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Strong convergence theorems for common solutions of a family of nonexpansive mappings and an accretive operator

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Abstract
In this paper, common solutions of a family of nonexpansive mappings and an accretive operator are investigated based on a viscosity iterative method. Strong convergence theorems for common solutions are established in a Banach space.
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1 Introduction

Fixed point theory has emerged as an effective and powerful tool for studying a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity and optimization; see [1–12] and the references therein. The computation of solutions is important in the study of many real world problems. The well-known convex feasibility problem which captures applications in various disciplines such as image restoration and radiation therapy treatment planning is to find a point in the intersection of common fixed point sets of a family of nonlinear mappings; see, for example, [13–24] and the references therein.

The aim of this paper is to investigate a common solution problem of a family of nonexpansive mappings and an accretive operator based on a viscosity iterative method. The organization of this article is as follows. In Section 2, we provide some necessary preliminaries. In Section 3, a viscosity iterative method is discussed. Strong convergence theorems of common solutions are established in a reflexive and strictly convex Banach space E which enjoys weakly continuous duality mappings.

2 Preliminaries

Throughout this paper, we assume that E is a real Banach space. Let C be a nonempty, closed and convex subset of E, and let $T : C \to C$ be a mapping. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in C : Tx = x\}$.

Recall that $T : C \to C$ is nonexpansive iff

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

$T : C \to C$ is a contraction iff there exists a constant $\alpha \in (0,1)$ such that

$$\|f(x) - f(y)\| \le \alpha \|x - y\|, \quad \forall x, y \in C.$$

We use Π_C to denote the collection of all contractions on C. That is, $\Pi_C := \{f | f : C \to C \text{ a contraction}\}$.

The Picard iterative algorithm is an efficient algorithm to study contractions. However, the Picard iterative algorithm fails to converge to fixed points of nonexpansive mappings even that their fixed point sets are not empty. One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping. More precisely, take $t \in (0,1)$ and define a mapping $T_t : C \to C$ by

$$T_t x = tf(u) + (1-t)Tx, \quad \forall x \in C,$$

where $u \in C$ is a fixed element and f is a contraction on C with the constant α. It is easy to see that T_t is a contraction with the constant α. Indeed, we have the following:

$$\begin{aligned} \|T_t x - T_t y\| &\le t\|f(x) - f(y)\| + (1-t)\|Tx - Ty\| \\ &\le t\alpha\|x - y\| + (1-t)\|x - y\| \\ &= \bigl[1 - t(1-\alpha)\bigr]\|x - y\|. \end{aligned}$$

Banach's contraction mapping principle guarantees that T_t has a unique fixed point. We denote the unique fixed point by x_t. Reich [25] proved that if E is a uniformly smooth Banach space, then x_t strongly converges to a fixed point of T, and the limit defines the (unique) sunny nonexpansive retraction from Π_C onto $F(T)$. Recently, Xu [26] further proved that the above results still hold in reflexive Banach spaces which have weakly continuous duality mappings.

Recall that the normal Mann iterative algorithm was introduced by Mann in 1953. Since then the construction of fixed points for nonexpansive mappings via the normal Mann iterative algorithm has been extensively investigated by many authors.

The normal Mann iterative algorithm generates a sequence $\{x_n\}$ in the following manner:

$$\forall x_1 \in C, \qquad x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, \quad \forall n \ge 1,$$

where the sequence $\{\alpha_n\}$ is in the interval $(0,1)$. If T is a nonexpansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by the normal Mann iterative algorithm converges weakly to a fixed point of T (this is also valid in a uniformly convex Banach space with the Fréchet differentiable norm). Since the Mann iterative algorithm only has weak convergence in infinite dimension spaces, many authors tried to modify the normal Mann iteration algorithm to have strong convergence for nonexpansive mappings.

Kim and Xu [27] considered the following iterative algorithm.

$$\begin{cases} x_0 \in C \quad \text{arbitrarily chosen}, \\ y_n = \beta_n x_n + (1 - \beta_n) T x_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) y_n, \quad n \geq 0, \end{cases}$$

where T is a nonexpansive mapping of C into itself, $u \in C$ is a given point, $\{\alpha_n\}$ and $\{\beta_n\}$ are two real number sequences in $(0,1)$. They proved that the sequence $\{x_n\}$ generated by the above iterative algorithm strongly converges to a fixed point of the mapping T provided that the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy appropriate conditions.

Recently, many authors have studied the following convex feasibility problem (CFP): $x \in \bigcap_{i=1}^{N} C_i$, where $N \geq 1$ is an integer, and each C_i is assumed to be the fixed point set of a nonexpansive mapping T_i, $i = 1, 2, \ldots, N$. There is a considerable investigation on CFP in the setting of Hilbert spaces which captures applications in various disciplines such as image restoration [28], computer tomography [29] and radiation therapy treatment planning [30].

In this paper, we consider the mapping W_n defined by

$$\begin{aligned} &U_{n,n+1} = I, \\ &U_{n,n} = \gamma_n T_n U_{n,n+1} + (1 - \gamma_n) I, \\ &U_{n,n-1} = \gamma_{n-1} T_{n-1} U_{n,n} + (1 - \gamma_{n-1}) I, \\ &\vdots \\ &U_{n,k} = \gamma_k T_k U_{n,k+1} + (1 - \gamma_k) I, \\ &U_{n,k-1} = \gamma_{k-1} T_{k-1} U_{n,k} + (1 - \gamma_{k-1}) I, \\ &\vdots \\ &U_{n,2} = \gamma_2 T_2 U_{n,3} + (1 - \gamma_2) I, \\ &W_n = U_{n,1} = \gamma_1 T_1 U_{n,2} + (1 - \gamma_1) I, \end{aligned} \tag{2.1}$$

where $\{\gamma_1\}, \{\gamma_2\}, \ldots$ are real numbers such that $0 \leq \gamma_n \leq 1$ and $T_1, T_2, \ldots$ are nonexpansive mappings of C into itself. Nonexpansivity of each T_i ensures the nonexpansivity of W_n.

We have the following lemmas which are important to prove our main results.

Lemma 2.1 [31] *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and let $\gamma_1, \gamma_2, \ldots$ be real numbers such that $0 < \gamma_n \leq b < 1$, where b is some real number, for any $n \geq 1$. Then, for every $x \in C$ and $k \in N$, the limit $\lim_{n\to\infty} U_{n,k} x$ exists.*

Using Lemma 2.1, one can define the mapping W of C into itself as follows.

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1} x, \quad \forall x \in C. \tag{2.2}$$

Such a mapping W is called the *W-mapping* generated by $T_1, T_2, \ldots$ and $\gamma_1, \gamma_2, \ldots$. Throughout this paper, we will assume that $0 < \gamma_n \leq b < 1$ for all $n \geq 1$.

Lemma 2.2 [31] *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and let $\gamma_1, \gamma_2, \ldots$ be real numbers such that $0 < \gamma_n \leq b < 1$ for any $n \geq 1$. Then $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$.*

Let I denote the identity operator on E. An operator $A \subset E \times E$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \bigcup\{Az : z \in D(A)\}$ is said to be *accretive* if for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, there exists $j(x_1 - x_2) \in J(x_1 - x_2)$ such that

$$\langle y_1 - y_2, j(x_1 - x_2)\rangle \geq 0.$$

An accretive operator A is said to be m-accretive if $R(I + rA) = E$ for all $r > 0$. In a real Hilbert space, an operator A is m-accretive if and only if A is maximal monotone. In this paper, we use $A^{-1}(0)$ to denote the set of zero points of A. Interest in accretive operators, which is an important class of nonlinear operators, stems mainly from their firm connection with equations of evolution.

For an accretive operator A, we can define a nonexpansive single-valued mapping $J_r : R(I + rA) \to D(A)$ by $J_r = (I + rA)^{-1}$ for each $r > 0$, which is called the *resolvent* of A. One of classical methods of studying the problem $0 \in Ax$, where $A \subset E \times E$ is an accretive operator, is the following:

$$x_0 \in E, \qquad x_{n+1} = J_{r_n} x_n, \quad \forall n \geq 0,$$

where $J_{r_n} = (I + r_n A)^{-1}$ and $\{r_n\}$ is a sequence of positive real numbers. Recently, different regularization iterative methods have been employed to treat zero points of accretive operators in the framework of Banach spaces; see [32–36] and the references therein.

In this paper, we investigate common fixed point problems of a family of nonexpansive mappings generated in (2.1) and a zero point problem of an accretive operator based on a viscosity approximation method. Strong convergence theorems of common fixed points are established in a Banach space. In order to prove our main results, we need the following definitions and lemmas.

Recall that if C and D are nonempty subsets of a Banach space E such that C is nonempty closed convex and $D \subset C$, then a map $Q : C \to D$ is sunny provided that $Q(x + t(x - Q(x))) = Q(x)$ for all $x \in C$ and $t \geq 0$ whenever $x + t(x - Q(x)) \in C$. A sunny nonexpansive retraction is a sunny retraction, which is also nonexpansive. Sunny nonexpansive retractions play an important role in our argument. They are characterized as follows: If E is a smooth Banach space, then $Q : C \to D$ is a sunny nonexpansive retraction if and only if the following inequality holds:

$$\langle x - Qx, J(y - Qx)\rangle \leq 0 \quad \text{for all } x \in C \text{ and } y \in D.$$

Chen and Zhu [34] showed that if E is a reflexive Banach space and has a weakly continuous duality, then there is a sunny nonexpansive retraction from Π_C onto $F(T)$ and it can be constructed as follows.

Lemma 2.3 [32] *Let E be a reflexive Banach space which has a weakly continuous duality mapping $J_\varphi(x)$. Let C be closed convex subset of E and let $T : C \to C$ be a nonexpansive* *mapping. Let $f : C \to C$ be a contractive mapping with $F(f) \neq \emptyset$. For any $t \in (0,1)$, let $\{x_t\}$ be defined by $x_t = tf(x_t) + (1-t)Tx_t$, where T is a nonexpansive mapping. Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 0^+$ and, in this case, $\{x_t\}$ converges, as $t \to 0^+$, strongly to a fixed point of T.*

Lemma 2.4 *Under the condition of Lemma* 2.3, *we define the mapping $Q : \Pi_C \to F(T)$ by*

$$Q(f) := \lim_{t\to 0} x_t, \quad f \in \Pi_C. \tag{2.3}$$

Then the mapping Q is a sunny nonexpansive retraction from Π_C onto $F(T)$.

Proof From Theorem 3.1 of [34], for all $t \in (0,1)$ and $p \in F(T)$, we have

$$\bigl\langle x_t - f(x_t), J_\varphi(x_t - p)\bigr\rangle \le 0.$$

Letting $t \to 0$, we have

$$\bigl\langle (I-f)Q(f), J_\varphi\bigl(Q(f) - p\bigr)\bigr\rangle \le 0.$$

Since $J_\varphi(x) = \frac{\varphi(\|x\|)}{\|x\|}J(x)$ for any $x \neq 0$, we have

$$\bigl\langle (I-f)Q(f), J\bigl(Q(f) - p\bigr)\bigr\rangle \le 0.$$

This completes the proof. □

Recall that a *gauge* is a continuous strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. The duality mapping $J_\varphi : X \to X^*$ associated to a gauge φ is defined by

$$J_\varphi(x) = \bigl\{x^* \in X^* : \langle x, x^*\rangle = \|x\|\varphi\bigl(\|x\|\bigr), \bigl\|x^*\bigr\| = \varphi\bigl(\|x\|\bigr)\bigr\}, \quad \forall x \in X.$$

Following Browder [37], we say that a Banach space E has a *weakly continuous duality mapping* if there exists a gauge φ for which the duality mapping $J_\varphi(x)$ is single-valued and weak-to-weak* sequentially continuous (*i.e.*, if $\{x_n\}$ is a sequence in E weakly convergent to a point x, then the sequence $J_\varphi(x_n)$ converges weakly* to J_φ). It is known that l^p has a weakly continuous duality mapping with a gauge function $\varphi(t) = t^{p-1}$ for all $1 < p < \infty$. Set

$$\Phi(t) = \int_0^t \varphi(\tau)\, d_\tau, \quad \forall t \ge 0.$$

Then

$$J_\varphi(x) = \partial\Phi\bigl(\|x\|\bigr), \quad \forall x \in X,$$

where ∂ denotes the sub-differential in the sense of convex analysis.

The first part of the next lemma is an immediate consequence of the sub-differential inequality and the proof of the second part can be found in [38].

Lemma 2.5 *Assume that a Banach space E has a weakly continuous duality mapping J_φ with a gauge φ.*

(i) *For all $x, y \in E$, the following inequality holds*:

$$\Phi(\|x+y\|) \le \Phi(\|x\|) + \langle y, J_\varphi(x+y)\rangle.$$

In particular, for all $x, y \in E$

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, J(x+y)\rangle.$$

(ii) *Assume that a sequence $\{x_n\}$ in E converges weakly to a point $x \in E$. Then the following identity holds*:

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|y - x\|), \quad \forall x, y \in E.$$

Lemma 2.6 [39] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \ge 0$ and*

$$\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.7 [40] *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \le (1-\gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

The following lemma can be obtained from Chang *et al.* [23]. For the sake of completeness, we still give the proof.

Lemma 2.8 *Let C be a nonempty closed convex subset of a strictly convex Banach space E, let $\{T_i : C \to C\}$ be a family of infinitely nonexpansive mappings with $\bigcap_{i=1}^{\infty} F(T_i) \ne \emptyset$, let $\{\gamma_n\}$ be a real sequence such that $0 < \gamma_n \le b < 1$ for each $n \ge 1$. If K is any bounded subset of C, then $\lim_{n\to\infty} \sup_{x\in K} \|Wx - W_n x\| = 0$.*

Proof Let $p \in \bigcap_{i=1}^{\infty} F(T_i)$. Since K is a bounded subset of C, there exists an $M > 0$ such that $\sup_{x\in K} \|x - p\| \le M$. It follows that

$$\begin{aligned} \|W_{n+1}x - W_n x\| &= \|\gamma_1 T_1 U_{n+1,2}x + (1-\gamma_1)x - \gamma_1 T_1 U_{n,2}x - (1-\gamma_1)x\| \\ &\le \gamma_1 \|U_{n+1,2}x - U_{n,2}x\| \\ &= \gamma_1 \|\gamma_2 T_2 U_{n+1,3}x + (1-\gamma_2)x - \gamma_2 T_2 U_{n,3}x - (1-\gamma_2)x\| \end{aligned}$$

$$
\begin{aligned}
&\le \gamma_1\gamma_2\|U_{n+1,3}x - U_{n,3}x\| \\
&\vdots \\
&\le \prod_{i=1}^{n} \gamma_i\|U_{n+1,n+1}x - U_{n,n+1}x\| \\
&\le \prod_{i=1}^{n+1} \gamma_i\bigl(\|T_{n+1}x - p\| + \|p - x\|\bigr) \\
&\le 2\prod_{i=1}^{n+1} \gamma_i M.
\end{aligned}
$$

Since $0 < \gamma_n \le b < 1$, for any given $\epsilon > 0$, there exists a positive integer n_0 such that

$$b^{n_0+1} \le \frac{\epsilon(1-b)}{2M}.$$

For any positive integers $m > n > n_0$, we find that

$$
\begin{aligned}
\|W_m x - W_n x\| &\le \sum_{j=n}^{m-1} \|W_j x - W_j x\| \\
&\le 2M \sum_{j=n}^{m-1} \prod_{i=1}^{j+1} \gamma_i \\
&\le 2M \sum_{j=n}^{m-1} b^{j+1} \\
&\le \frac{2Mb^{n+1}}{1-b} \\
&\le \epsilon, \quad \forall x \in K.
\end{aligned}
$$

Letting $m \to \infty$, we find that

$$\|Wx - W_n x\| \le \epsilon, \quad \forall n \ge n_0.$$

This implies that $\lim_{n\to\infty} \sup_{x\in K} \|Wx - W_n x\| = 0$. $\square$

3 Main results

Theorem 3.1 *Let E be a reflexive and strictly convex Banach space E which enjoys a weakly continuous duality map $J_\varphi(x)$ with gauge φ and let A be an m-accretive operator in E with the domain $D(A)$. Assume that $\overline{D(A)}$ is convex. Let T_i be a nonexpansive mapping from $C =: \overline{D(A)}$ into itself for $i \in \mathbb{Z}^+$. Let $f \in \Pi_C$ with the coefficient* $(0 < \alpha < 1)$ *and $J_r = (I + rA)^{-1}$ for some $r > 0$. Assume that $\Omega := F(J_r W) = F(J_r) \cap F(W) \neq \emptyset$, where W is a mapping defined by* (2.2). *Let $\{x_n\}$ be a sequence generated in the following iterative algorithm*:

$$
\begin{cases}
x_0 \in C, \\
y_n = \beta_n x_n + (1-\beta_n) J_r W_n x_n, \\
x_{n+1} = \alpha_n f(y_n) + (1-\alpha_n) y_n, \quad n \ge 0,
\end{cases}
$$

where W_n is generated in (2.1), $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are real number sequences in* $(0,1)$ *satisfying the following restrictions*:

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$;
(b) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ *strongly converges to* $Q(f) \in \Omega$, *where* $Q : \Pi_C \to \Omega$ *is defined by* (2.3).

Proof First we prove that sequences $\{x_n\}$ and $\{y_n\}$ are bounded. Fixing $p \in \Omega$, we see that

$$\begin{aligned}\|y_n - p\| &\le \beta_n\|x_n - p\| + (1-\beta_n)\|J_r W_n x_n - p\| \\ &\le \|x_n - p\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1} - p\| &= \big\|\alpha_n\big(f(y_n) - p\big) + (1-\alpha_n)(y_n - p)\big\| \\ &\le \alpha_n\big\|f(y_n) - p\big\| + (1-\alpha_n)\|y_n - p\| \\ &\le \big[1 - \alpha_n(1-\alpha)\big]\|x_n - p\| + \alpha_n\big\|f(p) - p\big\| \\ &\le \max\left\{\|x_n - p\|, \frac{\|f(p) - p\|}{1-\alpha}\right\}.\end{aligned}$$

This in turn implies that

$$\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{\|p - f(p)\|}{1-\alpha}\right\},$$

which gives that the sequence $\{x_n\}$ is bounded, so is $\{y_n\}$.

Next, we prove that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Putting $l_n = \frac{x_{n+1} - \beta_n x_n}{1-\beta_n}$, we have

$$x_{n+1} = (1-\beta_n)l_n + \beta_n x_n. \tag{3.1}$$

In the light of

$$\begin{aligned}l_{n+1} - l_n &= \frac{\alpha_{n+1}f(y_{n+1}) + (1-\alpha_{n+1})y_{n+1} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n f(y_n) + (1-\alpha_n)y_n - \beta_n x_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}(f(y_{n+1}) - y_{n+1})}{1-\beta_{n+1}} - \frac{\alpha_n(f(y_n) - y_n)}{1-\beta_n} \\ &\quad + J_r W_{n+1}x_{n+1} - J_r W_{n+1}x_n + J_r W_{n+1}x_n - J_r W_n x_n,\end{aligned} \tag{3.2}$$

we obtain that

$$\begin{aligned}\|l_{n+1} - l_n\| &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big\|f(y_{n+1}) - y_{n+1}\big\| + \frac{\alpha_n}{1-\beta_n}\big\|y_n - f(y_n)\big\| \\ &\quad + \|x_{n+1} - x_n\| + \|J_r W_{n+1}x_n - J_r W_n x_n\| \\ &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big\|f(y_{n+1}) - y_{n+1}\big\| + \frac{\alpha_n}{1-\beta_n}\big\|y_n - f(y_n)\big\| \\ &\quad + \|x_{n+1} - x_n\| + \|W_{n+1}x_n - W_n x_n\|.\end{aligned} \tag{3.3}$$

Since T_i and $U_{n,i}$ are nonexpansive, we have

$$\begin{aligned}
\|W_{n+1}x_n - W_nx_n\| &= \|\gamma_1 T_1 U_{n+1,2}x_n - \gamma_1 T_1 U_{n,2}x_n\| \\
&\le \gamma_1 \|U_{n+1,2}x_n - U_{n,2}x_n\| \\
&= \gamma_1 \|\gamma_2 T_2 U_{u+1,3}x_n - \gamma_2 T_2 U_{n,3}x_n\| \\
&\le \gamma_1\gamma_2 \|U_{u+1,3}x_n - U_{n,3}x_n\| \\
&\le \cdots \\
&\le \gamma_1\gamma_2\cdots\gamma_n \|U_{n+1,n+1}x_n - U_{n,n+1}x_n\| \\
&\le M_1 \prod_{i=1}^{n} \gamma_i,
\end{aligned} \tag{3.4}$$

where $M_1 \ge 0$ is an appropriate constant such that

$$\|U_{n+1,n+1}x_n - U_{n,n+1}x_n\| \le M_1$$

for all $n \ge 0$. Substituting (3.4) into (3.3), we have

$$\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\| \le \frac{\alpha_{n+1}}{1-\beta_{n+1}} \|f(y_{n+1}) - y_{n+1}\| + \frac{\alpha_n}{1-\beta_n} \|y_n - f(y_n)\| + M_1 \prod_{i=1}^{n} \gamma_i.$$

In view of conditions (a) and (b), we get that

$$\limsup_{n\to\infty} \bigl(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|\bigr) \le 0.$$

We can obtain from Lemma 2.6 that $\lim_{n\to\infty} \|l_n - x_n\| = 0$ easily. On the other hand, we see from (3.1) that

$$x_{n+1} - x_n = (1-\beta_n)(l_n - x_n).$$

This implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.5}$$

Next, we prove that $\lim_{n\to\infty} \|J_r W x_n - x_n\| = 0$. In view of

$$x_{n+1} - y_n = \alpha_n\bigl(f(y_n) - y_n\bigr),$$

we obtain that

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{3.6}$$

On the other hand, we have

$$\|y_n - x_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|.$$

In view of (3.5) and (3.6), we have

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.7}$$

Notice that

$$\begin{aligned}\|J_r W_n x_n - x_n\| &\le \|x_n - y_n\| + \|y_n - J_r W_n x_n\| \\ &\le \|x_n - y_n\| + \beta_n \|x_n - J_r W_n x_n\|.\end{aligned}$$

This implies that

$$(1-\beta_n)\|J_r W_n x_n - x_n\| \le \|x_n - y_n\|.$$

From condition (b) and (3.7), we obtain that

$$\lim_{n\to\infty} \|J_r W_n x_n - x_n\| = 0. \tag{3.8}$$

On the other hand, we have

$$\begin{aligned}\|J_r W x_n - x_n\| &\le \|J_r W x_n - J_r W_n x_n\| + \|J_r W_n x_n - x_n\| \\ &\le \|W x_n - W_n x_n\| + \|J_r W_n x_n - x_n\|.\end{aligned}$$

In view of Lemma 2.8, we find that

$$\lim_{n\to\infty} \|W x_n - W_n x_n\| = 0.$$

This in turn implies that

$$\lim_{n\to\infty} \|J_r W x_n - x_n\| = 0. \tag{3.9}$$

Next, we show $x_n \to Q(f)$ as $n \to \infty$. To show it, we first prove that

$$\limsup_{n\to\infty} \langle (I-f)Q(f), J_\varphi(Q(f) - x_n)\rangle \le 0. \tag{3.10}$$

In view of Lemma 2.4, we have the sunny nonexpansive retraction $Q : \Pi_C \to \Omega$. Take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle (I-f)Q(f), J_\varphi(Q(f) - x_n)\rangle = \lim_{k\to\infty} \langle (I-f)Q(f), J_\varphi(Q(f) - x_{n_k})\rangle. \tag{3.11}$$

Since E is reflexive, we may further assume that $x_{n_k} \rightharpoonup \bar{x}$ for some $\bar{x} \in C$. Since J_φ is weakly continuous, we obtain from Lemma 2.5 that

$$\limsup_{n\to\infty} \Phi(\|x_{n_k} - x\|) = \limsup_{n\to\infty} \Phi(\|x_{n_k} - \bar{x}\|) + \Phi(\|x - \bar{x}\|), \quad \forall x \in E.$$

Put

$$g(x) = \limsup_{k\to\infty} \Phi\big(\|x_{n_k} - x\|\big), \quad \forall x \in E.$$

It follows that

$$g(x) = g(\bar{x}) + \Phi\big(\|x - \bar{x}\|\big), \quad \forall x \in E.$$

With the aid of (3.9), we arrive at

$$\begin{aligned} g(J_r W\bar{x}) &= \limsup_{k\to\infty} \Phi\big(\|x_{n_k} - J_r W\bar{x}\|\big) = \limsup_{k\to\infty} \Phi\big(\|J_r W x_{n_k} - J_r W\bar{x}\|\big) \\ &\le \limsup_{k\to\infty} \Phi\big(\|x_{n_k} - \bar{x}\|\big) \\ &= g(\bar{x}). \end{aligned} \tag{3.12}$$

Notice that

$$g(J_r W\bar{x}) = g(\bar{x}) + \Phi\big(\|J_r W\bar{x} - \bar{x}\|\big). \tag{3.13}$$

From (3.12) and (3.13), we find that

$$\Phi\big(\|J_r W\bar{x} - \bar{x}\|\big) \le 0.$$

This implies that $J_r W\bar{x} = \bar{x}$. And hence $\bar{x} \in F(J_r W)$. That is, $\bar{x} \in \Omega$. Since Q is the sunny nonexpansive retraction from Π_C onto F, we have from (3.11)

$$\limsup_{n\to\infty} \big\langle (I - f)Q(f), J_\varphi\big(Q(f) - x_n\big)\big\rangle = \big\langle (I - f)Q(f), J_\varphi\big(Q(f) - \bar{x}\big)\big\rangle \le 0.$$

This shows that (3.10) holds. It follows from Lemma 2.5 that

$$\begin{aligned} &\Phi\big\|x_{n+1} - Q(f)\big\| \\ &\quad = \Phi\Big(\big\|\alpha_n\big(f(x_n) - f\big(Q(f)\big)\big) + \alpha_n\big(f\big(Q(f)\big) - Q(f)\big) + (1 - \alpha_n)\big(y_n - Q(f)\big)\big\|\Big) \\ &\quad \le \Phi\big(\alpha_n\big\|f(x_n) - f\big(Q(f)\big)\big\| + (1 - \alpha_n)\big\|y_n - Q(f)\big\|\big) \\ &\qquad + \alpha_n\big\langle f\big(Q(f)\big) - Q(f), J_\varphi\big(x_{n+1} - Q(f)\big)\big\rangle \\ &\quad \le \Phi\big(\big(1 - \alpha_n(1 - \alpha)\big)\big\|x_n - Q(f)\big\|\big) \\ &\qquad + \alpha_n\big\langle f\big(Q(f)\big) - Q(f), J_\varphi\big(x_{n+1} - Q(f)\big)\big\rangle \\ &\quad \le \big(1 - \alpha_n(1 - \alpha)\big)\Phi\big(\big\|x_n - Q(f)\big\|\big) \\ &\qquad + \alpha_n\big\langle f\big(Q(f)\big) - Q(f), J_\varphi\big(x_{n+1} - Q(f)\big)\big\rangle. \end{aligned}$$

We find that $\|x_n - Q(f)\| \to 0$ as $n \to \infty$ from Lemma 2.7. That is, $x_n \to Q(f)$. This completes the proof. □

Remark 3.2 Taking $T_i = I$, the identity mapping, $\forall i \geq 1$, we see that $W_n = I$. Then the strict convexity of E in Theorem 3.1 may not be needed.

Corollary 3.3 *Let E be a reflexive Banach space E which enjoys a weakly continuous duality map $J_\varphi(x)$ with gauge φ and A be an m-accretive operator in E with the domain $D(A)$. Assume that $\overline{D(A)}$ is convex. Let $f \in \Pi_{\overline{D(A)}}$ with the coefficient* $(0 < \alpha < 1)$ *and $J_r = (I + rA)^{-1}$ for some $r > 0$. Assume that $A^{-1}(0) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following iterative algorithm*:

$$\begin{cases} x_0 \in A^{-1}(0), \\ y_n = \beta_n x_n + (1 - \beta_n) J_r x_n, \\ x_{n+1} = \alpha_n f(y_n) + (1 - \alpha_n) y_n, \quad n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real number sequences in $(0,1)$ *satisfying the following restrictions*:
(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$;
(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
Then $\{x_n\}$ strongly converges to $Q(f) \in A^{-1}(0)$, where $Q : \Pi_{\overline{D(A)}} \to A^{-1}(0)$ is defined by (2.3).

If $f(x) = u$, where u is a fixed element in $\overline{D(A)}$, then Theorem 3.1 is reduced to the following.

Corollary 3.4 *Let E be a reflexive and strictly convex Banach space E which enjoys a weakly continuous duality map $J_\varphi(x)$ with gauge φ and A be an m-accretive operator in E with the domain $D(A)$. Assume that $\overline{D(A)}$ is convex. Let T_i be a nonexpansive mapping from $C =: \overline{D(A)}$ into itself for $i \in \mathbb{Z}^+$. Let $J_r = (I + rA)^{-1}$ for some $r > 0$. Assume that $\Omega := F(J_r W) = F(J_r) \cap F(W) \neq \emptyset$, where W is a mapping defined by* (2.2). *Let $\{x_n\}$ be a sequence generated in the following iterative algorithm*:

$$\begin{cases} x_0 \in C, \\ y_n = \beta_n x_n + (1 - \beta_n) J_r W_n x_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) y_n, \quad n \geq 0, \end{cases}$$

where W_n is generated in (2.1), *$\{\alpha_n\}$ and $\{\beta_n\}$ are real number sequences in* $(0,1)$ *satisfying the following restrictions*:
(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$;
(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
Then $\{x_n\}$ strongly converges to $Q(u) \in \Omega$, where $Q : \Pi_C \to \Omega$ is defined by (2.3).

If $A = I$, then Theorem 3.1 is reduced to the following.

Corollary 3.5 *Let E be a reflexive and strictly convex Banach space E which enjoys a weakly continuous duality map $J_\varphi(x)$ with gauge φ and let C be a closed and convex subset of E. Let T_i be a nonexpansive mapping from C into itself for $i \in \mathbb{Z}^+$. Let $f \in \Pi_C$ with the coefficient* $(0 < \alpha < 1)$. *Assume that $\Omega := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the* *following iterative algorithm*:

$$\begin{cases} x_0 \in C, \\ y_n = \beta_n x_n + (1-\beta_n) W_n x_n, \\ x_{n+1} = \alpha_n f(y_n) + (1-\alpha_n) y_n, \quad n \geq 0, \end{cases}$$

where W_n is generated in (2.1), $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are real number sequences in* $(0,1)$ *satisfying the following restrictions*:

(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$;

(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ *strongly converges to* $Q(f) \in \Omega$, *where* $Q : \Pi_C \to \Omega$ *is defined by* (2.3).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

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