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)x_n = x_n - \lambda A x_n,$$

and

$$P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)y_n = y_n - \lambda A y_n.$$

So, scheme (4.2) reduces to (4.31). Therefore, the desired result follows from Corollary 4.3. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors participated in the design of this work and performed equally. All authors read and approved the final manuscript.

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