Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Fixed point theory of cyclical generalized contractive conditions in partial metric spaces

Chi-Ming Chen*

*Correspondence: ming@mail.nhcue.edu.tw
Department of Applied Mathematics, National Hsinchu University of Education, Hsinchu, Taiwan

Abstract
The purpose of this paper is to study fixed point theorems for a mapping satisfying the cyclical generalized contractive conditions in complete partial metric spaces. Our results generalize or improve many recent fixed point theorems in the literature.
MSC: 47H10; 54C60; 54H25; 55M20

Keywords: fixed point; cyclic $\mathcal{CW}$-contraction; cyclic $\mathcal{MK}$-contraction; partial metric space

1 Introduction and preliminaries

Throughout this paper, by $\mathbb{R}^+$, we denote the set of all nonnegative real numbers, while $\mathbb{N}$ is the set of all natural numbers. Let (X,d) be a metric space, D be a subset of X and $f : D \to X$ be a map. We say f is contractive if there exists $\alpha \in [0,1)$ such that for all $x,y \in D$,

$$d(fx,fy) \leq \alpha \cdot d(x,y).$$

The well-known Banach fixed point theorem asserts that if $D = X$, f is contractive and (X,d) is complete, then f has a unique fixed point in X. It is well known that the Banach contraction principle [1] is a very useful and classical tool in nonlinear analysis. Also, this principle has many generalizations. For instance, in 1969, Boyd and Wong [2] introduced the notion of Φ-contraction. A mapping $f : X \to X$ on a metric space is called Φ-contraction if there exists an upper semi-continuous function $\Phi : [0,\infty) \to [0,\infty)$ such that

$$d(fx,fy) \leq \Phi\big(d(x,y)\big) \quad \text{for all } x,y \in X.$$

In 1994, Mattews [3] introduced the following notion of partial metric spaces.

Definition 1 [3] A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x,y,z \in X$,

(p_1) $x = y$ if and only if $p(x,x) = p(x,y) = p(y,y)$;
(p_2) $p(x,x) \leq p(x,y)$;

© 2013 Chen; licensee Springer. This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/2.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

(p_3) $p(x,y) = p(y,x)$;
(p_4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X.

Remark 1 It is clear that if $p(x,y) = 0$, then from (p_1) and (p_2), $x = y$. But if $x = y$, $p(x,y)$ may not be 0.

Each partial metric p on X generates a $\mathcal{T}_0$ topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\gamma) : x \in X, \gamma > 0\}$, where $B_p(x,\gamma) = \{y \in X : p(x,y) < p(x,x) + \gamma\}$ for all $x \in X$ and $\gamma > 0$. If p is a partial metric on X, then the function $d_p : X \times X \to \mathbb{R}^+$ given by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a metric on X.

We recall some definitions of a partial metric space as follows.

Definition 2 [3] Let (X,p) be a partial metric space. Then

(1) a sequence $\{x_n\}$ in a partial metric space (X,p) converges to $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$;
(2) a sequence $\{x_n\}$ in a partial metric space (X,p) is called a Cauchy sequence if and only if $\lim_{m,n\to\infty} p(x_m,x_n)$ exists (and is finite);
(3) a partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{m,n\to\infty} p(x_m,x_n)$;
(4) a subset A of a partial metric space (X,p) is closed if whenever $\{x_n\}$ is a sequence in A such that $\{x_n\}$ converges to some $x \in X$, then $x \in A$.

Remark 2 The limit in a partial metric space is not unique.

Lemma 1 [3, 4]

(a) *$\{x_n\}$ is a Cauchy sequence in a partial metric space (X,p) if and only if it is a Cauchy sequence in the metric space (x,d_p);*
(b) *a partial metric space (X,p) is complete if and only if the metric space (X,d_p) is complete. Furthermore, $\lim_{n\to\infty} d_p(x_n,x) = 0$ if and only if $p(x,x) = \lim_{n\to\infty} p(x_n,x) = \lim_{n\to\infty} p(x_n,x_m)$.*

In 2003, Kirk, Srinivasan and Veeramani [5] introduced the following notion of the cyclic representation.

Definition 3 [5] Let X be a nonempty set, $m \in \mathbb{N}$ and $f : X \to X$ be an operator. Then $X = \bigcup_{i=1}^m A_i$ is called a cyclic representation of X with respect to f if

(1) A_i, $i = 1,2,\ldots,m$ are nonempty subsets of X;
(2) $f(A_1) \subset A_2, f(A_2) \subset A_3, \ldots, f(A_{m-1}) \subset A_m, f(A_m) \subset A_1$.

Kirk, Srinivasan and Veeramani [5] also proved the following theorem.

Theorem 1 [5] *Let (X,d) be a complete metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$, be closed nonempty subsets of X and $X = \bigcup_{i=1}^m A_i$. Suppose that f satisfies the following condition*:

$$d(fx,fy) \le \psi\big(d(x,y)\big), \quad \textit{for all } x \in A_i, y \in A_{i+1}, i \in \{1,2,\ldots,m\},$$

where $\psi : [0,\infty) \to [0,\infty)$ is upper semi-continuous from the right and $0 \le \psi(t) < t$ for $t > 0$. Then f has a fixed point $z \in \bigcap_{i=1}^n A_i$.

Recently, the fixed theorems for an operator $f : X \to X$ defined on a metric space X with a cyclic representation of X with respect to f have appeared in the literature (see, *e.g.*, [6–8]). In 2010, Păcurar and Rus [7] introduced the following notion of a cyclic weaker φ-contraction.

Definition 4 [7] Let (X,d) be a metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ be closed nonempty subsets of X and $X = \bigcup_{i=1}^m A_i$. An operator $f : X \to X$ is called a cyclic weaker φ-contraction if

(1) $X = \bigcup_{i=1}^m A_i$ is a cyclic representation of X with respect to f;
(2) there exists a continuous, non-decreasing function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(t) > 0$ for $t \in (0,\infty)$ and $\varphi(0) = 0$ such that

$$d(fx,fy) \le d(x,y) - \varphi\big(d(x,y)\big)$$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1,2,\ldots,m$, where $A_{m+1} = A_1$.

And Păcurar and Rus [7] proved the following main theorem.

Theorem 2 [7] *Let (X,d) be a complete metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ be closed nonempty subsets of X and $X = \bigcup_{i=1}^m A_i$. Suppose that f is a cyclic weaker φ-contraction. Then f has a fixed point $z \in \bigcap_{i=1}^n A_i$.*

In the recent years, fixed point theory has developed rapidly on cyclic contraction mappings, see [9–15].

The purpose of this paper is to study fixed point theorems for a mapping satisfying the cyclical generalized contractive conditions in complete partial metric spaces. Our results generalize or improve many recent fixed point theorems in the literature.

2 Fixed point theorems (I)

In the section, we denote by Ψ the class of functions $\psi : \mathbb{R}^{+3} \to \mathbb{R}^+$ satisfying the following conditions:

(ψ_1) ψ is an increasing and continuous function in each coordinate;
(ψ_2) for $t \in \mathbb{R}^+$, $\psi(t,t,t) \le t$, $\psi(t,0,0) \le t$ and $\psi(0,0,t) \le t$.

Next, we denote by Θ the class of functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

(φ_1) φ is continuous and non-decreasing;
(φ_2) for $t > 0$, $\varphi(t) > 0$ and $\varphi(0) = 0$.

And we denote by Φ the class of functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

(ϕ_1) ϕ is continuous;
(ϕ_2) for $t > 0$, $\phi(t) > 0$ and $\phi(0) = 0$.

We now state a new notion of cyclic $\mathcal{CW}$-contractions in partial metric spaces as follows.

Definition 5 Let (X,p) be a partial metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ be nonempty subsets of X and $Y = \bigcup_{i=1}^m A_i$. An operator $f : Y \to Y$ is called a cyclic $\mathcal{CW}$-contraction if

(1) $\bigcup_{i=1}^m A_i$ is a cyclic representation of Y with respect to f;
(2) for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$,

$$\varphi\bigl(p(fx,fy)\bigr) \le \psi\bigl(\varphi\bigl(p(x,y)\bigr), \varphi\bigl(p(x,fx)\bigr), \varphi\bigl(p(y,fy)\bigr)\bigr) - \phi\bigl(M(x,y)\bigr), \tag{2.1}$$

where $\psi \in \Psi$, $\varphi \in \Theta$, $\phi \in \Phi$, and $M(x,y) = \max\{p(x,y), p(x,fx), p(y,fy)\}$.

Theorem 3 *Let (X,p) be a complete partial metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ be nonempty closed subsets of X and $Y = \bigcup_{i=1}^m A_i$. Let $f : Y \to Y$ be a cyclic $\mathcal{CW}$-contraction. Then f has a unique fixed point $z \in \bigcap_{i=1}^m A_i$.*

Proof Given x_0 and let $x_{n+1} = fx_n = f^n x_0$ for $n = 0, 1, 2, \ldots$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, then we finished the proof. Suppose that $x_{n+1} \neq x_n$ for any $n = 0, 1, 2, \ldots$. Notice that for any $n \ge 0$, there exists $i_n \in \{1, 2, \ldots, m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_n+1}$.

Step 1. We will prove that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0, \quad \text{that is,} \quad \lim_{n\to\infty} d_p(x_n, x_{n+1}) = 0.$$

Using (2.1), we have

$$\begin{aligned}
\varphi\bigl(p(x_n, x_{n+1})\bigr) &= \varphi\bigl(p(fx_{n-1}, fx_n)\bigr) \\
&\le \psi\bigl(\varphi\bigl(p(x_{n-1}, x_n)\bigr), \varphi\bigl(p(x_{n-1}, fx_{n-1})\bigr), \varphi\bigl(p(x_n, fx_n)\bigr)\bigr) - \phi\bigl(M(x_{n-1}, x_n)\bigr) \\
&= \psi\bigl(\varphi\bigl(p(x_{n-1}, x_n)\bigr), \varphi\bigl(p(x_{n-1}, x_n)\bigr), \varphi\bigl(p(x_n, x_{n+1})\bigr)\bigr) - \phi\bigl(M(x_{n-1}, x_n)\bigr),
\end{aligned}$$

where

$$\begin{aligned}
M(x_{n-1}, x_n) &= \max\bigl\{p(x_{n-1}, x_n), p(x_{n-1}, fx_{n-1}), p(x_n, fx_n)\bigr\} \\
&= \max\bigl\{p(x_{n-1}, x_n), p(x_{n-1}, x_n), p(x_n, x_{n+1})\bigr\}.
\end{aligned}$$

If $M(x_{n-1}, x_n) = p(x_n, x_{n+1})$, then

$$\begin{aligned}
\varphi\bigl(p(x_n, x_{n+1})\bigr) &\le \psi\bigl(\varphi\bigl(p(x_n, x_{n+1})\bigr), \varphi\bigl(p(x_n, x_{n+1})\bigr), \varphi\bigl(p(x_n, x_{n+1})\bigr)\bigr) - \phi\bigl(p(x_n, x_{n+1})\bigr) \\
&\le \varphi\bigl(p(x_n, x_{n+1})\bigr) - \phi\bigl(p(x_n, x_{n+1})\bigr),
\end{aligned}$$

which implies that $\phi(p(x_n, x_{n+1})) = 0$, and hence $p(x_n, x_{n+1}) = 0$. This contradicts our initial assumption.

From the above argument, we have that for each $n \in \mathbb{N}$,

$$\varphi\big(p(x_n, x_{n+1})\big) \leq \varphi\big(p(x_{n-1}, x_n)\big) - \phi\big(p(x_{n-1}, x_n)\big), \tag{2.2}$$

and

$$p(x_n, x_{n+1}) < p(x_{n-1}, x_n).$$

And since the sequence $\{p(x_n, x_{n+1})\}$ is decreasing, it must converge to some $\eta \geq 0$. Taking limit as $n \to \infty$ in (2.2) and by the continuity of φ and ϕ, we get

$$\varphi(\eta) \leq \varphi(\eta) - \phi(\eta),$$

and so we conclude that $\phi(\eta) = 0$ and $\eta = 0$. Thus, we have

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{2.3}$$

By (p_2), we also have

$$\lim_{n\to\infty} p(x_n, x_n) = 0. \tag{2.4}$$

Since $d_p(x,y) \leq 2p(x,y) - p(x,x) - p(y,y)$ for all $x, y \in X$, using (2.3) and (2.4), we obtain that

$$\lim_{n\to\infty} d_p(x_n, x_{n+1}) = 0. \tag{2.5}$$

Step 2. We show that $\{x_n\}$ is a Cauchy sequence in the metric space (Y, d_p). We claim that the following result holds.

Claim For every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that if $r, q \geq n$ with $r - q = 1 \bmod m$, then $d_p(x_r, x_q) < \varepsilon$.

Suppose the above statement is false. Then there exists $\epsilon > 0$ such that for any $n \in \mathbb{N}$, there are $r_n, q_n \in \mathbb{N}$ with $r_n > q_n \geq n$ with $r_n - q_n = 1 \bmod m$ satisfying

$$d_p(x_{q_n}, x_{r_n}) \geq \epsilon.$$

Now, we let $n > 2m$. Then corresponding to $q_n \geq n$ use, we can choose r_n in such a way it is the smallest integer with $r_n > q_n \geq n$ satisfying $r_n - q_n = 1 \bmod m$ and $d_p(x_{q_n}, x_{r_n}) \geq \epsilon$. Therefore, $d_p(x_{q_n}, x_{r_n - m}) \leq \epsilon$ and

$$\begin{aligned}
\epsilon &\leq d_p(x_{q_n}, x_{r_n}) \\
&\leq d_p(x_{q_n}, x_{r_n - m}) + \sum_{i=1}^{m} d_p(x_{r_{n-i}}, x_{r_{n-i+1}}) \\
&< \epsilon + \sum_{i=1}^{m} d_p(x_{r_{n-i}}, x_{r_{n-i+1}}).
\end{aligned}$$

Letting $n \to \infty$, we obtain that

$$\lim_{n\to\infty} d_p(x_{q_n}, x_{r_n}) = \epsilon. \tag{2.6}$$

On the other hand, we can conclude that

$$\begin{aligned}\epsilon &\le d_p(x_{q_n}, x_{r_n})\\ &\le d_p(x_{q_n}, x_{q_{n+1}}) + d_p(x_{q_{n+1}}, x_{r_{n+1}}) + d_p(x_{r_{n+1}}, x_{r_n})\\ &\le d_p(x_{q_n}, x_{q_{n+1}}) + d_p(x_{q_{n+1}}, x_{q_n}) + d_p(x_{q_n}, x_{r_n}) + d_p(x_{r_n}, x_{r_{n+1}}) + d_p(x_{r_{n+1}}, x_{r_n}).\end{aligned}$$

Letting $n \to \infty$, we obtain that

$$\lim_{n\to\infty} d_p(x_{q_{n+1}}, x_{r_{n+1}}) = \epsilon. \tag{2.7}$$

Since $d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ and using (2.4), (2.6) and (2.7), we have that

$$\lim_{n\to\infty} p(x_{q_n}, x_{r_n}) = \frac{\epsilon}{2}, \tag{2.8}$$

and

$$\lim_{n\to\infty} p(x_{q_{n+1}}, x_{r_{n+1}}) = \frac{\epsilon}{2}. \tag{2.9}$$

Since x_{q_n} and x_{r_n} lie in different adjacently labeled sets A_i and A_{i+1} for certain $1 \le i \le m$, by using the fact that f is a cyclic $\mathcal{CW}$-contraction, we have

$$\begin{aligned}\varphi\bigl(p(fx_{q_n+1}, fx_{r_n+1})\bigr) &= \varphi\bigl(p(fx_{q_n}, fx_{r_n})\bigr)\\ &\le \psi\bigl(\varphi\bigl(p(x_{q_n}, x_{r_n})\bigr), \varphi\bigl(p(x_{q_n}, fx_{q_n})\bigr), \varphi\bigl(p(x_{r_n}, fx_{r_n})\bigr)\bigr)\\ &\quad - \phi\bigl(M(x_{q_n}, x_{r_n})\bigr)\\ &= \psi\bigl(\varphi\bigl(p(x_{q_n}, x_{r_n})\bigr), \varphi\bigl(p(x_{q_n}, x_{q_n+1})\bigr), \varphi\bigl(p(x_{r_n}, x_{r_n+1})\bigr)\bigr)\\ &\quad - \phi\bigl(M(x_{q_n}, x_{r_n})\bigr),\end{aligned}$$

where

$$M(x_{q_n}, x_{r_n}) = \max\bigl\{p(x_{q_n}, x_{r_n}), p(x_{q_n}, x_{q_n+1}), p(x_{r_n}, x_{r_n+1})\bigr\}.$$

Thus, letting $n \to \infty$, we can conclude that

$$\varphi\left(\frac{\epsilon}{2}\right) \le \psi\left(\varphi\left(\frac{\epsilon}{2}\right), \varphi(0), \varphi(0)\right) - \phi\left(\frac{\epsilon}{2}\right) \le \varphi\left(\frac{\epsilon}{2}\right) - \phi\left(\frac{\epsilon}{2}\right),$$

which implies $\phi(\frac{\epsilon}{2}) = 0$, that is, $\epsilon = 0$. So, we get a contradiction. Therefore, our claim is proved.

In the sequel, we will show that $\{x_n\}$ is a Cauchy sequence in the metric space (Y, d_p). Let $\varepsilon > 0$ be given. By our claim, there exists $n_1 \in \mathbb{N}$ such that if $r, q \geq n_1$ with $r - q = 1 \bmod m$, then

$$d_p(x_r, x_q) \leq \frac{\varepsilon}{2}.$$

Since $\lim_{n\to\infty} d_p(x_n, x_{n+1}) = 0$, there exists $n_2 \in \mathbb{N}$ such that

$$d_p(x_n, x_{n+1}) \leq \frac{\varepsilon}{2m}$$

for any $n \geq n_2$.

Let $r, q \geq \max\{n_1, n_2\}$ and $r > q$. Then there exists $k \in \{1, 2, \dots, m\}$ such that $r - q = k \bmod m$. Therefore, $r - q + j = 1 \bmod m$ for $j = m - k + 1$, and so we have

$$\begin{aligned} d_p(x_q, x_r) &\leq d_p(x_q, x_{r+j}) + d_p(x_{r+j}, x_{r+j-1}) + \cdots + d_p(x_{r-1}, x_r) \\ &\leq \frac{\varepsilon}{2} + j \times \frac{\varepsilon}{2m} \\ &\leq \frac{\varepsilon}{2} + m \times \frac{\varepsilon}{2m} = \varepsilon. \end{aligned}$$

Thus, $\{x_n\}$ is a Cauchy sequence in the metric space (Y, d_p).

Step 3. We show that f has a fixed point ν in $\bigcap_{i=1}^m A_i$.

Since Y is closed, the subspace (Y, p) is complete. Then from Lemma 1, we have that (Y, d_p) is complete. Thus, there exists $\nu \in X$ such that

$$\lim_{n\to\infty} d_p(x_n, \nu) = 0.$$

And it follows from Lemma 1 that we have

$$p(\nu, \nu) = \lim_{n\to\infty} p(x_n, \nu) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{2.10}$$

On the other hand, since the sequence $\{x_n\}$ is a Cauchy sequence in the metric space (Y, d_p), we also have

$$\lim_{n\to\infty} d_p(x_n, x_m) = 0.$$

Since $d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$, we can deduce that

$$\lim_{n\to\infty} p(x_n, x_m) = 0. \tag{2.11}$$

Since $Y = \bigcup_{i=1}^m A_i$ is a cyclic representation of X with respect to f, the sequence $\{x_n\}$ has infinite terms in each A_i for $i \in \{1, 2, \dots, m\}$. Now, for all $i = 1, 2, \dots, m$, we may take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \in A_{i-1}$ and also all converge to ν. Using (2.10) and (2.11), we have

$$p(\nu, \nu) = \lim_{n\to\infty} p(x_n, \nu) = \lim_{n\to\infty} p(x_{n_k}, \nu) = 0.$$

By (2.1),

$$\begin{aligned}\varphi\big(p(x_{n_{k+1}},f\nu)\big) &= \varphi\big(p(fx_{n_k},f\nu)\big)\\ &\le \psi\big(\varphi\big(p(x_{n_k},\nu)\big),\varphi\big(p(x_{n_k},fx_{n_k})\big),\varphi\big(p(\nu,f\nu)\big)\big) - \phi\big(M(x_{n_k},\nu)\big)\\ &= \psi\big(\varphi\big(p(x_{n_k},\nu)\big),\varphi\big(p(x_{n_k},x_{n_k+1})\big),\varphi\big(p(\nu,f\nu)\big)\big) - \phi\big(M(x_{n_k},\nu)\big),\end{aligned}$$

where

$$M(x_{n_k},\nu) = \max\big\{p(x_{n_k},\nu),p(x_{n_k},x_{n_k+1}),p(\nu,f\nu)\big\}.$$

Letting $k \to \infty$, we have

$$\begin{aligned}\varphi\big(p(\nu,f\nu)\big) &\le \psi\big(\varphi(0),\varphi(0),\varphi\big(p(\nu,f\nu)\big)\big) - \phi\big(p(\nu,f\nu)\big)\\ &\le \varphi\big(p(\nu,f\nu)\big) - \phi\big(p(\nu,f\nu)\big),\end{aligned}$$

which implies $\phi(p(\nu,f\nu)) = 0$, that is, $p(\nu,f\nu) = 0$. So, $\nu = f\nu$.

Step 4. Finally, to prove the uniqueness of the fixed point, suppose that μ, ν are fixed points of f. Then using the inequality (2.1), we obtain that

$$\begin{aligned}\varphi\big(p(\mu,\nu)\big) = \varphi\big(p(f\mu,f\nu)\big) &\le \psi\big(\varphi\big(p(\mu,\nu)\big),\varphi\big(p(\mu,f\mu)\big),\varphi\big(p(\nu,f\nu)\big)\big)\\ &\quad - \phi\big(M(\mu,\nu)\big),\end{aligned}$$

where

$$M(\mu,\nu) = \max\big\{p(\mu,\nu),p(\mu,f\mu),p(\nu,f\nu)\big\} = p(\mu,\nu).$$

So, we also deduce that

$$\begin{aligned}\varphi\big(p(\mu,\nu)\big) &\le \psi\big(\varphi\big(p(\mu,\nu),0,0\big)\big)\\ &\le \varphi\big(p(\mu,\nu)\big) - \phi\big(p(\mu,\nu)\big),\end{aligned}$$

which implies that $\phi(p(\mu,\nu)) = 0$, and hence $p(\mu,\nu) = 0$, that is, $\mu = \nu$. So, we complete the proof. □

The following provides an example for Theorem 3.

Example 1 Let $X = [0,1]$ and $A = [0,1]$, $B = [0,\frac{1}{2}]$, $C = [0,\frac{1}{4}]$. We define the partial metric p on X by

$$p(x,y) = \max\{x,y\} \quad \text{for all } x,y \in X,$$

and define the function $f : X \to X$ by

$$f(x) = \frac{x^2}{1+x} \quad \text{for all } x \in X.$$

Now, we let $\varphi, \phi : \mathbb{R}^+ \to \mathbb{R}^+$ and $\psi : \mathbb{R}^{+3} \to \mathbb{R}^+$ be

$$\varphi(t) = 2t, \qquad \phi(t) = \frac{2t}{5(1+t)} \quad \text{and} \quad \psi(t) = \frac{4}{5} \cdot \max\{t_1, t_2, t_3\}.$$

Then f is a cyclic $\mathcal{CW}$-contraction and 0 is the unique fixed point.

Proof We claim that f is a cyclic $\mathcal{CW}$-contraction.

(1) Note that $f(A) = [0, \frac{1}{2}] \subset B$, $f(B) = [0, \frac{1}{6}] \subset C$ and $f(C) = [0, \frac{1}{20}] \subset A$. Thus, $A \cup B \cup C$ is a cyclic representation of X with respect to f;

(2) For $x \in A$ and $y \in B$ (or, $x \in B$ and $y \in C$), without loss of generality, we may assume that $x \geq y$, then we have

$$\begin{aligned}
&\varphi\bigl(p(fx, fy)\bigr) = \varphi\left(p\left(\frac{x^2}{1+x}, \frac{y^2}{1+y}\right)\right) = \varphi\left(\frac{x^2}{1+x}\right) = \frac{2x^2}{1+x}, \\
&\psi\bigl(\varphi\bigl(p(x,y)\bigr), \varphi\bigl(p(x,fx)\bigr), \varphi\bigl(p(y,fy)\bigr)\bigr) \\
&\quad = \psi\left(\varphi\bigl(p(x,y)\bigr), \varphi\left(p\left(x, \frac{x^2}{1+x}\right)\right), \varphi\left(p\left(y, \frac{y^2}{1+y}\right)\right)\right) \\
&\quad = \psi\bigl(\varphi(x), \varphi(x), \varphi(y)\bigr) \\
&\quad = \psi(2x, 2x, 2y) = \frac{8x}{5},
\end{aligned}$$

and

$$\begin{aligned}
&\phi\bigl(\max\bigl\{p(x,y), p(x,fx), p(y,fy)\bigr\}\bigr) \\
&\quad = \phi\left(\max\left\{p(x,y), p\left(x, \frac{x^2}{1+x}\right), p\left(y, \frac{y^2}{1+y}\right)\right\}\right) \\
&\quad = \phi\bigl(\max\{x, x, y\}\bigr) = \frac{2x}{5(1+x)}.
\end{aligned}$$

Since

$$\frac{2x^2}{1+x} \leq \frac{8x}{5} - \frac{2x}{5(1+x)},$$

we have

$$\begin{aligned}
\varphi\bigl(p(fx, fy)\bigr) \leq{} & \psi\bigl(\varphi\bigl(p(x,y)\bigr), \varphi\bigl(p(x,fx)\bigr), \varphi\bigl(p(y,fy)\bigr)\bigr) \\
& - \phi\bigl(\max\bigl\{p(x,y), p(x,fx), p(y,fy)\bigr\}\bigr).
\end{aligned}$$

On the other hand, for $x \in C$ and $y \in A$, without loss of generality, we may assume that $x \leq y$, then it is easy to get the above inequality.

Note that Example 1 satisfies all of the hypotheses of Theorem 3, and we get that 0 is the unique fixed point. □

3 Fixed point theorems (II)

In this article, we also recall the notion of a Meir-Keeler function (see [16]). A function $\phi : [0, \infty) \to [0, \infty)$ is said to be a Meir-Keeler function if for each $\eta > 0$, there exists $\delta > 0$

such that for $t \in [0,\infty)$ with $\eta \le t < \eta + \delta$, we have $\phi(t) < \eta$. We now introduce a new notion of a weaker Meir-Keeler function $\phi : [0,\infty) \to [0,\infty)$ in a partial metric space (X,p) as follows.

Definition 6 Let (X,p) be a partial metric space. We call $\phi : [0,\infty) \to [0,\infty)$ a weaker Meir-Keeler function in X if for each $\eta > 0$, there exists $\delta > 0$ such that for $x, y \in X$ with $\eta \le p(x,y) < \eta + \delta$, there exists $n_0 \in \mathbb{N}$ such that $\phi^{n_0}(p(x,y)) < \eta$.

In the section, we denote by Φ the class of weaker Meir-Keeler functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ in a partial metric space in (X,p) satisfying the following conditions:

(ϕ_1) $\phi(t) > 0$ for $t > 0$, $\phi(0) = 0$;
(ϕ_2) $\{\phi^n(t)\}_{n\in\mathbb{N}}$ is decreasing;
(ϕ_3) for $t_n \in [0,\infty)$,
  (a) if $\lim_{n\to\infty} t_n = \gamma > 0$, then $\lim_{n\to\infty} \phi(t_n) < \gamma$ and
  (b) if $\lim_{n\to\infty} t_n = 0$, then $\lim_{n\to\infty} \phi(t_n) = 0$.

And we denote by the class Ψ of functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ a continuous function satisfying $\psi(t) > 0$ for $t > 0$, $\psi(0) = 0$.

First, we state a new notion of cyclic $\mathcal{MK}$-contractions in partial metric spaces as follows.

Definition 7 Let (X,p) be a partial metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ be nonempty subsets of X and $Y = \bigcup_{i=1}^m A_i$. An operator $f : Y \to Y$ is called a cyclic $\mathcal{MK}$-contraction if

(1)
$\bigcup_{i=1}^m A_i$ is a cyclic representation of Y with respect to f;
(2)
for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$,

$$p(fx, fy) \le \phi\bigl(p(x,y)\bigr) - \psi\bigl(p(x,y)\bigr), \tag{3.1}$$

where
$A_{m+1} = A_1$, $\phi \in \Phi$ and $\psi \in \Psi$.

Theorem 4 *Let (X,p) be a complete partial metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ be nonempty closed subsets of X and $Y = \bigcup_{i=1}^m A_i$. Let $f : Y \to Y$ be a cyclic $\mathcal{MK}$-contraction. Then f has a unique fixed point $z \in \bigcap_{i=1}^m A_i$.*

Proof Given x_0 and let $x_{n+1} = fx_n = f^n x_0$, for $n = 0, 1, 2, \ldots$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0+1} = x_{n_0}$, then we finished the proof. Suppose that $x_{n+1} \ne x_n$ for any $n = 0, 1, 2, \ldots$. Notice that for any $n \ge 0$, there exists $i_n \in \{1, 2, \ldots, m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_n+1}$. Then by (3.1), we have

$$p(x_n, x_{n+1}) = p(fx_{n-1}, fx_n) \le \phi\bigl(p(x_{n-1}, x_n)\bigr) - \psi\bigl(p(x_{n-1}, x_n)\bigr).$$

Step 1. We will prove that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0, \quad \text{that is,} \quad \lim_{n\to\infty} d_p(x_n, x_{n+1}) = 0.$$

Since f is a cyclic $\mathcal{MK}$-contraction, we can conclude that

$$\begin{aligned} p(x_n, x_{n+1}) &\leq \phi\big(p(x_{n-1}, x_n)\big) \\ &\leq \phi(\phi\big(p(x_{n-2}, x_{n-1})\big) = \phi^2\big(p(x_{n-2}, x_{n-1})\big) \\ &\leq \cdots \\ &\leq \phi^n\big(p(x_0, x_1)\big). \end{aligned}$$

Since $\{\phi^n(p(x_0, x_1))\}_{n\in\mathbb{N}}$ is decreasing, it must converge to some $\eta \geq 0$. We claim that $\eta = 0$. On the contrary, assume that $0 < \eta$. Then by the definition of a weaker Meir-Keeler function ϕ, there exists $\delta > 0$ such that for $x_0, x_1 \in X$ with $\eta \leq p(x_0, x_1) < \delta + \eta$, there exists $n_0 \in \mathbb{N}$ such that $\phi^{n_0}(p(x_0, x_1)) < \eta$. Since $\lim_{n\to\infty} \phi^n(p(x_0, x_1)) = \eta$, there exists $k_0 \in \mathbb{N}$ such that $\eta \leq \phi^k(p(x_0, x_1)) < \delta + \eta$, for all $k \geq k_0$. Thus, we conclude that $\phi^{k_0+n_0}(p(x_0, x_1)) < \eta$. So, we get a contradiction. Therefore, $\lim_{n\to\infty} \phi^n(p(x_0, x_1)) = 0$, and so we have

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{3.2}$$

By (p_2), we also have

$$\lim_{n\to\infty} p(x_n, x_n) = 0. \tag{3.3}$$

Since $d_p(x, y) \leq 2p(x, y) - p(x, x) - p(y, y)$ for all $x, y \in X$, using (3.2) and (3.3), we obtain that

$$\lim_{n\to\infty} d_p(x_n, x_{n+1}) = 0. \tag{3.4}$$

Step 2. We show that $\{x_n\}$ is a Cauchy sequence in the metric space (Y, d_p). We claim that the following result holds.

Claim For every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that if $r, q \geq n$ with $r - q = 1 \bmod m$, then $d_p(x_r, x_q) < \varepsilon$.

Suppose the above statement is false. Then there exists $\epsilon > 0$ such that for any $n \in \mathbb{N}$, there are $r_n, q_n \in \mathbb{N}$ with $r_n > q_n \geq n$ with $r_n - q_n = 1 \bmod m$ satisfying

$$d_p(x_{q_n}, x_{r_n}) \geq \epsilon.$$

Now, we let $n > 2m$. Then corresponding to $q_n \geq n$ use, we can choose r_n in such a way it is the smallest integer with $r_n > q_n \geq n$ satisfying $r_n - q_n = 1 \bmod m$ and $d_p(x_{q_n}, x_{r_n}) \geq \epsilon$. Therefore, $d_p(x_{q_n}, x_{r_n - m}) \leq \epsilon$ and

$$\begin{aligned} \epsilon &\leq d_p(x_{q_n}, x_{r_n}) \\ &\leq d_p(x_{q_n}, x_{r_n - m}) + \sum_{i=1}^{m} d_p(x_{r_{n-i}}, x_{r_{n-i+1}}) \\ &< \epsilon + \sum_{i=1}^{m} d_p(x_{r_{n-i}}, x_{r_{n-i+1}}). \end{aligned}$$

Letting $n \to \infty$, we obtain that

$$\lim_{n\to\infty} d_p(x_{q_n}, x_{r_n}) = \epsilon. \tag{3.5}$$

On the other hand, we can conclude that

$$\begin{aligned}\epsilon &\leq d_p(x_{q_n}, x_{r_n}) \\ &\leq d_p(x_{q_n}, x_{q_{n+1}}) + d_p(x_{q_{n+1}}, x_{r_{n+1}}) + d_p(x_{r_{n+1}}, x_{r_n}) \\ &\leq d_p(x_{q_n}, x_{q_{n+1}}) + d_p(x_{q_{n+1}}, x_{q_n}) + d_p(x_{q_n}, x_{r_n}) + d_p(x_{r_n}, x_{r_{n+1}}) + d_p(x_{r_{n+1}}, x_{r_n}).\end{aligned}$$

Letting $n \to \infty$, we obtain that

$$\lim_{n\to\infty} d_p(x_{q_{n+1}}, x_{r_{n+1}}) = \epsilon. \tag{3.6}$$

Since $d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ and using (3.5) and (3.6), we have that

$$\lim_{n\to\infty} p(x_{q_n}, x_{r_n}) = \frac{\epsilon}{2}, \tag{3.7}$$

and

$$\lim_{n\to\infty} p(x_{q_{n+1}}, x_{r_{n+1}}) = \frac{\epsilon}{2}. \tag{3.8}$$

Since x_{q_n} and x_{r_n} lie in different adjacently labeled sets A_i and A_{i+1} for certain $1 \leq i \leq m$, by using the fact that f is a cyclic $\mathcal{MK}$-contraction, we have

$$p(x_{q_{n+1}}, x_{r_{n+1}}) = p(fx_{q_n}, fx_{r_n}) \leq \phi\big(p(x_{q_n}, x_{r_n})\big) - \psi\big(p(x_{q_n}, x_{r_n})\big).$$

Letting $n \to \infty$, by using the condition ϕ_3 of the function ϕ, we obtain that

$$\frac{\epsilon}{2} \leq \frac{\epsilon}{2} - \psi\left(\frac{\epsilon}{2}\right),$$

and consequently, $\psi(\frac{\epsilon}{2}) = 0$. By the definition of a function ψ, we get $\epsilon = 0$ which is a contraction. Therefore, our claim is proved.

In the sequel, we will show that $\{x_n\}$ is a Cauchy sequence in the metric space (Y, d_p). Let $\varepsilon > 0$ be given. By our claim, there exists $n_1 \in \mathbb{N}$ such that if $r, q \geq n_1$ with $r - q = 1 \bmod m$, then

$$d_p(x_r, x_q) \leq \frac{\varepsilon}{2}.$$

Since $\lim_{n\to\infty} d_p(x_n, x_{n+1}) = 0$, there exists $n_2 \in \mathbb{N}$ such that

$$d_p(x_n, x_{n+1}) \leq \frac{\varepsilon}{2m}$$

for any $n \geq n_2$.

Let $r, q \geq \max\{n_1, n_2\}$ and $r > q$. Then there exists $k \in \{1, 2, \ldots, m\}$ such that $r - q = k \bmod m$. Therefore, $r - q + j = 1 \bmod m$ for $j = m - k + 1$, and so we have

$$\begin{aligned} d_p(x_q, x_r) &\leq d_p(x_q, x_{r+j}) + d_p(x_{r+j}, x_{r+j-1}) + \cdots + d_p(x_{r-1}, x_r) \\ &\leq \frac{\varepsilon}{2} + j \times \frac{\varepsilon}{2m} \\ &\leq \frac{\varepsilon}{2} + m \times \frac{\varepsilon}{2m} = \varepsilon. \end{aligned}$$

Thus, $\{x_n\}$ is a Cauchy sequence in the metric space (Y, d_p).

Step 3. We show that f has a fixed point ν in $\bigcap_{i=1}^m A_i$.

Since Y is closed, the subspace (Y, p) is complete. Then from Lemma 1, we have that (Y, d_p) is complete. Thus, there exists $\nu \in X$ such that

$$\lim_{n\to\infty} d_p(x_n, \nu) = 0.$$

And it follows from Lemma 1 that we have

$$p(\nu, \nu) = \lim_{n\to\infty} p(x_n, \nu) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{3.9}$$

On the other hand, since the sequence $\{x_n\}$ is a Cauchy sequence in the metric space (Y, d_p), we also have

$$\lim_{n\to\infty} d_p(x_n, x_m) = 0.$$

Since $d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$, we can deduce that

$$\lim_{n\to\infty} p(x_n, x_m) = 0. \tag{3.10}$$

Since $Y = \bigcup_{i=1}^m A_i$ is a cyclic representation of X with respect to f, the sequence $\{x_n\}$ has infinite terms in each A_i for $i \in \{1, 2, \ldots, m\}$. Now, for all $i = 1, 2, \ldots, m$, we may take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \in A_{i-1}$ and also all converge to ν. Using (3.9) and (3.10), we have

$$p(\nu, \nu) = \lim_{n\to\infty} p(x_n, \nu) = \lim_{n\to\infty} p(x_{n_k}, \nu) = 0.$$

By (3.1),

$$\begin{aligned} p(x_{n_{k+1}}, f\nu) &= p(fx_{n_k}, f\nu) \\ &\leq \phi\bigl(p(x_{n_k}, \nu)\bigr) - \psi\bigl(p(x_{n_k}, \nu)\bigr) \\ &\leq \phi\bigl(p(x_{n_k}, \nu)\bigr). \end{aligned}$$

Letting $k \to \infty$, we have

$$p(\nu, f\nu) \leq 0,$$

and so $\nu = f\nu$.

Step 4. Finally, to prove the uniqueness of the fixed point, let μ be another fixed point of f in $\bigcap_{i=1}^m A_i$. By the cyclic character of f, we have $\mu, \nu \in \bigcap_{i=1}^n A_i$. Since f is a cyclic weaker $\mathcal{MK}$-contraction, we have

$$\begin{aligned} p(\nu,\mu) &= p(\nu, f\mu) \\ &= \lim_{n\to\infty} p(x_{n_{k+1}}, f\mu) \\ &= \lim_{n\to\infty} p(fx_{n_k}, f\mu) \\ &\le \lim_{n\to\infty} \bigl[\phi\bigl(p(x_{n_k},\mu)\bigr) - \psi\bigl(p(x_{n_k},\mu)\bigr)\bigr] \\ &\le p(\nu,\mu) - \psi\bigl(p(\nu,\mu)\bigr), \end{aligned}$$

and we can conclude that

$$\psi\bigl(p(\nu,\mu)\bigr) = 0,$$

which implies $p(\nu,\mu) = 0$. So, we have $\mu = \nu$. We complete the proof. □

The following provides an example for Theorem 4.

Example 2 Let $X = [0,1]$ and $A = [0,1]$, $B = [0,\frac{1}{2}]$, $C = [0,\frac{1}{4}]$. We define the partial metric p on X by

$$p(x,y) = \max\{x,y\} \quad \text{for all } x,y \in X,$$

and define the function $f : X \to X$ by

$$f(x) = \frac{x^2}{1+x} \quad \text{for all } x \in X.$$

Now, we let $\psi, \phi : \mathbb{R}^+ \to \mathbb{R}^+$ be

$$\phi(t) = \frac{4t}{5} \quad \text{and} \quad \psi(t) = \frac{t}{5(1+t)}.$$

Then f is a cyclic $\mathcal{MK}$-contraction and 0 is the unique fixed point.

By Theorem 4, it is easy to get the following corollary.

Corollary 1 *Let (X,p) be a complete partial metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ be nonempty closed subsets of X, $Y = \bigcup_{i=1}^m A_i$ and let $f : Y \to Y$. Assume that*

(1) *$\bigcup_{i=1}^m A_i$ is a cyclic representation of Y with respect to f;*

(2) *for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$,*

$$p(fx, fy) \le \phi\bigl(p(x,y)\bigr),$$

where $A_{m+1} = A_1$ and $\phi \in \Phi$.

Then f has a unique fixed point $z \in \bigcap_{i=1}^m A_i$.

Competing interests
The author declares that they have no competing interests.

Acknowledgements
The authors would like to thank referee(s) for many useful comments and suggestions for the improvement of the paper.

Received: 6 November 2012 Accepted: 13 January 2013 Published: 28 January 2013

References

1. Banach, S: Sur les operations dans les ensembles abstraits et leur application aux equations integerales. Fundam. Math. **3**, 133-181 (1922)
2. Boyd, DW, Wong, SW: On nonlinear contractions. Proc. Am. Math. Soc. **20**, 458-464 (1969)
3. Mattews, SG: Partial metric topology. In: Proc. 8th Summer of Conference on General Topology and Applications. Ann. New York Acad. Sci., vol. 728, pp. 183-197 (1994)
4. Oltra, S, Valero, O: Banach's fixed point theorem for partial metric spaces. Rend. Ist. Mat. Univ. Trieste **36**, 17-26 (2004)
5. Kirk, WA, Srinivasan, PS, Veeramani, P: Fixed points for mappings satisfying cyclical contractive conditions. Fixed Point Theory Appl. **4**(1), 79-89 (2003)
6. Karapinar, E: Fixed point theory for cyclic weaker ϕ-contraction. Appl. Math. Lett. **24**(6), 822-825 (2011)
7. Păcurar, M, Rus, IA: Fixed point theory for cyclic φ-contractions. Nonlinear Anal. **72**(3-4), 2683-2693 (2010)
8. Rus, IA: Cyclic representations and fixed points. Ann. "Tiberiu Popoviciu" Sem. Funct. Equ. Approx. Convexity **3**, 171-178 (2005)
9. Agarwal, RP, Alghamdi, MA, Shahzad, N: Fixed point theory for cyclic generalized contractions in partial metric spaces. Fixed Point Theory Appl. **2012**, Article ID 40 (2012)
10. Di Bari, C, Suzuki, T, Vetro, C: Best proximity for cyclic Meir-Keeler contractions. Nonlinear Anal. **69**, 3790-3794 (2008)
11. Karpagam, S, Agrawal, S: Best proximity point theorems for p-cyclic Meir-Keeler contractions. Fixed Point Theory Appl. **2009**, Article ID 197308 (2009)
12. Karpagam, S, Agrawal, S: Best proximity point theorems for cyclic orbital Meir-Keeler contraction maps. Nonlinear Anal. **74**, 1040-1046 (2010)
13. Karapinar, E, Erhan, IM: Best proximity point on different type contraction. Inf. Sci. Appl. Math. **5**, 342-353 (2011)
14. Karapinar, E, Erhan, IM: Fixed point theorem for cyclic maps on partial metric spaces. Inf. Sci. Appl. Math. **6**, 239-244 (2012)
15. Karapinar, E, Erhan, IM: Cyclic contractions and fixed point theorems. Filomat **26**, 777-782 (2012)
16. Meir, A, Keeler, E: A theorem on contraction mappings. J. Math. Anal. Appl. **28**, 326-329 (1969)

doi:10.1186/1687-1812-2013-17
Cite this article as: Chen: **Fixed point theory of cyclical generalized contractive conditions in partial metric spaces.** *Fixed Point Theory and Applications* 2013 **2013**:17.

Submit your manuscript to a SpringerOpen journal and benefit from:

► Convenient online submission
► Rigorous peer review
► Immediate publication on acceptance
► Open access: articles freely available online
► High visibility within the field
► Retaining the copyright to your article

Submit your next manuscript at ► springeropen.com