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Best proximity point results for generalized contractions in metric spaces

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Dedicated to Professor Wataru Takahashi on the occasion of his seventieth birthday

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Abstract
In this paper, we first introduce a cyclic generalized contraction map in metric spaces and give an existence result for a best proximity point of such mappings in the setting of a uniformly convex Banach space. Then we give an existence and uniqueness best proximity point theorem for non-self proximal generalized contractions. Moreover, an algorithm is exhibited to determine such a unique best proximity point. Some examples are also given to support our main results. Our results extend and improve certain recent results in the literature.
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1 Introduction and preliminaries

Fixed point theory is indispensable for solving various equations of the form $Tx = x$ for self-mappings T defined on subsets of metric spaces. Given nonempty subsets A and B of a metric space and a non-self mapping $T : A \to B$, the equation $Tx = x$ does not necessarily have a solution, which is known as a fixed point of the mapping T. However, in such circumstances, it may be speculated to determine an element x for which the error $d(x, Tx)$ is minimum, in which case x and Tx are in close proximity to each other. Best approximation theorems and best proximity point theorems are relevant in this perspective. One of the most interesting results in this direction is due to Fan [1] and can be stated as follows.

Theorem F *Let K be a nonempty compact convex subset of a normed space E and let $T : K \to E$ be a continuous non-self-mapping. Then there exists an x such that $\|x - Tx\| = d(K, Tx) = \inf\{\|Tx - u\| : u \in K\}$.*

Many generalizations and extensions of this theorem appeared in the literature (see [2–6] and references therein).

On the other hand, though best approximation theorems ensure the existence of approximation solutions, such results need not yield optimal solutions. But, best proximity point theorems provide sufficient conditions that assure the existence of approximate solutions which are optimal as well. A best proximity point theorem furnishes sufficient conditions that ascertain the existence of an optimal solution to the problem of globally minimizing the error $d(x, Tx)$, and hence the existence of a consummate approximate solution to the equation $Tx = x$. Indeed, in view of the fact that $d(x, Tx) \geq d(A, B)$ for all x, a best proximity point theorem offers sufficient conditions for the existence of an element x, called a best proximity point of the mapping T, satisfying the condition that $d(x, Tx) = d(A, B)$. Further, it is interesting to observe that best proximity point theorems also emerge as a natural generalization of fixed point theorems for a best proximity point reduces to a fixed point if the mapping under consideration is a self-mapping. Best proximity point theory of cyclic contraction maps has been studied by many authors; see [7–15] and references therein. Investigation of several variants of contractions for the existence of a best proximity point can be found in [16–19]. Best proximity point theorems for multivalued mappings are available in [20, 21].

2 Best proximity points for cyclic generalized contractions

Let A and B be nonempty subsets of a metric space (X, d), $T : A \cup B \to A \cup B$, $T(A) \subseteq B$ and $T(B) \subseteq A$. We say that

(a) T is cyclic contraction [10] if

$$d(Tx, Ty) \leq kd(x, y) + (1 - k)d(A, B) \quad \text{for all } x \in A, y \in B$$

for some $k \in [0, 1)$, where

$$d(A, B) = \inf\bigl\{d(x, y) : x \in A, y \in B\bigr\}.$$

(b) $x \in A \cup B$ is a best proximity point for T if $d(x, Tx) = d(A, B)$.

We first introduce the following new class of cyclic generalized contraction maps.

Definition 2.1 Let A and B be nonempty subsets of a metric space (X, d). A map $T : A \cup B \to A \cup B$ is a cyclic generalized contraction map if $T(A) \subseteq B$, $T(B) \subseteq A$ and

$$d(Tx, Ty) \leq \alpha\bigl(d(x, y)\bigr)d(x, y) + \bigl(1 - \alpha\bigl(d(x, y)\bigr)\bigr)d(A, B) \tag{2.1}$$

for each $x \in A$ and $y \in B$, where $\alpha : [d(A, B), \infty) \to [0, 1)$ satisfies $\limsup_{s \to t^+} \alpha(s) < 1$ for each $t \in (d(A, B), \infty)$.

If $\alpha(t) = k$ for each $t \in [d(A, B), \infty)$, where $k \in [0, 1)$ is constant, then T is a cyclic contraction.

A Banach space X is said to be uniformly convex if there exists a strictly increasing function $\delta : (0, 2] \to [0, 1]$ such that the following implication holds for all $x_1, x_2, p \in X$, $R > 0$ and $r \in [0, 2R]$:

$$\|x_i - p\| \leq R, \quad i = 1, 2 \quad \text{and} \quad \|x_1 - x_2\| \geq r \quad \Rightarrow \quad \left\|\frac{x_1 + x_2}{2} - p\right\| \leq \left(1 - \delta\left(\frac{r}{R}\right)\right)R.$$

Theorem 2.1 (Geraghty [22]) *Let (X, d) be a complete metric space and let $T : X \to X$ be a map satisfying*

$$d(Tx, Ty) \leq \alpha\bigl(d(x, y)\bigr)d(x, y) \quad \textit{for each } x, y \in X,$$

where $\alpha : [0,\infty) \to [0,1)$ *satisfies* $\limsup_{s\to t^+} \alpha(s) < 1$ *for each* $t \in (0,\infty)$. *Then* T *has a fixed point*.

Now, we are ready to state our main result in this section.

Theorem 2.2 *Let* A *and* B *be nonempty closed and convex subsets of a uniformly convex Banach space* X *and let* $T : A \cup B \to A \cup B$ *be a cyclic generalized contraction map. Then* T *has a best proximity point.*

Proof Suppose that $d(A,B) = 0$, then the theorem follows from the above mentioned Geraghty fixed point theorem. Therefore, we may assume that $d(A,B) > 0$. Let $x_0 \in A$ and let $x_{n+1} = Tx_n$ for each $n \in \mathbb{N}$. Then from (2.1) we have

$$\begin{aligned} &\|x_{2m+1} - x_{2n}\| \\ &\quad \le \alpha\bigl(\|x_{2m} - x_{2n-1}\|\bigr)\|x_{2m} - x_{2n-1}\| + \bigl(1 - \alpha\bigl(\|x_{2m} - x_{2n-1}\|\bigr)\bigr)d(A,B) \end{aligned} \tag{2.2}$$

for each $m, n \in \mathbb{N}$. Since $\alpha(\|x_{2m} - x_{2n-1}\|) < 1$ and $\|x_{2m} - x_{2n-1}\| \ge d(A,B)$, so we have

$$\begin{aligned} &\alpha\bigl(\|x_{2m} - x_{2n-1}\|\bigr)\|x_{2m} - x_{2n-1}\| + \bigl(1 - \alpha\bigl(\|x_{2m} - x_{2n-1}\|\bigr)\bigr)d(A,B) \\ &\quad \le \|x_{2m} - x_{2n-1}\|. \end{aligned} \tag{2.3}$$

From (2.2) and (2.3), we get

$$\|x_{2m+1} - x_{2n}\| \le \|x_{2m} - x_{2n-1}\| \tag{2.4}$$

for each $m, n \in \mathbb{N}$. Then from (2.4) we get $\|x_n - x_{n+1}\| \le \|x_{n-1} - x_n\|$ for each $n \in \mathbb{N}$, and so $\{\|x_n - x_{n+1}\|\}$ is a nonnegative nonincreasing sequence in $\mathbb{R}$. Hence $\{\|x_n - x_{n+1}\|\}$ converges to some real number $r_0 \ge d(A,B)$. On the contrary, assume that $r_0 > d(A,B)$. Since $\limsup_{s\to r_0^+} \alpha(s) < 1$ and $\alpha(r_0) < 1$, there exist $r \in (0,1)$ and $\epsilon > 0$ such that $\alpha(s) \le r$ for all $s \in [r_0, r_0 + \epsilon]$. We can take $N_0 \in \mathbb{N}$ such that $r_0 \le \|x_n - x_{n+1}\| \le r_0 + \epsilon$ for all $n \ge N_0$. Then

$$\alpha\bigl(\|x_n - x_{n+1}\|\bigr) \le r \quad \text{for } n \ge N_0.$$

Let $m \in \{n, n-1\}$ and let $n \ge N_0$. Then from (2.2) and the above inequality, we get (note that $\|x_{2m} - x_{2n-1}\| - d(A,B) \ge 0$)

$$\begin{aligned} \|x_{2m+1} - x_{2n}\| &\le \alpha\bigl(\|x_{2m} - x_{2n-1}\|\bigr)\bigl(\|x_{2m} - x_{2n-1}\| - d(A,B)\bigr) + d(A,B) \\ &\le r\bigl(\|x_{2m} - x_{2n-1}\| - d(A,B)\bigr) + d(A,B) = r\|x_{2m} - x_{2n-1}\| + (1-r)d(A,B) \end{aligned}$$

for each $n \ge N_0$ and $m \in \{n, n-1\}$. So, we get

$$\|x_n - x_{n+1}\| \le r\|x_{n-1} - x_n\| + (1-r)d(A,B) \quad \text{for } n \ge 2N_0. \tag{2.5}$$

Letting $n \to \infty$, (2.5) implies $r_0 = \lim_{n\to\infty} \|x_n - x_{n+1}\| \le d(A,B)$, a contradiction. Then

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = d(A,B). \tag{2.6}$$

Now, we show that

$$\lim_{n\to\infty} \|x_{2n+2} - x_{2n}\| = 0 \tag{2.7}$$

and

$$\lim_{n\to\infty} \|x_{2n+3} - x_{2n+1}\| = 0. \tag{2.8}$$

To show that $\lim_{n\to\infty} \|x_{2n+2} - x_{2n}\| = 0$, on the contrary, assume that there exists $\epsilon_0 > 0$ such that for each $k \in \mathbb{N}$ there exists $n_k > k$ such that

$$\|x_{2n_k+2} - x_{2n_k}\| \geq \epsilon_0. \tag{2.9}$$

Choose $0 < \gamma < 1$ such that $\frac{\epsilon_0}{\gamma} > d(A,B)$ and choose ϵ such that

$$0 < \epsilon < \min\left\{\frac{\epsilon_0}{\gamma} - d(A,B), \frac{d(A,B)\delta(\gamma)}{1-\delta(\gamma)}\right\}.$$

By (2.6) there exists N_1 such that

$$\|x_{2n_k+2} - x_{2n_k+1}\| \leq d(A,B) + \epsilon \quad \text{for all } n_k \geq N_1. \tag{2.10}$$

Also, there exists N_2 such that

$$\|x_{2n_k} - x_{2n_k+1}\| \leq d(A,B) + \epsilon \quad \text{for all } n_k \geq N_2. \tag{2.11}$$

Let $N = \max\{N_1, N_2\}$. It follows from (2.9)-(2.11) and the uniform convexity of X that

$$\left\|\frac{x_{2n_k+2} + x_{2n_k}}{2} - x_{2n_k+1}\right\| \leq \left(1 - \delta\left(\frac{\epsilon_0}{d(A,B)+\epsilon}\right)\right)\bigl(d(A,B)+\epsilon\bigr)$$

for all $n_k \geq N$. As $\frac{x_{2n_k+2}+x_{2n_k}}{2} \in A$, the choice of ϵ and the fact that δ is strictly increasing imply that

$$\left\|\frac{x_{2n_k+2} + x_{2n_k}}{2} - x_{2n_k+1}\right\| < d(A,B) \quad \text{for all } n_k \geq N,$$

a contradiction. A similar argument shows $\|x_{2n+3} - x_{2n+1}\| \to 0$ as $n \to \infty$. Hence, (2.7) and (2.8) hold.

Now we show that for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m > n \geq N$,

$$\|x_{2m} - x_{2n+1}\| < d(A,B) + \epsilon. \tag{2.12}$$

On the contrary, assume that there exists $\epsilon_1 > 0$ such that for each $k \geq 1$ there is $m_k > n_k \geq k$ satisfying

$$\|x_{2m_k} - x_{2n_k+1}\| \geq d(A,B) + \epsilon_1 \tag{2.13}$$

and

$$\|x_{2(m_k-1)} - x_{2n_k+1}\| < d(A,B) + \epsilon_1. \tag{2.14}$$

It follows from (2.13), (2.14) and the triangle inequality that

$$\begin{aligned} d(A,B) + \epsilon_1 &\leq \|x_{2m_k} - x_{2n_k+1}\| \\ &\leq \|x_{2m_k} - x_{2(m_k-1)}\| + \|x_{2(m_k-1)} - x_{2n_k+1}\| \\ &< \|x_{2m_k} - x_{2(m_k-1)}\| + d(A,B) + \epsilon_1. \end{aligned}$$

Letting $k \to \infty$, (2.7) implies

$$\lim_{k\to\infty} \|x_{2m_k} - x_{2n_k+1}\| = d(A,B) + \epsilon_1. \tag{2.15}$$

Let $t_0 = d(A,B) + \epsilon_1$. Since $\limsup_{s\to t_0^+} \alpha(s) < 1$ and $\alpha(t_0) < 1$, there exist $r' \in (0,1)$ and $\epsilon' > 0$ such that $\alpha(s) \leq r'$ for all $s \in [t_0, t_0 + \epsilon']$. Thanks to (2.15), we can take $K_0 \in \mathbb{N}$ such that $t_0 \leq \|x_{2m_k} - x_{2n_k+1}\| \leq t_0 + \epsilon'$ for all $k \geq K_0$. Then

$$\alpha\big(\|x_{2m_k} - x_{2n_k+1}\|\big) \leq r' \quad \text{for } k \geq K_0,$$

and so from (2.2) we get

$$\|x_{2m_k+1} - x_{2n_k+2}\| \leq r'\|x_{2m_k} - x_{2n_k+1}\| + \big(1 - r'\big)d(A,B) \quad \text{for } k \geq K_0. \tag{2.16}$$

From (2.4) and (2.16), we get

$$\begin{aligned} \|x_{2m_k} - x_{2n_k+1}\| &\leq \|x_{2m_k} - x_{2m_k+2}\| + \|x_{2m_k+2} - x_{2n_k+3}\| + \|x_{2n_k+3} - x_{2n_k+1}\| \\ &\leq \|x_{2m_k} - x_{2m_k+2}\| + \|x_{2m_k+1} - x_{2n_k+2}\| + \|x_{2n_k+3} - x_{2n_k+1}\| \\ &\leq \|x_{2m_k} - x_{2m_k+2}\| + \|x_{2n_k+3} - x_{2n_k+1}\| \\ &\quad + r'\|x_{2m_k} - x_{2n_k+1}\| + \big(1 - r'\big)d(A,B) \end{aligned} \tag{2.17}$$

for each $k \geq K_0$. Letting $k \to \infty$ and using (2.7), (2.8), (2.15) and (2.17), we get

$$d(A,B) + \epsilon_1 \leq r'\big(d(A,B) + \epsilon_1\big) + \big(1 - r'\big)d(A,B) = d(A,B) + r'\epsilon_1,$$

a contradiction. Thus (2.12) holds.

Now we show that $\{x_{2n}\}$ is a Cauchy sequence in A. To show the claim, we assume the contrary. Then there exists $\epsilon_2 > 0$ such that for each $k \geq 1$, there exist $p_k > q_k \geq k$ such that

$$\|x_{2p_k} - x_{2q_k}\| \geq \epsilon_2. \tag{2.18}$$

Choose $0 < \gamma < 1$ such that $\frac{\epsilon_2}{\gamma} > d(A,B)$ and choose $\epsilon > 0$ such that

$$\epsilon < \min\left\{\frac{\epsilon_2}{\gamma} - d(A,B), \frac{d(A,B)\delta(\gamma)}{1 - \delta(\gamma)}\right\}.$$

By (2.6) there exists N_1 such that

$$\|x_{2n_k} - x_{2n_k+1}\| < d(A,B) + \epsilon \quad \text{for all } n_k \geq N_1. \tag{2.19}$$

By (2.12) there exists N_2 such that

$$\|x_{2m_k} - x_{2n_k+1}\| < d(A,B) + \epsilon \quad \text{for all } m_k > n_k \geq N_2. \tag{2.20}$$

Let $N = \max\{N_1, N_2\}$. It follows from (2.18)-(2.20) and the uniform convexity of X that

$$\left\| \frac{x_{2m_k} + x_{2n_k}}{2} - x_{2n_k+1} \right\| \leq \left(1 - \delta\left(\frac{\epsilon_2}{d(A,B) + \epsilon}\right)\right)\bigl(d(A,B) + \epsilon\bigr) \quad \text{for all } m_k > n_k \geq N.$$

By the choice of ϵ and the fact that δ is strictly increasing, we have

$$\left\| \frac{x_{2m_k} + x_{2n_k}}{2} - x_{2n_k+1} \right\| < d(A,B) \quad \text{for all } m_k > n_k \geq N,$$

a contradiction. Thus $\{x_{2n}\}$ is a Cauchy sequence in A. Now the completeness of X and the closedness of A imply that

$$\lim_{n\to\infty} x_{2n} = x \in A. \tag{2.21}$$

Since (note that $x \in A$ and $x_{2n-1} \in B$)

$$d(A,B) \leq \|x - x_{2n-1}\| \leq \|x - x_{2n}\| + \|x_{2n} - x_{2n-1}\| \quad \text{for each } n \in \mathbb{N},$$

it follows from (2.6) and (2.21) that

$$\lim_{n\to\infty} \|x - x_{2n-1}\| = d(A,B). \tag{2.22}$$

Since

$$\begin{aligned} d(A,B) &\leq \|x_{2n} - Tx\| = \|Tx_{2n-1} - Tx\| \\ &\leq \alpha\bigl(\|x_{2n-1} - x\|\bigr)\|x_{2n-1} - x\| + \bigl(1 - \alpha\bigl(\|x_{2n-1} - x\|\bigr)\bigr)d(A,B) \\ &\leq \|x_{2n-1} - x\| \end{aligned} \tag{2.23}$$

for each $n \in \mathbb{N}$, then from (2.21)-(2.23) we get $\|x - Tx\| = d(A,B)$. Therefore, T has a best proximity point. □

Now we illustrate our main result by the following example.

Example 2.1 Consider the uniformly convex Banach space $X = \mathbb{R}^2$ with Euclidean metric. Let $A := \{(0,x) : 0 \leq x\}$ and $B := \{(2,y) : 0 \leq y\}$. Then A and B are nonempty closed and convex subsets of X and $d(A,B) = 2$.

Let $T : A \cup B \to A \cup B$ be defined as

$$T(0,x) = \left(2, \frac{x}{2}\right) \quad \text{and} \quad T(2,y) = \left(0, \frac{y}{2}\right) \quad \text{for each } x, y \geq 0.$$

We show that T is a generalized cyclic contraction map with $\alpha(t) = \frac{1}{2}$ for $t \in [2,\infty)$. To show the claim, notice first that the function $f(t) = \sqrt{4+t^2} - 2$, $t \in [0,\infty)$ is convex, $f(0) = 0$ and so $f(\frac{t}{2}) \leq \frac{1}{2}f(t)$ for $t \in [0,\infty)$. For each $x, y \in [0,\infty)$, we have

$$\begin{aligned} d\big(T(0,x), T(2,y)\big) &= \sqrt{4 + \left(\frac{|x-y|}{2}\right)^2} \\ &\leq 2 + \frac{1}{2}\big(\sqrt{4 + \big(|x-y|\big)^2} - 2\big) \\ &= \frac{1}{2}\big(\sqrt{4 + \big(|x-y|\big)^2}\big) + \frac{1}{2}2 \\ &= \frac{1}{2}d\big((0,x),(2,y)\big) + \frac{1}{2}d(A,B). \end{aligned}$$

Thus all of the hypotheses of Theorem 2.2 are satisfied and then T has a best proximity point ($(0,0)$ is a best proximity point of T in A).

Now we provide the following example to show that Theorem 2.2 is an essential extension of Theorem 3.10 of Eldred and Veeramani [10].

Example 2.2 Consider the uniformly convex Banach space $X = \mathbb{R}^2$ with Euclidean metric. Let $A := \{(0,x) : 0 \leq x\}$ and $B := \{(2,y) : 0 \leq y\}$. Then A and B are nonempty closed and convex subsets of X and $d(A,B) = 2$.

Let $T : A \cup B \to A \cup B$ be defined as

$$T(0,x) = \big(2, \ln(1+x)\big) \quad \text{and} \quad T(2,y) = \big(0, \ln(1+y)\big) \quad \text{for each } x, y \geq 0.$$

We first show that T is not a cyclic contraction map. To show the claim, on the contrary, assume that there exists $k \in [0,1)$ such that

$$d\big(T(0,x), T(2,y)\big) \leq kd\big((0,x),(2,y)\big) + (1-k)d(A,B)$$

for each $x, y \in [0,\infty)$. Then

$$\sqrt{4 + \big(\ln(1+x) - \ln(1+y)\big)^2} \leq k\sqrt{4 + (x-y)^2} + 2(1-k)$$

for each $x, y \in [0,\infty)$. Letting $y = 0$, we get

$$\frac{\sqrt{4 + (\ln(1+x))^2} - 2}{\sqrt{4+x^2} - 2} \leq k$$

for each $x \in (0,\infty)$. Then

$$1 = \lim_{x\to 0^+} \frac{\sqrt{4 + (\ln(1+x))^2} - 2}{\sqrt{4+x^2} - 2} \leq k,$$

a contradiction. Now, we show that T is a cyclic generalized contraction, where $\alpha(t) = \frac{\sqrt{4+(\ln(1+\sqrt{t^2-4}))^2}-2}{t-2}$ for $t \in (2,\infty)$. Notice first that the function $f(t) = \ln(1+t) : [0,\infty) \to$

$[0,\infty)$ is increasing and concave and so is subadditive, that is, $f(r+s) \le f(r) + f(s)$ for each $r,s \in [0,\infty)$. For each $x,y \in [0,\infty)$ with $x \ne y$, we have (note that for $x = y$ we have $d(T(0,x),T(2,y)) = d((0,x),(2,y)) = 2 = d(A,B)$ and so (2.1) trivially holds for $\alpha(2) = 0$)

$$\begin{aligned} d\bigl(T(0,x),T(2,y)\bigr) &= \sqrt{4 + \bigl(\ln(1+x) - \ln(1+y)\bigr)^2} \\ &\le \sqrt{4 + \bigl(\ln\bigl(1+|x-y|\bigr)\bigr)^2} \\ &= \frac{\sqrt{4 + (\ln(1+|x-y|))^2} - 2}{\sqrt{4+(x-y)^2} - 2}\bigl(\sqrt{4+(x-y)^2} - 2\bigr) + 2, \end{aligned}$$

$$\begin{aligned} &\alpha\bigl(\sqrt{4+(x-y)^2}\bigr)\bigl(\sqrt{4+(x-y)^2} - 2\bigr) + 2 \\ &\quad = \alpha\bigl(d\bigl((0,x),(2,y)\bigr)\bigr)d\bigl((0,x),(2,y)\bigr) + 2\bigl(1 - \alpha\bigl(d\bigl((0,x),(2,y)\bigr)\bigr)\bigr). \end{aligned}$$

Thus all of the hypotheses of Theorem 2.2 are satisfied and then T has a best proximity point ($(0,0)$ is a best proximity point of T in A). But since T is not a cyclic contraction, we cannot invoke the main result of [10] to show the existence of the best proximity point for T.

3 Best proximity points for generalized contraction

Given nonempty subsets A and B of a metric space, we recall the following notations and notions, which will be used in the sequel.

$$\begin{aligned} &d(A,B) = \inf\bigl\{d(x,y) : x \in A \text{ and } y \in B\bigr\}, \\ &A_0 = \bigl\{x \in A : d(x,y) = d(A,B) \text{ for some } y \in B\bigr\}, \\ &B_0 = \bigl\{y \in B : d(x,y) = d(A,B) \text{ for some } x \in A\bigr\}. \end{aligned}$$

The set B is said to be approximatively compact with respect to A if every sequence $\{y_n\}$ in B, satisfying the condition that $d(x,y_n) \to d(x,B)$ for some x in A, has a convergent subsequence. It is trivial to note that every set is approximatively compact with respect to itself, and that every compact set B is approximatively compact with respect to A.

A mapping $T : A \to B$ is said to be a proximal contraction if there exists a non-negative number $\alpha < 1$ such that for all u_1, u_2, x_1, x_2 in A,

$$d(u_1, Tx_1) = d(A,B) = d(u_2, Tx_2) \quad \Rightarrow \quad d(u_1,u_2) \le \alpha d(x_1,x_2).$$

To establish our results, we introduce the following new class of proximal contractions.

Definition 3.1 Let $T : A \to B$, $g : A \to A$ be two maps. Let $\varphi : [0,\infty) \to [0,\infty)$ satisfy

$$\varphi(0) = 0, \qquad \varphi(t) < t, \quad \text{and} \quad \limsup_{s \to t^+} \varphi(s) < t \quad \text{for each } t > 0.$$

Then T is said to be a (φ,g)-proximal contraction if

$$d(u_1, Tx_1) = d(A,B) = d(u_2, Tx_2) \quad \Rightarrow \quad d(u_1,u_2) \le \varphi\bigl(d(gx_1,gx_2)\bigr)$$

for all u_1, u_2, x_1, x_2 in A.

Now, we are ready to state our first main result in this section.

Theorem 3.1 *Let A and B be nonempty closed subsets of a complete metric space (X,d) such that B is approximately compact with respect to A. Moreover, assume that A_0 and B_0 are nonempty. Let $T : A \to B$ and $g : A \to A$ satisfy the following conditions.*

(a) *T is a (φ,g)-proximal contraction,*
(b) *$T(A_0) \subseteq B_0$,*
(c) *g is a one-to-one continuous map such that $g^{-1} : g(A) \to A$ is uniformly continuous,*
(d) *$A_0 \subseteq g(A_0)$.*

Then there exists a unique element $x \in A$ such that $d(gx, Tx) = d(A,B)$. Further, for any fixed element $x_0 \in A_0$, the sequence $\{x_n\}$ defined by $d(gx_{n+1}, Tx_n) = d(A,B)$ converges to x.

Proof Let x_0 be a fixed element in A_0. Since $T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, then there exists an element $x_1 \in A_0$ such that $d(gx_1, Tx_0) = d(A,B)$. Proceeding in this manner, having chosen $x_n \in A_0$, we can find $x_{n+1} \in A_0$ satisfying

$$d(gx_{n+1}, Tx_n) = d(A,B) \quad \text{for each } n \in \mathbb{N}. \tag{3.1}$$

Since T is a (φ,g)-proximal contraction, then from (3.1) we have

$$d(gx_{n+1}, gx_{n+2}) \leq \varphi\bigl(d(gx_n, gx_{n+1})\bigr) \quad \text{for each } n \in \mathbb{N}. \tag{3.2}$$

We shall show that $\{gx_n\}$ is a Cauchy sequence. Let $\delta_n = d(gx_n, gx_{n+1})$. From (3.2) we get that the sequence $\{\delta_n\}$ is non-increasing (note that $\varphi(t) \leq t$ for all $t \geq 0$). Therefore, there is some $\delta \geq 0$ such that $\lim_{n\to\infty} \delta_n = \delta^+$. We show that $\delta = 0$. Suppose, to the contrary, that $\delta > 0$. Then from (3.2) we get

$$\delta = \lim_{n\to\infty} \delta_{n+1} \leq \limsup_{n\to\infty} \varphi(\delta_n) < \delta,$$

a contradiction. Thus $\delta = 0$, that is,

$$\lim_{n\to\infty} \delta_n = 0. \tag{3.3}$$

Suppose, to the contrary, that $\{gx_n\}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$ and two subsequences of integers $\{l(k)\}$ and $\{m(k)\}$, $m(k) > l(k) \geq k$ with

$$r_k = d(gx_{l(k)}, gx_{m(k)}) \geq \epsilon \quad \text{for } k \in \{1, 2, \ldots\}. \tag{3.4}$$

We may also assume

$$d(gx_{l(k)}, gx_{m(k)-1}) < \epsilon \quad \text{for } k \in \{1, 2, \ldots\}, \tag{3.5}$$

by choosing $m(k)$ to be the smallest number exceeding $l(k)$ for which (3.4) holds. From (3.4), (3.5) and by the triangle inequality,

$$\epsilon \leq r_k \leq d(gx_{l(k)}, gx_{m(k)-1}) + d(gx_{m(k)-1}, gx_{m(k)}) < \epsilon + \delta_{m(k)-1}.$$

Taking the limit as $k \to \infty$, we get (note that $\lim_{k\to\infty} \delta_{m(k)-1} = 0$)

$$\lim_{k\to\infty} r_k = \epsilon^+. \tag{3.6}$$

By the triangle inequality

$$\begin{aligned} r_k &= d(gx_{l(k)}, gx_{m(k)}) \\ &\le d(gx_{l(k)}, gx_{l(k)+1}) + d(gx_{l(k)+1}, gx_{m(k)+1}) + d(gx_{m(k)+1}, gx_{m(k)}) \\ &= \delta_{l(k)} + \delta_{m(k)} + d(gx_{l(k)+1}, gx_{m(k)+1}). \end{aligned} \tag{3.7}$$

From (3.2), we have

$$d(gx_{l(k)+1}, gx_{m(k)+1}) \le \varphi\big(d(gx_{l(k)}, gx_{m(k)})\big) = \varphi(r_k). \tag{3.8}$$

Then from (3.7) and (3.8), we have

$$r_k \le \delta_{l(k)} + \delta_{m(k)} + \varphi(r_k).$$

Letting $k \to \infty$ and using (3.2) and (3.6), we get

$$\epsilon \le \limsup_{k\to\infty} \varphi(r_k) < \epsilon,$$

a contradiction. Therefore $\{gx_n\}$ is a Cauchy sequence. Since g^{-1} is uniformly continuous and $\{gx_n\}$ is a Cauchy sequence, then we get that $\{x_n\}$ is also a Cauchy sequence. Since X is complete and $A \subseteq X$ is closed, there exists $x \in A$ such that $\lim_{n\to\infty} x_n = x$. Further, it can be noted

$$\begin{aligned} d(gx, B) &\le d(gx, Tx_n) \\ &\le d(gx, gx_{n+1}) + d(gx_{n+1}, Tx_n) \\ &\le d(gx, gx_{n+1}) + d(A, B) \le d(gx, gx_{n+1}) + d(gx, B). \end{aligned}$$

Since g is continuous and $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} gx_n = gx$. Therefore from the above, $d(gx, Tx_n) \to d(gx, B)$ as $n \to \infty$. Since B is approximatively compact with respect to A, it follows that the sequence $\{Tx_n\}$ has a subsequence converging to some element $y \in B$. Thus $d(gx, y) = d(A, B)$ and hence $gx \in A_0$. Since $A_0 \subseteq g(A_0)$, $gx = gu$ for some $u \in A_0$. Therefore $x = u \in A_0$. Since $TA_0 \subseteq B_0$, then

$$d(z, Tx) = d(A, B) \quad \text{for some } z \in A. \tag{3.9}$$

From (3.1), (3.2) and (3.9), we have

$$d(gx_{n+1}, z) \le \varphi\big(d(gx, gx_{n+1})\big) \le d(gx, gx_{n+1}).$$

Therefore

$$z = \lim_{n\to\infty} gx_{n+1} = gx.$$

Hence

$$d(gx, Tx) = d(z, Tx) = d(A, B). \tag{3.10}$$

Suppose that there is another x^* such that

$$d\big(gx^*, Tx^*\big) = d(A, B). \tag{3.11}$$

Then from (3.10) and (3.11) we get

$$d\big(gx, gx^*\big) \le \varphi\big(d\big(gx, gx^*\big)\big),$$

which implies that $x = x^*$. □

The following theorem, which is the main result of Sadiq Basha [18], is immediate.

Theorem 3.2 *Let A and B be nonempty closed subsets of a complete metric space (X, d) such that B is approximately compact with respect to A. Moreover, assume that A_0 and B_0 are nonempty. Let $T : A \to B$ and $g : A \to A$ satisfy the following conditions.*

(a) *T is a proximal contraction,*
(b) *$T(A_0) \subseteq B_0$,*
(c) *g is an isometry,*
(d) *$A_0 \subseteq g(A_0)$.*

Then there exists a unique element $x \in A$ such that $d(gx, Tx) = d(A, B)$. Further, for any fixed element $x_0 \in A_0$, the sequence $\{x_n\}$ defined by $d(gx_{n+1}, Tx_n) = d(A, B)$ converges to x.

Now we illustrate our best proximity point theorem by the following example.

Example 3.1 Consider the complete metric space $X = [0, 1] \times [0, 1]$ with Euclidean metric. Let $A := \{(0, x) : 0 \le x \le 1\}$ and $B = \{(1, y) : 0 \le y \le 1\}$. Then $d(A, B) = 1$, $A_0 = A$ and $B_0 = B$. Let $g : A \to A$ be defined as $g(0, x) = (0, \frac{2x}{1+x})$. Then g is a one-to-one continuous map, $g^{-1} : A \to A$ is uniformly continuous and $g(A_0) = A_0$.

Let $T : A \to B$ be defined as $T(0, x) = (1, \frac{x}{4})$. Let $\varphi(t) = \frac{t}{2}$ for each $t \ge 0$. Then it is easy to see that T is (φ, g)-proximal contraction. So, all the hypotheses of Theorem 3.1 are satisfied. Further, it is easy to see that $(0, 0)$ is the unique element satisfying the conclusion of Theorem 3.1. However, we cannot invoke the above mentioned Theorem 3.2 of Sadiq Basha to show the existence of a best proximity point because g is not an isometry.

The following are immediate consequences of Theorem 3.1.

Theorem 3.3 *Let A and B be nonempty closed subsets of a complete metric space (X, d) such that B is compact. Moreover, assume that B_0 is nonempty. Let $T : A \to B$ and $g : A \to A$ satisfy the following conditions.*

(a) *T is a (φ, g)-proximal contraction,*
(b) *$T(A_0) \subseteq B_0$,*
(c) *g is a one-to-one continuous map such that $g^{-1} : g(A) \to A$ is uniformly continuous,*
(d) *$A_0 \subseteq g(A_0)$.*

Then there exists a unique element $x \in A$ such that $d(gx, Tx) = d(A,B)$. Further, for any fixed element $x_0 \in A_0$, the sequence $\{x_n\}$ defined by $d(gx_{n+1}, Tx_n) = d(A,B)$ converges to x.

Theorem 3.4 *Let (X,d) be a complete metric space and let T, $g : X \to X$ satisfy the following conditions.*

(a) *T is a (φ,g)-contraction,*
(b) *$g : X \to X$ is a one-to-one, onto continuous map such that g^{-1} is uniformly continuous.*

Then there exists a unique element $x \in X$ such that $gx = Tx$, that is, (T,g) has a coincidence point x. Further, for any fixed element $x_0 \in X$, the sequence $\{x_n\}$ defined by $gx_{n+1} = Tx_n$ converges to x.

Theorem 3.5 *Let (X,d) be a complete metric space and let $T : X \to X$ be a φ-contraction. Then T has a unique fixed point $x \in X$. Further, for any fixed element $x_0 \in X$, the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ converges to x.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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