Liu *Fixed Point Theory and Applications* 2013, **2013**:147
http://www.fixedpointtheoryandapplications.com/content/2013/1/147

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Quadruple fixed point theorems in partially ordered metric spaces with mixed g-monotone property

Xiao-lan Liu*

*Correspondence:
stellalwp@163.com
School of Science, Sichuan University of Science and Engineering, Zigong, Sichuan 643000, China

Abstract
We prove quadruple fixed point theorems in partially ordered metric spaces with mixed g-monotone property. Also, we state some examples to show that our results are real generalization of known ones in quadruple fixed point theory.

MSC: 46T99; 54H25; 47H10; 54E50

Keywords: quadruple coincidence point; quadruple common fixed point; ordered metric spaces; mixed g-monotone property

1 Introduction

In 1987, the notion of coupled fixed point was introduced by Guo and Lakshmikantham [1]. Later, Bhaskar and Lakshmikantham [2] introduced the concept of mixed monotone property for contractive operators of the form $F: X \to X$, where X is a partially ordered metric space, and then established some coupled fixed point theorems. They also illustrated these results by proving the existence and uniqueness of the solution for a periodic boundary value problem. Recently, Lakshmikantham and Ćirić in [3] defined a g-monotone property and proved coupled coincidence and coupled common fixed point results for nonlinear mappings satisfying certain contractive conditions in partially ordered metric spaces. They also proved related fixed point theorems. Many authors focused on coupled fixed point theory and proved remarkable results (see [4–17]).

Very recently, Berinde and Borcut [18] introduced the concept of triple fixed point and proved some tripled point theorems by virtue of mixed monotone mappings. Their contributions generalize and extend Bhaskar and Lakshmikantham's research for nonlinear mappings. The notion of fixed point of order $N \geq 3$ was first introduced by Samet and Vetro [19]. Karapinar used the concept of quadruple fixed point and proved some fixed point theorems on the topic [20]. Following this study, a quadruple fixed point is developed and some related fixed point theorems are obtained in [21–24]. Recently, Karapinar *et al.* [25] have proved a number of quadruple fixed point theorems under ϕ-contractive conditions for a mapping $F: X^4 \to X$ in ordered metric spaces.

Let us recall some basic definitions from [21].

Definition 1.1 (See [21]) Let X be a nonempty set and let $F : X^4 \to X$ be a given mapping. An element $(x, y, z, w) \in X \times X \times X \times X$ is called a quadruple fixed point of F if

$$F(x,y,z,w) = x, \qquad F(y,z,w,x) = y, \qquad F(z,w,x,y) = z \quad \text{and} \quad F(w,x,y,z) = w.$$

Let (X, d) be a metric space. The mapping $\bar{d} : X^4 \to X$, given by

$$\bar{d}\bigl((x,y,z,w),(u,v,h,l)\bigr) = d(x,y) + d(y,v) + d(z,h) + d(w,l),$$

defines a metric on X^4, which will be denoted for convenience by d.

Definition 1.2 (See [21]) Let $(X, \leq)$ be a partially ordered set and let $F : X^4 \to X$ be a mapping. We say that F has the mixed monotone property if $F(x,y,z,w)$ is monotone nondecreasing in x and z and is monotone non-increasing in y and w; that is, for any $x, y, z, w \in X$,

$$\begin{aligned}
&x_1, x_2 \in X, \quad x_1 \leq x_2 \quad \text{implies} \quad F(x_1,y,z,w) \leq F(x_2,y,z,w),\\
&y_1, y_2 \in X, \quad y_1 \leq y_2 \quad \text{implies} \quad F(x,y_2,z,w) \leq F(x,y_1,z,w),\\
&z_1, z_2 \in X, \quad z_1 \leq z_2 \quad \text{implies} \quad F(x,y,z_1,w) \leq F(x,y,z_2,w) \quad \text{and}\\
&w_1, w_2 \in X, \quad w_1 \leq w_2 \quad \text{implies} \quad F(x,y,z,w_2) \leq F(x,y,z,w_1).
\end{aligned}$$

In this article, we establish some quadruple coincidence and common fixed point theorems for $F : X^4 \to X$ and $g : X \to X$ satisfying nonlinear contractions in partially ordered metric spaces. Also, some examples are given to support our results.

2 Preliminary

We start this section with the following definitions.

Definition 2.1 Let $(X, \leq)$ be a partially ordered set. Let $F : X^4 \to X$ and $g : X \to X$. The mapping F is said to have the mixed g-monotone property if for any $x, y, z, w \in X$,

$$\begin{aligned}
&x_1, x_2 \in X, \quad gx_1 \leq gx_2 \quad \Rightarrow \quad F(x_1,y,z,w) \leq F(x_2,y,z,w),\\
&y_1, y_2 \in X, \quad gy_1 \leq gy_2 \quad \Rightarrow \quad F(x,y_1,z,w) \geq F(x,y_2,z,w),\\
&z_1, z_2 \in X, \quad gz_1 \leq gz_2 \quad \Rightarrow \quad F(x,y,z_1,w) \leq F(x,y,z_2,w) \quad \text{and}\\
&w_1, w_2 \in X, \quad gw_1 \leq gw_2 \quad \Rightarrow \quad F(x,y,z,w_1) \geq F(x,y,z,w_2).
\end{aligned}$$

Definition 2.2 Let $F : X^4 \to X$ and $g : X \to X$. An element (x, y, z, w) is called a quadruple coincidence point of F and g if

$$F(x,y,z,w) = gx, \qquad F(y,z,w,x) = gy, \qquad F(z,w,x,y) = gz \quad \text{and} \quad F(w,x,y,z) = gw.$$

Definition 2.3 Let $F : X^4 \to X$ and $g : X \to X$. An element (x, y, z, w) is called a quadruple common fixed point of F and g if

$$\begin{aligned}
&F(x,y,z,w) = gx = x, \qquad F(y,z,w,x) = gy = y,\\
&F(z,w,x,y) = gz = z \quad \text{and} \quad F(w,x,y,z) = gw = w.
\end{aligned}$$

Definition 2.4 Let X be a nonempty set. Then we say that the mappings $F : X^4 \to X$ and $g : X \to X$ are commutative if for all $x, y, z, w \in X$,

$$g\big(F(x,y,z,w)\big) = F(gx, gy, gz, gw).$$

Let Φ denote all the functions $\phi : [0,\infty) \to [0,\infty)$ which satisfy that $\lim_{t\to r} \phi(t) > 0$ for all $r > 0$ and $\lim_{t\to 0+} \phi(t) = 0$.

Let Ψ denote all the functions $\psi : [0,\infty) \to [0,\infty)$ which satisfy
(i) $\psi(t) = 0$ if and only if $t = 0$,
(ii) ψ is continuous and nondecreasing,
(iii) $\psi(s + t) \le \psi(s) + \psi(t)$, $\forall s, t \in [0,\infty)$.

Examples of typical functions ϕ and ψ are given in [4]. The aim of this paper is to prove the following theorem.

3 Main results

Now, we present the main results of this paper.

Theorem 3.1 *Let $(X, \le)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose that $F : X^4 \to X$ and $g : X \to X$ are such that F is continuous and has the mixed g-monotone property. Assume also that there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$\begin{aligned}\psi\big(d\big(F(x,y,z,w), F(u,v,h,l)\big)\big) \le{}& \frac{1}{4}\psi\big(d(gx,gu) + d(gy,gv) + d(gz,gh) + d(gw,gl)\big)\\ &- \phi\big(d(gx,gu) + d(gy,gv) + d(gz,gh) + d(gw,gl)\big) \qquad (1)\end{aligned}$$

for any $x, y, z, w, u, v, h, l \in X$, for which $gx \le gu$, $gv \ge gy$, $gz \le gh$, and $gl \ge gw$. Suppose that $F(X^4) \subset g(X)$, g is continuous and commutes with F. If there exist $x_0, y_0, z_0, w_0 \in X$ such that

$$\begin{aligned}&gx_0 \le F(x_0, y_0, z_0, w_0), \qquad gy_0 \ge F(y_0, z_0, w_0, x_0),\\ &gz_0 \le F(z_0, w_0, x_0, y_0) \quad \text{and} \quad gw_0 \ge F(w_0, x_0, y_0, z_0),\end{aligned}$$

then there exist $x, y, z, w \in X$ such that

$$F(x,y,z,w) = gx, \qquad F(y,z,w,x) = gy, \qquad F(z,w,x,y) = gz \quad \text{and} \quad F(w,x,y,z) = gw,$$

that is, F and g have a quadruple coincidence point.

Proof Let $x_0, y_0, z_0, w_0 \in X$ such that

$$\begin{aligned}&gx_0 \le F(x_0, y_0, z_0, w_0), \qquad gy_0 \ge F(y_0, z_0, w_0, x_0),\\ &gz_0 \le F(z_0, w_0, x_0, y_0), \qquad gw_0 \ge F(w_0, x_0, y_0, z_0).\end{aligned}$$

Since $F(X^4) \subset g(X)$, then we can choose $x_1, y_1, z_1, w_1 \in X$ such that

$$\begin{aligned}&gx_1 = F(x_0, y_0, z_0, w_0), \qquad gy_1 = F(y_0, z_0, w_0, x_0),\\ &gz_1 = F(z_0, w_0, x_0, y_0), \qquad gw_1 = F(w_0, x_0, y_0, z_0).\end{aligned} \qquad (2)$$

Taking into account $F(X^4) \subset g(X)$, by continuing this process, we can construct sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{w_n\}$ in X such that

$$\begin{aligned} &gx_{n+1} = F(x_n, y_n, z_n, w_n), \qquad gy_{n+1} = F(y_n, z_n, w_n, x_n), \\ &gz_{n+1} = F(z_n, w_n, x_n, y_n), \qquad gw_{n+1} = F(w_n, x_n, y_n, z_n). \end{aligned} \tag{3}$$

We shall show that

$$\begin{aligned} &gx_n \le gx_{n+1}, \qquad gy_{n+1} \ge gy_n, \\ &gz_n \le gz_{n+1} \quad \text{and} \quad gw_{n+1} \ge gw_n \quad \text{for } n = 0, 1, 2, \ldots. \end{aligned} \tag{4}$$

For this purpose, we use the mathematical induction. Since $gx_0 \le F(x_0, y_0, z_0, w_0)$, $gy_0 \ge F(y_0, z_0, w_0, x_0)$, $gz_0 \le F(z_0, w_0, x_0, y_0)$, and $gw_0 \ge F(w_0, x_0, y_0, z_0)$, then by (2), we get

$$gx_0 \le gx_1, \qquad gy_1 \ge gy_0, \qquad gz_0 \le gz_1 \quad \text{and} \quad gw_1 \ge gw_0,$$

that is, (4) holds for $n = 0$. We presume that (4) holds for some $n > 0$. As F has the mixed g-monotone property and $gx_n \le gx_{n+1}$, $gy_{n+1} \ge gy_n$, $gz_n \le gz_{n+1}$, and $gw_{n+1} \ge gw_n$, we obtain

$$\begin{aligned} gx_{n+1} &= F(x_n, y_n, z_n, w_n) \le F(x_{n+1}, y_n, z_n, w_n) \\ &\le F(x_{n+1}, y_n, z_{n+1}, w_n) \le F(x_{n+1}, y_{n+1}, z_{n+1}, w_n) \\ &\le F(x_{n+1}, y_{n+1}, z_{n+1}, w_{n+1}) = gx_{n+2}, \\ gy_{n+2} &= F(y_{n+1}, z_{n+1}, w_{n+1}, x_{n+1}) \le F(y_{n+1}, z_n, w_{n+1}, x_{n+1}) \\ &\le F(y_n, z_n, w_{n+1}, x_{n+1}) \le F(y_n, z_n, w_n, x_{n+1}) \\ &\le F(y_n, z_n, w_n, x_n) = gy_{n+1}, \\ gz_{n+1} &= F(z_n, w_n, x_n, y_n) \le F(z_{n+1}, w_n, x_n, y_n) \\ &\le F(z_{n+1}, w_n, x_n, y_{n+1}) \le F(z_{n+1}, w_n, x_{n+1}, y_{n+1}) \\ &\le F(z_{n+1}, w_{n+1}, x_{n+1}, y_{n+1}) = gz_{n+2}, \\ gw_{n+2} &= F(w_{n+1}, x_{n+1}, y_{n+1}, z_{n+1}) \le F(w_{n+1}, x_n, y_{n+1}, z_{n+1}) \\ &\le F(w_n, x_n, y_{n+1}, z_{n+1}) \le F(w_n, x_n, y_n, z_{n+1}) \\ &\le F(w_n, x_n, y_n, z_n) = gw_{n+1}. \end{aligned}$$

Thus, (4) holds for any $n \in N$. Assume, for some $n \in N$, that

$$gx_n = gx_{n+1}, \qquad gy_n = gy_{n+1}, \qquad gz_n = gz_{n+1} \quad \text{and} \quad gw_n = gw_{n+1},$$

then, by (3), (x_n, y_n, z_n, w_n) is a quadruple coincidence point of F and g. From now on, assume for any $n \in N$ that at least $gx_n \ne gx_{n+1}$ or $gy_n \ne gy_{n+1}$, or $gz_n \ne gz_{n+1}$, or $gw_n \ne gw_{n+1}$.

Due to (1)-(4), we have

$$\begin{aligned} \psi\big(d(gx_{n+1}, gx_{n+2})\big) &= \psi\big(d\big(F(x_n, y_n, z_n, w_n), F(x_{n+1}, y_{n+1}, z_{n+1}, w_{n+1})\big)\big) \\ &\le \frac{1}{4}\psi\big(d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) + d(gz_n, gz_{n+1}) + d(gw_n, gw_{n+1})\big) \end{aligned}$$

$$
\begin{aligned}
&\quad - \phi\big(d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) + d(gz_n, gz_{n+1}) \\
&\quad + d(gw_n, gw_{n+1})\big), \qquad (5)\\
\psi\big(d(gy_{n+1}, gy_{n+2})\big) &= \psi\big(d\big(F(y_n, z_n, w_n, x_n), F(y_{n+1}, z_{n+1}, w_{n+1}, x_{n+1})\big)\big) \\
&\le \frac{1}{4}\psi\big(d(gy_n, gy_{n+1}) + d(gz_n, gz_{n+1}) + d(gw_n, gw_{n+1}) + d(gx_n, gx_{n+1})\big) \\
&\quad - \phi\big(d(gy_n, gy_{n+1}) + d(gz_n, gz_{n+1}) + d(gw_n, gw_{n+1}) \\
&\quad + d(gx_n, gx_{n+1})\big), \qquad (6)\\
\psi\big(d(gz_{n+1}, gz_{n+2})\big) &= \psi\big(d\big(F(z_n, w_n, x_n, y_n), F(z_{n+1}, w_{n+1}, x_{n+1}, y_{n+1})\big)\big) \\
&\le \frac{1}{4}\psi\big(d(gz_n, gz_{n+1}) + d(gw_n, gw_{n+1}) + d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})\big) \\
&\quad - \phi\big(d(gz_n, gz_{n+1}) + d(gw_n, gw_{n+1}) + d(gx_n, gx_{n+1}) \\
&\quad + d(gy_n, gy_{n+1})\big), \qquad (7)\\
\psi\big(d(gw_{n+1}, gw_{n+2})\big) &= \psi\big(d\big(F(w_n, x_n, y_n, z_n), F(w_{n+1}, x_{n+1}, y_{n+1}, z_{n+1})\big)\big) \\
&\le \frac{1}{4}\psi\big(d(gw_n, gw_{n+1}) + d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) + d(gz_n, gz_{n+1})\big) \\
&\quad - \phi\big(d(gw_n, gw_{n+1}) + d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) \\
&\quad + d(gz_n, gz_{n+1})\big). \qquad (8)
\end{aligned}
$$

Due to (5)-(8), we conclude that

$$
\begin{aligned}
&\psi\big(d(gx_{n+1}, gx_{n+2})\big) + \psi\big(d(gy_{n+1}, gy_{n+2})\big) + \psi\big(d(gz_{n+1}, gz_{n+2})\big) + \psi\big(d(w_{n+1}, w_{n+2})\big) \\
&\quad \le \psi\big(d(gz_n, gz_{n+1}) + d(gw_n, gw_{n+1}) + d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})\big) \\
&\qquad - 4\phi\big(d(gz_n, gz_{n+1}) + d(gw_n, gw_{n+1}) + d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})\big). \qquad (9)
\end{aligned}
$$

From the property (iii) of ψ, we have

$$
\begin{aligned}
&\psi\big(d(gx_{n+1}, gx_{n+2})\big) + d(gy_{n+1}, gy_{n+2}) + d(gz_{n+1}, gz_{n+2}) + d(gw_{n+1}, gw_{n+2}) \\
&\quad \le \psi\big(d(gx_{n+1}, gx_{n+2})\big) + \psi\big(d(gy_{n+1}, gy_{n+2})\big) + \psi\big(d(gz_{n+1}, gz_{n+2})\big) \\
&\qquad + \psi\big(d(gw_{n+1}, gw_{n+2})\big). \qquad (10)
\end{aligned}
$$

Combining with (9) and (10), we get that

$$
\begin{aligned}
&\psi\big(d(gx_{n+1}, gx_{n+2})\big) + d(gy_{n+1}, gy_{n+2}) + d(gz_{n+1}, gz_{n+2}) + d(gw_{n+1}, gw_{n+2}) \\
&\quad \le \psi\big(d(gz_n, gz_{n+1}) + d(gw_n, gw_{n+1}) + d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})\big) \\
&\qquad - 4\phi\big(d(gz_n, gz_{n+1}) + d(gw_n, gw_{n+1}) + d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})\big). \qquad (11)
\end{aligned}
$$

Set $\delta_n = d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}) + d(gz_n, gz_{n-1}) + d(gw_n, gw_{n-1})$. Then we have

$$
\psi(\delta_{n+2}) \le \psi(\delta_{n+1}) - 4\phi(\delta_{n+1}) \quad \text{for all } n, \qquad (12)
$$

which yields that $\psi(\delta_{n+2}) \le \psi(\delta_{n+1})$ for all n.

Since ψ is nondecreasing, we get that $\delta_{n+2} \leq \delta_{n+1}$ for all n. Hence $\{\delta_n\}$ is a non-increasing sequence. Since it is bounded below from 0, there is some $\delta \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \delta. \tag{13}$$

We shall show that $\delta = 0$. Suppose, on the contrary, that $\delta > 0$.

Letting $n \to \infty$ in (12) and having in mind that we suppose that $\lim_{t\to r} \phi(t) > 0$ for all $r > 0$ and $\lim_{t\to 0+} \phi(t) = 0$, we have

$$\psi(\delta) \leq \psi(\delta) - 4\phi(\delta) < \psi(\delta), \tag{14}$$

which is a contraction. Thus, $\delta = 0$, that is,

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \left[d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1}) + d(gz_n, gz_{n-1}) + d(gw_n, gw_{n-1})\right] = 0. \tag{15}$$

Now, we shall show that $\{gx_n\}$, $\{gy_n\}$, $\{gz_n\}$, and $\{gw_n\}$ are Cauchy sequences in the metric space (X, d). Assume the contrary, that is, one of the sequences $\{gx_n\}$, $\{gy_n\}$, $\{gz_n\}$ or $\{gw_n\}$ is not a Cauchy sequence, that is,

$$\begin{aligned} &\lim_{n,m\to\infty} d(gx_m, gx_n) \neq 0 \quad \text{or} \quad \lim_{n,m\to\infty} d(gy_m, gy_n) \neq 0, \quad \text{or} \\ &\lim_{n,m\to\infty} d(gz_m, gz_n) \neq 0, \quad \text{or} \quad \lim_{n,m\to\infty} d(gw_m, gw_n) \neq 0. \end{aligned}$$

This means that there exists $\epsilon > 0$, for which we can find subsequences $\{x_{n(k)}\}$, $\{x_{m(k)}\}$ of x_n and $\{y_{n(k)}\}$, $\{y_{m(k)}\}$ of y_n and $\{z_{n(k)}\}$, $\{z_{m(k)}\}$ of z_n and $\{w_{n(k)}\}$, $\{w_{m(k)}\}$ of w_n with $n(k) \geq m(k) \geq k$ such that

$$d(gx_{m(k)}, gx_{n(k)}) + d(gy_{m(k)}, gy_{n(k)}) + d(gz_{m(k)}, gz_{n(k)}) + d(gw_{m(k)}, gw_{n(k)}) \geq \epsilon. \tag{16}$$

In addition, by virtue of $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k) \geq k$ and satisfying (16). It follows that

$$\begin{aligned} &d(gx_{m(k)}, gx_{n(k)-1}) + d(gy_{m(k)}, gy_{n(k)-1}) \\ &\quad + d(gz_{m(k)}, gz_{n(k)-1}) + d(gw_{m(k)}, gw_{n(k)-1}) < \epsilon. \end{aligned} \tag{17}$$

By use of the triangle inequality, we have

$$d(gx_{m(k)}, gx_{n(k)}) \leq d(gx_{m(k)}, gx_{n(k)-1}) + d(gx_{n(k)-1}, gx_{n(k)}). \tag{18}$$

Similarly, we get that

$$d(gy_{m(k)}, gy_{n(k)}) \leq d(gy_{m(k)}, gy_{n(k)-1}) + d(gy_{n(k)-1}, gy_{n(k)}), \tag{19}$$

$$d(gz_{m(k)}, gz_{n(k)}) \leq d(gz_{m(k)}, gz_{n(k)-1}) + d(gz_{n(k)-1}, gz_{n(k)}), \tag{20}$$

$$d(gw_{m(k)}, gw_{n(k)}) \leq d(gw_{m(k)}, gw_{n(k)-1}) + d(gw_{n(k)-1}, gw_{n(k)}). \tag{21}$$

Adding both sides to (18), (19), (20), (21) and using (16) and (17), we have that

$$\begin{aligned}
\epsilon &\le d(gx_{m(k)}, gx_{n(k)}) + d(gy_{m(k)}, gy_{n(k)}) + d(gz_{m(k)}, gz_{n(k)}) + d(gw_{m(k)}, gw_{n(k)}) \\
&\le d(gx_{m(k)}, gx_{n(k)-1}) + d(gx_{n(k)-1}, gx_{n(k)}) + d(gy_{m(k)}, gy_{n(k)-1}) + d(gy_{n(k)-1}, gy_{n(k)}) \\
&\quad + d(gz_{m(k)}, gz_{n(k)-1}) + d(gz_{n(k)-1}, gz_{n(k)}) + d(gw_{m(k)}, gw_{n(k)-1}) + d(gw_{n(k)-1}, gw_{n(k)}) \\
&\le \epsilon + d(gx_{n(k)-1}, gx_{n(k)}) + d(gy_{n(k)-1}, gy_{n(k)}) + d(gz_{n(k)-1}, gz_{n(k)}) + d(gw_{n(k)-1}, gw_{n(k)}).
\end{aligned}$$

Letting $k \to \infty$ and by use of (15), we get

$$\begin{aligned}
\lim_{k\to\infty} \lambda_k &= \lim_{k\to\infty} \bigl[d(gx_{m(k)}, gx_{n(k)}) + d(gy_{m(k)}, gy_{n(k)}) + d(gz_{m(k)}, gz_{n(k)}) + d(gw_{m(k)}, gw_{n(k)})\bigr] \\
&= \epsilon.
\end{aligned}$$

Again, by the triangle inequality, we have

$$\begin{aligned}
\lambda_k &= d(gx_{m(k)}, gx_{n(k)}) + d(gy_{m(k)}, gy_{n(k)}) + d(gz_{m(k)}, gz_{n(k)}) + d(gw_{m(k)}, gw_{n(k)}) \\
&\le d(gx_{m(k)}, gx_{m(k)+1}) + d(gx_{m(k)+1}, gx_{n(k)+1}) + d(gx_{n(k)+1}, gx_{n(k)}) \\
&\quad + d(gy_{m(k)}, gy_{m(k)+1}) + d(gy_{m(k)+1}, gy_{n(k)+1}) + d(gy_{n(k)+1}, gy_{n(k)}) \\
&\quad + d(gz_{m(k)}, gz_{m(k)+1}) + d(gz_{m(k)+1}, gz_{n(k)+1}) + d(gz_{n(k)+1}, gz_{n(k)}) \\
&\quad + d(gw_{m(k)}, gw_{m(k)+1}) + d(gw_{m(k)+1}, gw_{n(k)+1}) + d(gw_{n(k)+1}, gw_{n(k)}) \\
&\le \delta_{m(k)+1} + \delta_{n(k)+1} + d(gx_{m(k)+1}, gx_{n(k)+1}) \\
&\quad + d(gy_{m(k)+1}, gy_{n(k)+1}) + d(gz_{m(k)+1}, gz_{n(k)+1}) + d(gw_{m(k)+1}, gw_{n(k)+1}).
\end{aligned} \tag{22}$$

Since $n(k) \ge m(k)$, then

$$\begin{aligned}
&gx_{m(k)} \le gx_{n(k)}, \qquad gy_{m(k)} \ge gy_{n(k)}, \\
&gz_{m(k)} \le gz_{n(k)}, \qquad gw_{m(k)} \ge gw_{n(k)}.
\end{aligned} \tag{23}$$

Hence from (1), (3), and (22), we get that

$$\begin{aligned}
\psi\bigl(d(gx_{m(k)+1}, gx_{n(k)+1})\bigr) &= \psi\bigl(d\bigl(F(x_{m(k)}, y_{m(k)}, z_{m(k)}, w_{m(k)}), F(x_{n(k)}, y_{n(k)}, z_{n(k)}, w_{n(k)})\bigr)\bigr) \\
&\le \frac{1}{4}\psi\bigl(d(gx_{m(k)}, gx_{n(k)}) + d(gy_{m(k)}, gy_{n(k)}) + d(gz_{m(k)}, gz_{n(k)}) \\
&\quad + d(gw_{m(k)}, gw_{n(k)})\bigr) - \phi\bigl(d(gx_{m(k)}, gx_{n(k)}) + d(gy_{m(k)}, gy_{n(k)}) \\
&\quad + d(gz_{m(k)}, gz_{n(k)}) + d(gw_{m(k)}, gw_{n(k)})\bigr),
\end{aligned} \tag{24}$$

$$\begin{aligned}
\psi\bigl(d(gy_{m(k)+1}, gy_{n(k)+1})\bigr) &= \psi\bigl(d\bigl(F(y_{m(k)}, z_{m(k)}, w_{m(k)}, x_{m(k)}), F(y_{n(k)}, z_{n(k)}, w_{n(k)}, x_{n(k)})\bigr)\bigr) \\
&\le \frac{1}{4}\psi\bigl(d(gy_{m(k)}, gy_{n(k)}) + d(gz_{m(k)}, gz_{n(k)}) + d(gw_{m(k)}, gw_{n(k)})\bigr) \\
&\quad + d(gx_{m(k)}, gx_{n(k)}) - \phi\bigl(+d(gy_{m(k)}, gy_{n(k)}) + d(gz_{m(k)}, gz_{n(k)}) \\
&\quad + d(gw_{m(k)}, gw_{n(k)}) + d(gx_{m(k)}, gx_{n(k)})\bigr),
\end{aligned} \tag{25}$$

$$\begin{aligned}\psi\big(d(gz_{m(k)+1}, gz_{n(k)+1})\big) &= \psi\big(d\big(F(z_{m(k)}, w_{m(k)}, x_{m(k)}, y_{m(k)}), F(z_{n(k)}, w_{n(k)}, x_{n(k)}, y_{n(k)})\big)\big)\\ &\le \frac{1}{4}\psi\big(d(gz_{m(k)}, gz_{n(k)}) + d(gw_{m(k)}, gw_{n(k)}) + d(gx_{m(k)}, gx_{n(k)})\\ &\quad + d(gy_{m(k)}, gy_{n(k)})\big) - \phi\big(d(gz_{m(k)}, gz_{n(k)}) + d(gw_{m(k)}, gw_{n(k)})\\ &\quad + d(gx_{m(k)}, gx_{n(k)}) + d(gy_{m(k)}, gy_{n(k)})\big), \end{aligned}\tag{26}$$

$$\begin{aligned}\psi\big(d(gw_{m(k)+1}, gw_{n(k)+1})\big) &= \psi\big(d\big(F(w_{m(k)}, x_{m(k)}, y_{m(k)}, z_{m(k)}), F(w_{n(k)}, x_{n(k)}, y_{n(k)}, z_{n(k)})\big)\big)\\ &\le \frac{1}{4}\psi\big(d(gw_{m(k)}, gw_{n(k)}) + d(gx_{m(k)}, gx_{n(k)}) + d(gy_{m(k)}, gy_{n(k)})\\ &\quad + d(gz_{m(k)}, gz_{n(k)})\big) - \phi\big(d(gw_{m(k)}, gw_{n(k)}) + d(gx_{m(k)}, gx_{n(k)})\\ &\quad + d(gy_{m(k)}, gy_{n(k)}) + d(gz_{m(k)}, gz_{n(k)})\big). \end{aligned}\tag{27}$$

Combining (22) and (24)-(27), we have that

$$\begin{aligned}\psi(\lambda_k) &\le \psi\big(\delta_{m(k)+1} + \delta_{n(k)+1} + d(gx_{m(k)+1}, gx_{n(k)+1}) + d(gy_{m(k)+1}, gy_{n(k)+1})\\ &\quad + d(gz_{m(k)+1}, gz_{n(k)+1}) + d(gw_{m(k)+1}, gw_{n(k)+1})\big)\\ &\le \psi(\delta_{m(k)+1} + \delta_{n(k)+1}) + \psi\big(d(gx_{m(k)+1}, gx_{n(k)+1}) + d(gy_{m(k)+1}, gy_{n(k)+1})\\ &\quad + d(gz_{m(k)+1}, gz_{n(k)+1}) + d(gw_{m(k)+1}, gw_{n(k)+1})\big)\\ &\le \psi(\delta_{m(k)+1}) + \psi(\delta_{n(k)+1}) + \psi\big(d(gx_{m(k)+1}, gx_{n(k)+1})\big) + \psi\big(d(gy_{m(k)+1}, gy_{n(k)+1})\big)\\ &\quad + \psi\big(d(gz_{m(k)+1}, gz_{n(k)+1})\big) + \psi\big(d(gw_{m(k)+1}, gw_{n(k)+1})\big)\\ &\le \psi(\delta_{m(k)+1}) + \psi(\delta_{n(k)+1}) + \psi(\lambda_k) - 4\phi(\lambda_k).\end{aligned}$$

Letting $k \to \infty$, we get a contradiction. This shows that $\{gx_n\}$, $\{gy_n\}$, $\{gz_n\}$, and $\{gw_n\}$ are Cauchy sequences in the metric space (X, d). Since (X, d) is complete, there exist $x, y, z, w \in X$ such that

$$\lim_{n\to\infty} gx_n = x, \qquad \lim_{n\to\infty} gy_n = y, \qquad \lim_{n\to\infty} gz_n = z \quad \text{and} \quad \lim_{n\to\infty} gw_n = w. \tag{28}$$

From (28) and the continuity of g, we have

$$\begin{aligned}&\lim_{n\to\infty} g(gx_n) = gx, \qquad \lim_{n\to\infty} g(gy_n) = gy,\\ &\lim_{n\to\infty} g(gz_n) = gz \quad \text{and} \quad \lim_{n\to\infty} g(gw_n) = gw.\end{aligned}\tag{29}$$

It follows from (3) and the commutativity of F and g that

$$g(gx_{n+1}) = g\big(F(x_n, y_n, z_n, w_n)\big) = F(gx_n, gy_n, gz_n, gw_n), \tag{30}$$

$$g(gy_{n+1}) = g\big(F(y_n, z_n, w_n, x_n)\big) = F(gy_n, gz_n, gw_n, gx_n), \tag{31}$$

$$g(gz_{n+1}) = g\big(F(z_n, w_n, x_n, y_n)\big) = F(gz_n, gw_n, gx_n, gy_n), \tag{32}$$

$$g(gw_{n+1}) = g\big(F(w_n, x_n, y_n, z_n)\big) = F(gw_n, gx_n, gy_n, gz_n). \tag{33}$$

Now we shall show that $gx = F(x, y, z, w)$, $gy = F(y, z, w, x)$, $gz = F(z, w, x, y)$, $gw = F(w, x, y, z)$.

By letting $n \to \infty$ in (30)-(33), by (28), (29), and the continuity of F, we obtain

$$\begin{aligned} gx &= \lim_{n\to\infty} g(gx_{n+1}) = \lim_{n\to\infty} F(gx_n, gy_n, gz_n, gw_n) \\ &= F\Big(\lim_{n\to\infty} gx_n, \lim_{n\to\infty} gy_n, \lim_{n\to\infty} gz_n, \lim_{n\to\infty} gw_n\Big) \\ &= F(x, y, z, w), \end{aligned} \tag{34}$$

$$\begin{aligned} gy &= \lim_{n\to\infty} g(gy_{n+1}) = \lim_{n\to\infty} F(gy_n, gz_n, gw_n, gx_n) \\ &= F\Big(\lim_{n\to\infty} gy_n, \lim_{n\to\infty} gz_n, \lim_{n\to\infty} gw_n, \lim_{n\to\infty} gx_n\Big) \\ &= F(y, z, w, x), \end{aligned} \tag{35}$$

$$\begin{aligned} gz &= \lim_{n\to\infty} g(gz_{n+1}) = \lim_{n\to\infty} F(gz_n, gw_n, gx_n, gy_n) \\ &= F\Big(\lim_{n\to\infty} gz_n, \lim_{n\to\infty} gw_n, \lim_{n\to\infty} gx_n, \lim_{n\to\infty} gy_n\Big) \\ &= F(z, w, x, y), \end{aligned} \tag{36}$$

$$\begin{aligned} gw &= \lim_{n\to\infty} g(gw_{n+1}) = \lim_{n\to\infty} F(gw_n, gx_n, gy_n, gz_n) \\ &= F\Big(\lim_{n\to\infty} gw_n, \lim_{n\to\infty} gx_n, \lim_{n\to\infty} gy_n, \lim_{n\to\infty} gz_n\Big) \\ &= F(w, x, y, z). \end{aligned} \tag{37}$$

We have shown that F and g have a quadruple coincidence point. □

In the following theorem, the continuity of F is removed. We state the following definition.

Definition 3.1 Let $(X, \leq)$ be a partially ordered metric space and d be a metric on X. We say that $(X, d, \leq)$ is regular if the following conditions hold:

(i) if a nondecreasing sequence $a_n \to a$, then $a_n \leq a$ for all n,
(ii) if a non-increasing sequence $b_n \to b$, then $b \leq b_n$ for all n.

Theorem 3.2 *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that $(X, d, \leq)$ is regular. Suppose that $F : X^4 \to X$ and $g : X \to X$ are such that F has the mixed g-monotone property. Assume also that there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$\begin{aligned} \psi\big(d\big(F(x,y,z,w), F(u,v,h,l)\big)\big) &\leq \frac{1}{4}\psi\big(d(gx,gu) + d(gy,gv) + d(gz,gh) + d(gw,gl)\big) \\ &\quad - \phi\big(d(gx,gu) + d(gy,gv) + d(gz,gh) + d(gw,gl)\big) \end{aligned}$$

for any $x, y, z, w, u, v, h, l \in X$, for which $gx \leq gu$, $gv \geq gy$, $gz \leq gh$, and $gl \geq gw$. Suppose that $F(X^4) \subset g(X)$, $(g(X), d)$ is a complete metric space. If there exist $x_0, y_0, z_0, w_0 \in X$ such that

$$gx_0 \leq F(x_0, y_0, z_0, w_0), \qquad gy_0 \geq F(y_0, z_0, w_0, x_0),$$

$$gz_0 \leq F(z_0, w_0, x_0, y_0), \qquad gw_0 \geq F(w_0, x_0, y_0, z_0),$$

then there exist $x, y, z, w \in X$ such that

$$F(x,y,z,w) = gx, \qquad F(y,z,w,x) = gy, \qquad F(z,w,x,y) = gz \quad \textit{and} \quad F(w,x,y,z) = gw,$$

that is, F and g have a quadruple coincidence point.

Proof Proceeding exactly as in Theorem 3.1, we have that $\{gx_n\}$, $\{gy_n\}$, $\{gz_n\}$, and $\{gw_n\}$ are Cauchy sequences in the complete metric space $(g(X), d)$. Then there exist $x, y, z, w \in X$ such that

$$gx_n \to gx, \qquad gy_n \to gy, \qquad gz_n \to gz \quad \text{and} \quad gw_n \to gw. \tag{38}$$

Since $\{gx_n\}$, $\{gz_n\}$ are nondecreasing and $\{gy_n\}$, $\{gw_n\}$ are non-increasing, then since $(X, d, \leq)$ is regular, we get that

$$gx_n \leq gx, \qquad gy_n \geq gy, \qquad gz_n \leq gz \quad \text{and} \quad gw_n \geq gw$$

for all n. If $gx_n = gx$, $gy_n = gy$, $gz_n = gz$, and $gw_n = gw$ for some $n > 0$, then $gx = gx_n \leq gx_{n+1} \leq gx = gx_n$, $gy \leq gy_{n+1} \leq gy_n = gy$, $gz = gz_n \leq gz_{n+1} \leq gz = gz_n$, and $gw \leq gw_{n+1} \leq gw_n = gw$, which implies that

$$\begin{aligned}
&gx_n = gx_{n+1} = F(x_n, y_n, z_n, w_n), \qquad gy_n = gy_{n+1} = F(y_n, z_n, w_n, x_n) \quad \text{and} \\
&gz_n = gz_{n+1} = F(z_n, w_n, x_n, y_n), \qquad gw_n = gw_{n+1} = F(w_n, x_n, y_n, z_n),
\end{aligned}$$

that is, (x_n, y_n, z_n, w_n) is a quadruple coincidence point of F and g. Then, we suppose that $(gx_n, gy_n, gz_n, gw_n) \neq (gx, gy, gz, gw)$ for all $n > 0$. By use of (1), consider now

$$\begin{aligned}
\psi\big(d\big(gx, F(x,y,z,w)\big)\big) &\leq \psi\big(d(gx, gx_{n+1}) + d\big(gx_{n+1}, F(x,y,z,w)\big)\big) \\
&= \psi\big(d(gx, gx_{n+1})\big) + \psi\big(d\big(F(x_n, y_n, z_n, w_n), F(x,y,z,w)\big)\big) \\
&\leq \psi\big(d(gx, gx_{n+1})\big) + \frac{1}{4}\psi\big(d(gx_n, gx) + d(gy_n, gy) \\
&\quad + d(gz_n, gz) + d(gw_n, gw)\big) \\
&\quad - \phi\big(d(gx_n, gx) + d(gy_n, gy) + d(gz_n, gz) + d(gw_n, gw)\big).
\end{aligned} \tag{39}$$

Letting $n \to \infty$ and by (38), then the right-hand side of (39) tends to 0, thus $\psi(d(gx, F(x, y, z, w))) = 0$. By the property (i) of ψ, we have $d(gx, F(x, y, z, w)) = 0$. It follows that $gx = F(x, y, z, w)$. Similarly, we can show that

$$gy = F(y,z,w,x), \qquad gz = F(z,w,x,y), \qquad gw = F(w,x,y,z).$$

Therefore, we have proved that F and g have a quadruple coincidence point. □

Corollary 3.1 *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose that $F : X^4 \to X$ and $g : X \to X$ are* *such that F is continuous and has the mixed g-monotone property. Assume also that there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$\psi\bigl(d\bigl(F(x,y,z,w),F(u,v,,h,l)\bigr)\bigr) \le \frac{1}{4}\psi\bigl(\max\bigl\{d(gx,gu),d(gy,gv),d(gz,gh),d(gw,gl)\bigr\}\bigr) - \phi\bigl(d(gx,gu)+d(gy,gv)+d(gz,gh)+d(gw,gl)\bigr)$$

for any $x,y,z,w,u,v,h,l \in X$, for which $gx \le gu$, $gv \ge gy$, $gz \le gh$, and $gl \ge gw$. Suppose that $F(X^4) \subset g(X)$, g is continuous and commutes with F. If there exist $x_0,y_0,z_0,w_0 \in X$ such that

$$gx_0 \le F(x_0,y_0,z_0,w_0), \qquad gy_0 \ge F(y_0,z_0,w_0,x_0),$$
$$gz_0 \le F(z_0,w_0,x_0,y_0), \qquad gw_0 \ge F(w_0,x_0,y_0,z_0),$$

then there exist $x,y,z,w \in X$ such that

$$F(x,y,z,w)=gx, \qquad F(y,z,w,x)=gy, \qquad F(z,w,x,y)=gz \quad \text{and} \quad F(w,x,y,z)=gw,$$

that is, F and g have a quadruple coincidence point.

Proof Since

$$\max\bigl\{d(gx,gu),d(gy,gv),d(gz,gh),d(gw,gl)\bigr\} \le d(gx,gu)+d(gy,gv)+d(gz,gh)+d(gw,gl),$$

then we apply Theorem 3.1, since ψ is assumed to be nondecreasing. □

Similarly, as an easy consequence of Theorem 3.2, we have the following corollary.

Corollary 3.2 *Let $(X,\le)$ be a partially ordered set and suppose there is a metric d on X such that $(X,d,\le)$ is regular. Suppose that $F: X^4 \to X$ and $g: X \to X$ are such that F has the mixed g-monotone property. Assume also that there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$\psi\bigl(d\bigl(F(x,y,z,w),F(u,v,h,l)\bigr)\bigr) \le \frac{1}{4}\psi\bigl(\max\bigl\{d(gx,gu),d(gy,gv),d(gz,gh),d(gw,gl)\bigr\}\bigr) - \phi\bigl(d(gx,gu)+d(gy,gv)+d(gz,gh)+d(gw,gl)\bigr)$$

for any $x,y,z,w,u,v,h,l \in X$, for which $gx \le gu$, $gv \ge gy$, $gz \le gh$, and $gl \ge gw$. Suppose that $F(X^4) \subset g(X)$, $(g(X),d)$ is a complete metric space. If there exist $x_0,y_0,z_0,w_0 \in X$ such that

$$gx_0 \le F(x_0,y_0,z_0,w_0), \qquad gy_0 \ge F(y_0,z_0,w_0,x_0),$$
$$gz_0 \le F(z_0,w_0,x_0,y_0), \qquad gw_0 \ge F(w_0,x_0,y_0,z_0),$$

then there exist $x,y,z,w \in X$ such that

$$F(x,y,z,w)=gx, \qquad F(y,z,w,x)=gy, \qquad F(z,w,x,y)=gz \quad \text{and} \quad F(w,x,y,z)=gw,$$

that is, F and g have a quadruple coincidence point.

Corollary 3.3 *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose that $F : X^4 \to X$ and $g : X \to X$ are such that F is continuous and has the mixed g-monotone property. Assume also that there exists $k \in [0, 1)$ such that*

$$d\bigl(F(x,y,z,w), F(u,v,h,l)\bigr) \leq \frac{k}{4}\bigl(d(gx,gu) + d(gy,gv) + d(gz,gh) + d(gw,gl)\bigr)$$

for any $x, y, z, w, u, v, h, l \in X$, for which $gx \leq gu$, $gv \geq gy$, $gz \leq gh$, and $gl \geq gw$. Suppose that $F(X^4) \subset g(X)$, g is continuous and commutes with F. If there exist $x_0, y_0, z_0, w_0 \in X$ such that

$$\begin{aligned} &gx_0 \leq F(x_0, y_0, z_0, w_0), \qquad gy_0 \geq F(y_0, z_0, w_0, x_0), \\ &gz_0 \leq F(z_0, w_0, x_0, y_0), \qquad gw_0 \geq F(w_0, x_0, y_0, z_0), \end{aligned}$$

then there exist $x, y, z, w \in X$ such that

$$F(x,y,z,w) = gx, \qquad F(y,z,w,x) = gy, \qquad F(z,w,x,y) = gz \quad \text{and} \quad F(w,x,y,z) = gw,$$

that is, F and g have a quadruple coincidence point.

Proof It is sufficient to set $\psi(t) = t$ and $\phi(t) = \frac{1-k}{4}t$ in Theorem 3.1. □

Corollary 3.4 *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that $(X, d, \leq)$ is regular. Suppose that $F : X^4 \to X$ and $g : X \to X$ are such that F has the mixed g-monotone property. Assume also that there exists $k \in [0, 1)$ such that*

$$d\bigl(F(x,y,z,w), F(u,v,h,l)\bigr) \leq \frac{k}{4}\bigl(d(gx,gu) + d(gy,gv) + d(gz,gh) + d(gw,gl)\bigr)$$

for any $x, y, z, w, u, v, h, l \in X$, for which $gx \leq gu$, $gv \geq gy$, $gz \leq gh$, and $gl \geq gw$. Suppose that $F(X^4) \subset g(X)$, $(g(X), d)$ is a complete metric space. If there exist $x_0, y_0, z_0, w_0 \in X$ such that

$$\begin{aligned} &gx_0 \leq F(x_0, y_0, z_0, w_0), \qquad gy_0 \geq F(y_0, z_0, w_0, x_0), \\ &gz_0 \leq F(z_0, w_0, x_0, y_0), \qquad gw_0 \geq F(w_0, x_0, y_0, z_0), \end{aligned}$$

then there exist $x, y, z, w \in X$ such that

$$F(x,y,z,w) = gx, \qquad F(y,z,w,x) = gy, \qquad F(z,w,x,y) = gz \quad \text{and} \quad F(w,x,y,z) = gw,$$

that is, F and g have a quadruple coincidence point.

Proof It is sufficient to set $\psi(t) = t$ and $\phi(t) = \frac{1-k}{4}t$ in Theorem 3.2. □

Corollary 3.5 *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose that $F : X^4 \to X$ and $g : X \to X$ are* *such that F is continuous and has the mixed g-monotone property. Assume also that there exists $k \in [0,1)$ such that*

$$d\bigl(F(x,y,z,w),F(u,v,h,l)\bigr) \le \frac{k}{4}\bigl(\max\bigl\{d(gx,gu),d(gy,gv),d(gz,gh),d(gw,gl)\bigr\}\bigr)$$

for any $x,y,z,w,u,v,h,l \in X$, for which $gx \le gu$, $gv \ge gy$, $gz \le gh$, and $gl \ge gw$. Suppose that $F(X^4) \subset g(X)$, g is continuous and commutes with F. If there exist $x_0,y_0,z_0,w_0 \in X$ such that

$$\begin{aligned} &gx_0 \le F(x_0,y_0,z_0,w_0), \qquad gy_0 \ge F(y_0,z_0,w_0,x_0), \\ &gz_0 \le F(z_0,w_0,x_0,y_0), \qquad gw_0 \ge F(w_0,x_0,y_0,z_0), \end{aligned}$$

then there exist $x,y,z,w \in X$ such that

$$F(x,y,z,w) = gx, \qquad F(y,z,w,x) = gy, \qquad F(z,w,x,y) = gz \quad \text{and} \quad F(w,x,y,z) = gw,$$

that is, F and g have a quadruple coincidence point.

Proof It suffices to remark that

$$\begin{aligned} &\max\bigl\{d(gx,gu),d(gy,gv),d(gz,gh),d(gw,gl)\bigr\} \\ &\quad \le d(gx,gu) + d(gy,gv) + d(gz,gh) + d(gw,gl). \end{aligned}$$

Then we apply Corollary 3.3. □

Corollary 3.6 *Let $(X,\le)$ be a partially ordered set and suppose there is a metric d on X such that $(X,d,\le)$ is regular. Suppose that $F: X^4 \to X$ and $g: X \to X$ are such that F has the mixed g-monotone property. Assume also that there exists $k \in [0,1)$ such that*

$$d\bigl(F(x,y,z,w),F(u,v,h,l)\bigr) \le \frac{k}{4}\bigl(\max\bigl\{d(gx,gu),d(gy,gv),d(gz,gh),d(gw,gl)\bigr\}\bigr)$$

for any $x,y,z,w,u,v,h,l \in X$, for which $gx \le gu$, $gv \ge gy$, $gz \le gh$, and $gl \ge gw$. Suppose that $F(X^4) \subset g(X)$, $(g(X),d)$ is a complete metric space. If there exist $x_0,y_0,z_0,w_0 \in X$ such that

$$\begin{aligned} &gx_0 \le F(x_0,y_0,z_0,w_0), \qquad gy_0 \ge F(y_0,z_0,w_0,x_0), \\ &gz_0 \le F(z_0,w_0,x_0,y_0), \qquad gw_0 \ge F(w_0,x_0,y_0,z_0), \end{aligned}$$

then there exist $x,y,z,w \in X$ such that

$$F(x,y,z,w) = gx, \qquad F(y,z,w,x) = gy, \qquad F(z,w,x,y) = gz \quad \text{and} \quad F(w,x,y,z) = gw,$$

that is, F and g have a quadruple coincidence point.

Remark 3.1 (i) Theorem 11 of Karapinar and Luong [21] is a particular case of Theorem 3.1 and Theorem 3.2 by taking $g = I_X$, respectively. Corollary 12 of Karapinar and

Luong [21] is a particular case of Theorem 3.1 and Theorem 3.2 by taking $g = I_X$, $\psi(t) = t$, $\phi(t) = \frac{1-k}{4}t$.

(ii) Theorem 2.3 of Karapinar [22] is a particular case of Theorem 3.1 and Theorem 3.2 by taking $g = I_X$ and $\psi(t) = t$, respectively. Corollary 2.4 of Karapinar [22] is a particular case of Theorem 3.1 and Theorem 3.2 by taking $g = I_X$, $\psi(t) = t$, $\phi(t) = \frac{1-k}{4}t$.

Now, we shall prove the existence and uniqueness of a quadruple common fixed point. For a product X^4 of a partial ordered set $(X, \leq)$, we define a partial ordering in the following way: For all $(x, y, z, w), (u, v, r, h) \in X^4$,

$$(x, y, z, w) \leq (u, v, r, h) \quad \Rightarrow \quad x \leq u, \qquad y \geq v, \qquad z \leq r \quad \text{and} \quad w \geq l. \tag{40}$$

We say that (x, y, z, w) and (u, v, r, l) are comparable if

$$(x, y, z, w) \leq (u, v, r, l) \quad \text{or} \quad (u, v, r, l) \geq (x, y, z, w).$$

Also, we say that (x, y, z, w) is equal to (u, v, r, l) if and only if $x = u$, $y = v$, $z = r$ and $w = l$.

Theorem 3.3 *In addition to the hypotheses of Theorem* 3.1, *suppose that for all* (x, y, z, w), $(u, v, r, l) \in X^4$, *there exists* $(a, b, c, d) \in X^4$ *such that*

$$\bigl(F(a, b, c, d), F(b, c, d, a), F(c, d, a, b), F(d, a, b, c)\bigr)$$

is comparable to

$$\bigl(F(x, y, z, w), F(y, z, w, x), F(z, w, x, y), F(w, x, y, z)\bigr)$$

and

$$\bigl(F(u, v, r, l), F(v, r, l, u), F(r, l, u, v), F(l, u, v, r)\bigr).$$

Then F and g have a unique quadruple common fixed point (x, y, z, w) *such that*

$$x = gx = F(x, y, z, w), \qquad y = gy = F(y, z, w, x),$$
$$z = gz = F(z, w, x, y), \qquad w = gw = F(w, x, y, z).$$

Proof The set of quadruple coincidence points of F and g is not empty due to Theorem 3.1. Assume now that (x, y, z, w) and (u, v, r, l) are two quadruple coincidence points of F and g, *i.e.*,

$$F(x, y, z, w) = gx, \qquad F(u, v, r, l) = gu,$$
$$F(y, z, w, x) = gy, \qquad F(v, r, l, u) = gv,$$
$$F(z, w, x, y) = gz, \qquad F(r, l, u, v) = gr,$$
$$F(w, x, y, z) = gw, \qquad F(l, u, v, r) = gl.$$

We shall show that (gx, gy, gz, gw) and (gu, gv, gr, gl) are equal. By assumption, there exists $(a, b, c, d) \in X^4$ such that $(F(a, b, c, d), F(b, c, d, a), F(c, d, a, b), F(d, a, b, c))$ is comparable to $(F(x,y,z,w),F(y,z,w,x),F(z,w,x,y),F(w,x,y,z))$ and $(F(u,v,r,l),F(v,r,l,u),F(r,l,u,v),F(l,u,v,r))$. Define sequences $\{ga_n\}$, $\{gb_n\}$, $\{gc_n\}$, and $\{gd_n\}$ such that $a_0 = a$, $b_0 = b$, $c_0 = c$, $d_0 = d$, and for any $n \geq 1$,

$$\begin{aligned} ga_n &= F(a_{n-1},b_{n-1},c_{n-1},d_{n-1}), \qquad gbn = F(b_{n-1},c_{n-1},d_{n-1},a_{n-1}), \\ gc_n &= F(c_{n-1},d_{n-1},a_{n-1},b_{n-1}), \qquad gd_n = F(d_{n-1},a_{n-1},b_{n-1},c_{n-1}) \end{aligned} \tag{41}$$

for all n. Further, set $x_0 = x$, $y_0 = y$, $z_0 = z$, $w_0 = w$ and $u_0 = u$, $v_0 = v$, $r_0 = r$, $l_0 = l$, and in the same way define the sequences $\{gx_n\}$, $\{gy_n\}$, $\{gz_n\}$, $\{gw_n\}$ and $\{gu_n\}$, $\{gv_n\}$, $\{gr_n\}$, $\{gl_n\}$. Then it is easy to see that

$$\begin{aligned} gx_n &= F(x_{n-1},y_{n-1},z_{n-1},w_{n-1}), \qquad gu_n = F(u_{n-1},v_{n-1},r_{n-1},l_{n-1}), \\ gy_n &= F(y_{n-1},z_{n-1},w_{n-1},x_{n-1}), \qquad gv_n = F(v_{n-1},r_{n-1},l_{n-1},u_{n-1}), \\ gz_n &= F(z_{n-1},w_{n-1},x_{n-1},y_{n-1}), \qquad gr_n = F(r_{n-1},l_{n-1},u_{n-1},v_{n-1}), \\ gw_n &= F(w_{n-1},x_{n-1},y_{n-1},z_{n-1}), \qquad gl_n = F(l_{n-1},u_{n-1},v_{n-1},r_{n-1}) \end{aligned} \tag{42}$$

for all $n \geq 1$.

Since $(F(x,y,z,w),F(y,z,w,x),F(z,w,x,y),F(w,x,y,z)) = (gx_1,gy_1,gz_1,gw_1) = (gx,gy,gz,gw)$ is comparable to

$$\bigl(F(a,b,c,d),F(b,c,d,a),F(c,d,a,b),F(d,a,b,c)\bigr) = (ga_1,gb_1,gc_1,gd_1),$$

then it is easy to show $(gx,gy,gz,gw) \geq (ga_1,gb_1,gc_1,gd_1)$. Recursively, we get that

$$(ga_n,gb_n,gc_n,gd_n) \leq (gx,gy,gz,gw) \quad \text{for all } n, \tag{43}$$

$$\begin{aligned} \psi\bigl(d(ga_{n+1},gx)\bigr) &= \frac{1}{4}\psi\bigl(d\bigl(F(a_n,b_n,c_n,d_n),F(x,y,z,w)\bigr)\bigr) \\ &\leq \psi\bigl(d(ga_n,gx)+d(gb_n,gy)+d(gc_n,gz)+d(gd_n,gw)\bigr) \\ &\quad - \phi\bigl(d(ga_n,gx)+d(gb_n,gy)+d(gc_n,gz)+d(gd_n,gw)\bigr), \end{aligned} \tag{44}$$

$$\begin{aligned} \psi\bigl(d(gy,gb_{n+1})\bigr) &= \frac{1}{4}\psi\bigl(d\bigl(F(y,z,w,x),F(b_n,c_n,d_n,a_n)\bigr)\bigr) \\ &\leq \psi\bigl(d(gy,gb_n)+d(gz,gc_n)+d(gw,gd_n)+d(gx,ga_n)\bigr) \\ &\quad - \phi\bigl(d(gy,gb_n)+d(gz,gc_n)+d(gw,gd_n)+d(gx,ga_n)\bigr), \end{aligned} \tag{45}$$

$$\begin{aligned} \psi\bigl(d(gc_{n+1},gz)\bigr) &= \frac{1}{4}\psi\bigl(d\bigl(F(c_n,d_n,a_n,b_n),F(z,w,x,y)\bigr)\bigr) \\ &\leq \psi\bigl(d(gc_n,gz)+d(gd_n,gw)+d(ga_n,gx)+d(gb_n,gy)\bigr) \\ &\quad - \phi\bigl(d(gc_n,gz)+d(gd_n,gw)+d(ga_n,gx)+d(gb_n,gy)\bigr) \end{aligned} \tag{46}$$

and

$$\begin{aligned} \psi\bigl(d(gw,gd_{n+1})\bigr) &= \frac{1}{4}d\bigl(F(w,x,y,z),F(d_n,a_n,b_n,c_n)\bigr) \\ &\leq \psi\bigl(d(gw,gd_n)+d(gx,ga_n)+d(gy,gb_n)+d(gz,gc_n)\bigr) \\ &\quad - \phi\bigl(d(gw,gd_n)+d(gx,ga_n)+d(gy,gb_n)+d(gz,gc_n)\bigr). \end{aligned} \tag{47}$$

From (44)-(47), it follows that

$$\begin{aligned}
&\psi\big(d(ga_{n+1},gx)\big)+\psi\big(d(gy,gb_{n+1})\big)+\psi\big(d(gc_{n+1},gz)\big)+\psi\big(d(gw,gd_{n+1})\big)\\
&\quad\le\psi\big(d(gw,gd_n)+d(gx,ga_n)+d(gy,gb_n)+d(gz,gc_n)\big)\\
&\qquad-4\phi\big(d(gw,gd_n)+d(gx,ga_n)+d(gy,gb_n)+d(gz,gc_n)\big).
\end{aligned}$$

By the property (iii) of ψ, we obtain that

$$\begin{aligned}
&\psi\big(d(ga_{n+1},gx)+d(gy,gb_{n+1})+d(gc_{n+1},gz)+d(gw,gd_{n+1})\big)\\
&\quad\le\psi\big(d(ga_{n+1},gx)\big)+\psi\big(d(gy,gb_{n+1})\big)+\psi\big(d(gc_{n+1},gz)\big)+\psi\big(d(gw,gd_{n+1})\big)\\
&\quad\le\psi\big(d(gw,gd_n)+d(gx,ga_n)+d(gy,gb_n)+d(gz,gc_n)\big)\\
&\qquad-4\phi\big(d(gw,gd_n)+d(gx,ga_n)+d(gy,gb_n)+d(gz,gc_n)\big).
\end{aligned}\tag{48}$$

Set $\sigma_n=d(ga_n,gx)+d(gy,gb_n)+d(gc_n,gz)+d(gw,gd_n)$. Then due to (48), we have

$$\psi(\sigma_{n+1})\le\psi(\sigma_n)-4\phi(\sigma_n)\quad\text{for all } n,\tag{49}$$

which implies that $\psi(\sigma_{n+1})\le\psi(\sigma_n)$. By the property of ψ, we obtain that $\sigma_{n+1}\le\sigma_n$. Thus, the sequence $\{\sigma_n\}$ is decreasing and bounded below from 0. Therefore, there exists $\sigma\ge 0$ such that

$$\lim_{n\to\infty}\sigma_n=\sigma.$$

Now, we shall show that $\sigma=0$. Suppose to the contrary that $\sigma>0$. Letting $n\to\infty$ in (49), we obtain that

$$\psi(\sigma)\le\psi(\sigma)-4\lim_{n\to\infty}\phi(\sigma_n)<\psi(\sigma),$$

which is a contradiction. It yields that $\sigma=0$. That is, $\lim_{n\to\infty}\sigma_n=0$.

Consequently, we have

$$\begin{aligned}
&\lim_{n\to\infty}d(ga_n,gx)=0, &&\lim_{n\to\infty}d(gy,gb_n)=0,\\
&\lim_{n\to\infty}d(gc_n,gz)=0, &&\lim_{n\to\infty}d(gw,gd_n)=0.
\end{aligned}\tag{50}$$

Similarly, we can prove that

$$\begin{aligned}
&\lim_{n\to\infty}d(ga_n,gu)=0, &&\lim_{n\to\infty}d(gv,gb_n)=0,\\
&\lim_{n\to\infty}d(gc_n,gr)=0, &&\lim_{n\to\infty}d(gl,gd_n)=0.
\end{aligned}\tag{51}$$

Combining (50) and (51) yields that (gx,gy,gz,gw) and (gu,gv,gr,gl) are equal.

Since $gx = F(x,y,z,w)$, $gy = F(y,z,w,x)$, $gz = F(z,w,x,y)$, and $gw = F(w,x,y,z)$, by the commutativity of F and g, we obtain that

$$\begin{aligned}
gx' &= g(gx) = g\bigl(F(x,y,z,w)\bigr) = F(gx,gy,gw,gz) = F\bigl(x',y',z',w'\bigr),\\
gy' &= g(gy) = g\bigl(F(y,z,w,x)\bigr) = F(gy,gz,gw,gx) = F\bigl(y',z',w',x'\bigr),\\
gz' &= g(gz) = g\bigl(F(z,w,x,y)\bigr) = F(gz,gw,gx,gy) = F\bigl(z',w',x',y'\bigr),\\
gw' &= g(gz) = g\bigl(F(w,x,y,z)\bigr) = F(gw,gx,gy,gz) = F\bigl(w',x',y',z'\bigr),
\end{aligned}$$

where $gx = x'$, $gy = y'$, $gz = z'$, and $gw = w'$. Thus, (x',y',z',w') is a quadruple coincidence point of F and g. Therefore, (gx',gy',gz',gw') and (gx,gy,gz,gw) are equal. We obtain that

$$gx' = gx = x', \qquad gy' = gy = y', \qquad gz' = gz = z', \qquad gw' = gw = w'.$$

Thus, (x',y',z',w') is a quadruple common fixed point of F and g. Its uniqueness follows from contraction (1). □

Example 3.1 Let $X = \mathbb{R}$ with the metric $d(x,y) = |x-y|$ for all $x,y \in X$ and the usual ordering. Let $F : X^4 \to X$ and $g : X \to X$ be given by

$$g(x) = \frac{3}{4}x, \qquad F(x,y,z,w) = \frac{x-y+z-w}{16} \quad \text{for all } x,y,z,w \in X.$$

Let $\psi,\phi : [0,\infty) \to [0,\infty)$ be given by

$$\psi(t) = \frac{1}{5}t \quad \text{and} \quad \phi(t) = \frac{t}{60} \quad \text{for all } t \in [0,\infty).$$

We will check that the condition (1) is satisfied for all $x,y,z,w,u,v,h,l \in X$ satisfying $gx \le gu$, $gv \le gy$, $gz \le gh$, $gl \le gw$. In this case, we have

$$\begin{aligned}
\psi\bigl(d\bigl(F(x,y,z,w),F(u,v,h,l)\bigr)\bigr) &= \frac{1}{5}\left[\frac{u-x}{16}+\frac{y-v}{16}+\frac{h-z}{16}+\frac{w-l}{16}\right]\\
&= \frac{1}{4}\left\{\frac{1}{5}\left[\frac{3(u-x)}{4}+\frac{3(y-v)}{4}+\frac{3(h-z)}{4}+\frac{3(w-l)}{4}\right]\right\}\\
&\quad - \frac{1}{30}\left[\frac{3(u-x)}{4}+\frac{3(y-v)}{4}+\frac{3(h-z)}{4}+\frac{3(w-l)}{4}\right]\\
&\le \frac{1}{4}\left\{\frac{1}{5}\left[\frac{3(u-x)}{4}+\frac{3(y-v)}{4}+\frac{3(h-z)}{4}+\frac{3(w-l)}{4}\right]\right\}\\
&\quad - \frac{1}{60}\left[\frac{3(u-x)}{4}+\frac{3(y-v)}{4}+\frac{3(h-z)}{4}+\frac{3(w-l)}{4}\right]\\
&= \frac{1}{4}\psi\bigl[d(gx,gu)+d(gy,gv)+d(gz,gh)+d(gw,gl)\bigr]\\
&\quad - \phi\bigl[d(gx,gu)+d(gy,gv)+d(gz,gh)+d(gw,gl)\bigr].
\end{aligned}$$

It is easy to check that all the conditions of Theorem 3.3 are satisfied and $(0,0,0,0)$ is the unique quadruple fixed point of F and g.

Competing interests
The author declares that they have no competing interests.

Acknowledgements
This work is partially supported by the Scientific Research Fund of Sichuan Provincial Education Department (12ZA098), Scientific Research Fund of Sichuan University of Science and Engineering (2012KY08).

Received: 31 October 2012 Accepted: 20 May 2013 Published: 5 June 2013

References
1. Guo, D, Lakshmikantham, V: Couple fixed points of nonlinear operators with applications. Nonlinear Anal. **11**, 623-632 (1987)
2. Bhaskar, TG, Lakshmikantham, V: Fixed point theory in partially ordered spaces and applications. Nonlinear Anal. **65**, 1379-1393 (2006)
3. Lakshmikantham, V, Ćirić, L: Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. Nonlinear Anal. **70**, 4341-4349 (2009)
4. Luong, NV, Thuan, NX: Coupled fixed points in partially ordered metric spaces and application. Nonlinear Anal. **74**, 983-992 (2011)
5. Samet, B: Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces. Nonlinear Anal. **74**(12), 4508-4517 (2010)
6. Karapinar, E: Coupled fixed point theorems for nonlinear contractions in cone metric spaces. Comput. Math. Appl. **59**(12), 3656-3668 (2010)
7. Karapinar, E: Couple fixed point on cone metric spaces. Gazi Univ. J. Sci. **24**(1), 51-58 (2011)
8. Choudhury, BS, Kundu, A: A coupled coincidence point result in partially ordered metric spaces for compatible mappings. Nonlinear Anal. **73**, 2524-2531 (2010)
9. Choudhury, BS, Metiya, N, Kundu, A: Coupled coincidence point theorems in ordered metric spaces. Ann. Univ. Ferrara **57**, 1-16 (2011)
10. Abbas, M, Khan, MA, Radenovi, S: Common coupled fixed point theorems in cone metric space for *w*-compatible mappings. Appl. Math. Comput. **217**, 195-203 (2010)
11. Nashine, HK, Shatanawi, W: Coupled common fixed point theorems for pair of commuting mappings in partially ordered complete metric spaces. Comput. Math. Appl. (2011). doi:10.1016/j.camwa.2011.06.042
12. Sabetghadam, F, Masiha, HP, Sanatpour, AH: Some coupled fixed point theorems in cone metric spaces. Fixed Point Theory Appl. **2009**, Article ID 125426 (2009)
13. Shatanawi, W: Partially ordered cone metric spaces and coupled fixed point results. Comput. Math. Appl. **60**, 2508-2515 (2010)
14. Shatanawi, W, Samet, B, Abbas, M: Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces. Math. Comput. Model. (2011). doi:10.1016/j.mcm.2011.08.042
15. Shatanawi, W: Some common coupled fixed point results in cone metric spaces. Int. J. Math. Anal. **4**, 2381-2388 (2010)
16. Shatanawi, W: Fixed point theorems for nonlinear weakly *C*-contractive mappings in metric spaces. Math. Comput. Model. **54**, 2816-2826 (2011)
17. Nieto, JJ, Rodriguez-Lopez, R: Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. Order **22**, 223-239 (2005)
18. Berinde, V, Borcut, M: Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces. Nonlinear Anal. **74**(15), 4889-4897 (2011)
19. Samet, B, Vetro, C: Coupled fixed point, f-invariant set and fixed point of N-order. Ann. Funct. Anal. **1**(2), 46-56 (2010)
20. Karapinar, E: Quartet fixed point for nonlinear contraction. http://arxiv.org/abs/1106.5472
21. Karapinar, E, Luong, NV: Quadruple fixed point theorems for nonlinear contractions. Comput. Math. Appl. **64**, 1839-1848 (2012)
22. Karapinar, E: Quadruple fixed point theorems for weak ϕ-contractions. ISRN Math. Anal. **2011**, Article ID 989423 (2011)
23. Karapinar, E, Berinde, V: Quadruple fixed point theorems for nonlinear contractions in partially ordered metric spaces. Banach J. Math. Anal. **6**(1), 74-89 (2012)
24. Karapinar, E: A new quartet fixed point theorem for nonlinear contractions. JP J. Fixed Point Theory Appl. **6**(2), 119-135 (2011)
25. Karapinar, E, Shatanawi, W, Mustafa, Z: Quadruple fixed point theorems under nonlinear contractive conditions in partially ordered metric spaces. J. Appl. Math. **2012**, Article ID 951912 (2012)

doi:10.1186/1687-1812-2013-147
Cite this article as: Liu: **Quadruple fixed point theorems in partially ordered metric spaces with mixed g-monotone property.** *Fixed Point Theory and Applications* 2013 **2013**:147.