Naraghirad and Yao *Fixed Point Theory and Applications* 2013, **2013**:141
http://www.fixedpointtheoryandapplications.com/content/2013/1/141

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Bregman weak relatively nonexpansive mappings in Banach spaces

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Abstract

In this paper, we introduce a new class of mappings called Bregman weak relatively nonexpansive mappings and propose new hybrid iterative algorithms for finding common fixed points of an infinite family of such mappings in Banach spaces. We prove strong convergence theorems for the sequences produced by the methods. Furthermore, we apply our method to prove strong convergence theorems of iterative algorithms for finding common fixed points of finitely many Bregman weak relatively nonexpansive mappings in reflexive Banach spaces. These algorithms take into account possible computational errors. We also apply our main results to solve equilibrium problems in reflexive Banach spaces. Finally, we study hybrid iterative schemes for finding common solutions of an equilibrium problem, fixed points of an infinite family of Bregman weak relatively nonexpansive mappings and null spaces of a γ-inverse strongly monotone mapping in 2-uniformly convex Banach spaces. Some application of our results to the solution of equations of Hammerstein-type is presented. Our results improve and generalize many known results in the current literature.

MSC: 47H10; 37C25

Keywords: Bregman function; uniformly convex function; uniformly smooth function; fixed point; strong convergence; Bregman weak relatively nonexpansive mapping

1 Introduction

The hybrid projection method was first introduced by Hangazeau in [1]. In a series of papers [2–12], authors investigated the hybrid projection method and proved strong and weak convergence theorems for the sequences produced by their method. The shrinking projection method, which is a generalization of the hybrid projection method, was first introduced by Takahashi *et al.* in [13]. Throughout this paper, we denote the set of real numbers and the set of positive integers by $\mathbb{R}$ and $\mathbb{N}$, respectively. Let E be a Banach space with the norm $\|\cdot\|$ and the dual space E^*. For any $x \in E$, we denote the value of $x^* \in E^*$ at x by $\langle x, x^*\rangle$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in E. We denote the strong convergence of $\{x_n\}_{n\in\mathbb{N}}$ to $x \in E$ as $n \to \infty$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is denoted by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon\right\}$$

for every ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. Let $S_E = \{x \in E : \|x\| = 1\}$. The norm of E is said to be *Gâteaux differentiable* if for each $x, y \in S_E$, the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.1}$$

exists. In this case, E is called *smooth*. If the limit (1.1) is attained uniformly for all $x, y \in S_E$, then E is called *uniformly smooth*. The Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ whenever $x, y \in S_E$ and $x \neq y$. It is well known that E is uniformly convex if and only if E^* is uniformly smooth. It is also known that if E is reflexive, then E is strictly convex if and only if E^* is smooth; for more details, see [14, 15].

Let C be a nonempty subset of E. Let $T : C \to E$ be a mapping. We denote the set of fixed points of T by $F(T)$, *i.e.*, $F(T) = \{x \in C : Tx = x\}$. A mapping $T : C \to E$ is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to E$ is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - y\| \le \|x - y\|$ for all $x \in C$ and $y \in F(T)$. The concept of nonexpansivity plays an important role in the study of Mann-type iteration [16] for finding fixed points of a mapping $T : C \to C$. Recall that the Mann-type iteration is given by the following formula:

$$x_{n+1} = \gamma_n Tx_n + (1 - \gamma_n)x_n, \quad x_1 \in C. \tag{1.2}$$

Here, $\{\gamma_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers in $[0, 1]$ satisfying some appropriate conditions. The construction of fixed points of nonexpansive mappings via Mann's algorithm [16] has been extensively investigated recently in the current literature (see, for example, [17] and the references therein). In [17], Reich proved the following interesting result.

Theorem 1.1 *Let C be a closed and convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm, let $T : C \to C$ be a nonexpansive mapping with a fixed point, and let γ_n be a sequence of real numbers such that $\gamma_n \in [0, 1]$ and $\sum_{n=1}^{\infty} \gamma_n(1 - \gamma_n) = \infty$. Then the sequence $\{x_n\}_{n\in\mathbb{N}}$ generated by Mann's algorithm* (1.2) *converges weakly to a fixed point of T.*

However, the convergence of the sequence $\{x_n\}_{n\in\mathbb{N}}$ generated by Mann's algorithm (1.2) is in general not strong (see a counterexample in [18]; see also [19]). Some attempts to modify the Mann iteration method (1.2) so that strong convergence is guaranteed have recently been made. Bauschke and Combettes [4] proposed the following modification of the Mann iteration method for a single nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 = x \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C_n : \|z - y_n\| \le \|z - x_n\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n\cap Q_n} x, \end{cases} \tag{1.3}$$

where C is a closed and convex subset of H, P_Q denotes the metric projection from H onto a closed and convex subset Q of H. They proved that if the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ is bounded above from one, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ generated by (1.3) converges strongly to $P_{F(T)}x$ as $n \to \infty$.

Let E be a smooth, strictly convex and reflexive Banach space and let J be a normalized duality mapping of E. Let C be a nonempty, closed and convex subset of E. The generalized projection Π_C from E onto C [20] is defined and denoted by

$$\Pi_C(x) = \arg\min_{y\in C} \phi(y,x),$$

where $\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$. Let C be a nonempty, closed and convex subset of a smooth Banach space E, let T be a mapping from C into itself. A point $p \in C$ is said to be an *asymptotic fixed point* [21] of T if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in C which converges weakly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote the set of all asymptotic fixed points of T by $\hat{F}(T)$. A point $p \in C$ is called a *strong asymptotic fixed point* of T if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in C which converges strongly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote the set of all strong asymptotic fixed points of T by $\tilde{F}(T)$.

Following Matsushita and Takahashi [22], a mapping $T : C \to C$ is said to be *relatively nonexpansive* if the following conditions are satisfied:

(1) $F(T)$ is nonempty;
(2) $\phi(u, Tx) \le \phi(u,x)$, $\forall u \in F(T)$, $x \in C$;
(3) $\hat{F}(T) = F(T)$.

In 2005, Matsushita and Takahashi [22] proved the following strong convergence theorem for relatively nonexpansive mappings in a Banach space.

Theorem 1.2 *Let E be a uniformly smooth and uniformly convex Banach space, let C be a nonempty, closed and convex subset of E, let T be a relatively nonexpansive mapping from C into itself, and let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $0 \le \alpha_n < 1$ and $\limsup_{n\to\infty} \alpha_n < 1$. Suppose that $\{x_n\}_{n\in\mathbb{N}}$ is given by*

$$\begin{cases} x_0 = x \in C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ H_n = \{z \in C_n : \phi(z,y_n) \le \phi(z,x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{H_n\cap W_n}x. \end{cases} \tag{1.4}$$

If $F(T)$ is nonempty, then $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\Pi_{F(T)}x$.

1.1 Some facts about gradient

For any convex function $g : E \to (-\infty, +\infty]$ we denote the domain of g by $\operatorname{dom} g = \{x \in E : g(x) < \infty\}$. For any $x \in \operatorname{int}\operatorname{dom} g$ and any $y \in E$, we denote by $g^0(x,y)$ the *right-hand derivative* of g at x in the direction y, that is,

$$g^0(x,y) = \lim_{t\downarrow 0} \frac{g(x+ty) - g(x)}{t}. \tag{1.5}$$

The function g is said to be *Gâteaux differentiable* at x if $\lim_{t\to 0} \frac{g(x+ty)-g(x)}{t}$ exists for any y. In this case, $g^0(x,y)$ coincides with $\nabla g(x)$, the value of the *gradient* ∇g of g at x. The function g is said to be *Gâteaux differentiable* if it is Gâteaux differentiable everywhere. The function g is said to be *Fréchet differentiable* at x if this limit is attained uniformly in $\|y\| = 1$. The function g is Fréchet differentiable at $x \in E$ (see, for example, [23, p.13] or [24, p.508]) if for all $\epsilon > 0$, there exists $\delta > 0$ such that $\|y - x\| \leq \delta$ implies that

$$\bigl|g(y) - g(x) - \bigl\langle y - x, \nabla g(x)\bigr\rangle\bigr| \leq \epsilon \|y - x\|.$$

The function g is said to be *Fréchet differentiable* if it is Fréchet differentiable everywhere. It is well known that if a continuous convex function $g : E \to \mathbb{R}$ is Gâteaux differentiable, then ∇g is norm-to-weak* continuous (see, for example, [23, Proposition 1.1.10]). Also, it is known that if g is Fréchet differentiable, then ∇g is norm-to-norm continuous (see [24, p.508]). The mapping ∇g is said to be *weakly sequentially continuous* if $x_n \rightharpoonup x$ as $n \to \infty$ implies that $\nabla g(x_n) \rightharpoonup^* \nabla g(x)$ as $n \to \infty$ (for more details, see [23, Theorem 3.2.4] or [24, p.508]). The function g is said to be *strongly coercive* if

$$\lim_{\|x_n\| \to \infty} \frac{g(x_n)}{\|x_n\|} = \infty.$$

It is also said to be *bounded on bounded subsets of* E if $g(U)$ is bounded for each bounded subset U of E. Finally, g is said to be *uniformly Fréchet differentiable* on a subset X of E if the limit (1.5) is attained uniformly for all $x \in X$ and $\|y\| = 1$.

Let $A : E \to 2^{E^*}$ be a set-valued mapping. We define the domain and range of A by $\operatorname{dom} A = \{x \in E : Ax \neq \varnothing\}$ and $\operatorname{ran} A = \bigcup_{x \in E} Ax$, respectively. The graph of A is denoted by $G(A) = \{(x, x^*) \in E \times E^* : x^* \in Ax\}$. The mapping $A \subset E \times E^*$ is said to be *monotone* [25] if $\langle x - y, x^* - y^*\rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in A$. It is also said to be *maximal monotone* [26] if its graph is not contained in the graph of any other monotone operator on E. If $A \subset E \times E^*$ is maximal monotone, then we can show that the set $A^{-1}0 = \{z \in E : 0 \in Az\}$ is closed and convex. A mapping $A : \operatorname{dom} A \subset E \to E^*$ is called γ-inverse strongly monotone if there exists a positive real number γ such that for all $x, y \in \operatorname{dom} A$, $\langle x - y, Ax - Ay\rangle \geq \gamma \|Ax - Ay\|^2$.

1.2 Some facts about Legendre functions

Let E be a reflexive Banach space. For any proper, lower semicontinuous and convex function $g : E \to (-\infty, +\infty]$, the *conjugate function* g^* of g is defined by

$$g^*\bigl(x^*\bigr) = \sup_{x \in E}\bigl\{\bigl\langle x, x^*\bigr\rangle - g(x)\bigr\}$$

for all $x^* \in E^*$. It is well known that $g(x) + g^*(x^*) \geq \langle x, x^*\rangle$ for all $(x, x^*) \in E \times E^*$. It is also known that $(x, x^*) \in \partial g$ is equivalent to

$$g(x) + g^*\bigl(x^*\bigr) = \bigl\langle x, x^*\bigr\rangle. \tag{1.6}$$

Here, ∂g is the subdifferential of g [27, 28]. We also know that if $g : E \to (-\infty, +\infty]$ is a proper, lower semicontinuous and convex function, then $g^* : E^* \to (-\infty, +\infty]$ is a proper, weak* lower semicontinuous and convex function; see [15] for more details on convex analysis.

Let $g : E \to (-\infty, +\infty]$ be a mapping. The function g is said to be:

(i) *essentially smooth*, if ∂g is both locally bounded and single-valued on its domain;

(ii) *essentially strictly convex*, if $(\partial g)^{-1}$ is locally bounded on its domain and g is strictly convex on every convex subset of $\operatorname{dom} \partial g$;

(iii) *Legendre*, if it is both essentially smooth and essentially strictly convex (for more details, we refer to [29, Definition 5.2]).

If E is a reflexive Banach space and $g : E \to (-\infty, +\infty]$ is a Legendre function, then in view of [30, p.83],

$$\nabla g^* = (\nabla g)^{-1}, \qquad \operatorname{ran} \nabla g = \operatorname{dom} g^* = \operatorname{int} \operatorname{dom} g^* \quad \text{and} \quad \operatorname{ran} \nabla g = \operatorname{int} \operatorname{dom} g.$$

Examples of Legendre functions are given in [29, 31]. One important and interesting Legendre function is $\frac{1}{s}\| \cdot \|^s$ $(1 < s < \infty)$, where the Banach space E is smooth and strictly convex and, in particular, a Hilbert space.

1.3 Some facts about Bregman distance

Let E be a Banach space and let E^* be the dual space of E. Let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Then the *Bregman distance* [32, 33] corresponding to g is the function $D_g : E \times E \to \mathbb{R}$ defined by

$$D_g(x, y) = g(x) - g(y) - \langle x - y, \nabla g(y) \rangle, \quad \forall x, y \in E. \tag{1.7}$$

It is clear that $D_g(x, y) \geq 0$ for all $x, y \in E$. In that case when E is a smooth Banach space, setting $g(x) = \|x\|^2$ for all $x \in E$, we obtain that $\nabla g(x) = 2Jx$ for all $x \in E$ and hence $D_g(x, y) = \phi(x, y)$ for all $x, y \in E$.

Let E be a Banach space and let C be a nonempty and convex subset of E. Let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Then we know from [34] that for $x \in E$ and $x_0 \in C$, $D_g(x_0, x) = \min_{y \in C} D_g(y, x)$ if and only if

$$\langle y - x_0, \nabla g(x) - \nabla g(x_0) \rangle \leq 0, \quad \forall y \in C. \tag{1.8}$$

Furthermore, if C is a nonempty, closed and convex subset of a reflexive Banach space E and $g : E \to \mathbb{R}$ is a strongly coercive Bregman function, then for each $x \in E$, there exists a unique $x_0 \in C$ such that

$$D_g(x_0, x) = \min_{y \in C} D_g(y, x).$$

The *Bregman projection* proj_C^g from E onto C is defined by $\operatorname{proj}_C^g(x) = x_0$ for all $x \in E$. It is also well known that proj_C^g has the following property:

$$D_g\bigl(y, \operatorname{proj}_C^g x\bigr) + D_g\bigl(\operatorname{proj}_C^g x, x\bigr) \leq D_g(y, x) \tag{1.9}$$

for all $y \in C$ and $x \in E$ (see [23] for more details).

1.4 Some facts about uniformly convex and totally convex functions

Let E be a Banach space and let $B_r := \{z \in E : \|z\| \leq r\}$ for all $r > 0$. Then a function $g : E \to \mathbb{R}$ is said to be *uniformly convex on bounded subsets of* E [35, pp.203, 221] if $\rho_r(t) > 0$ for all $r, t > 0$, where $\rho_r : [0, +\infty) \to [0, \infty]$ is defined by

$$\rho_r(t) = \inf_{x,y \in B_r, \|x-y\|=t, \alpha \in (0,1)} \frac{\alpha g(x) + (1-\alpha)g(y) - g(\alpha x + (1-\alpha)y)}{\alpha(1-\alpha)}$$

for all $t \geq 0$. The function ρ_r is called the gauge of uniform convexity of g. The function g is also said to be *uniformly smooth on bounded subsets of* E [35, pp.207, 221] if $\lim_{t \downarrow 0} \frac{\sigma_r(t)}{t} = 0$ for all $r > 0$, where $\sigma_r : [0, +\infty) \to [0, \infty]$ is defined by

$$\sigma_r(t) = \sup_{x \in B_r, y \in S_E, \alpha \in (0,1)} \frac{\alpha g(x + (1-\alpha)ty) + (1-\alpha)g(x - \alpha ty) - g(x)}{\alpha(1-\alpha)}$$

for all $t \geq 0$. The function g is said to be *uniformly convex* if the function $\delta_g : [0, +\infty) \to [0, +\infty]$, defined by

$$\delta_g(t) := \sup\left\{\frac{1}{2}g(x) + \frac{1}{2}g(y) - g\left(\frac{x+y}{2}\right) : \|y - x\| = t\right\},$$

satisfies that $\lim_{t \downarrow 0} \frac{\sigma_r(t)}{t} = 0$.

Remark 1.1 Let E be a Banach space, let $r > 0$ be a constant and let $g : E \to \mathbb{R}$ be a convex function which is uniformly convex on bounded subsets. Then

$$g\big(\alpha x + (1-\alpha)y\big) \leq \alpha g(x) + (1-\alpha)g(y) - \alpha(1-\alpha)\rho_r\big(\|x - y\|\big)$$

for all $x, y \in B_r$ and $\alpha \in (0, 1)$, where ρ_r is the gauge of uniform convexity of g.

Let $g : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. Recall that, in view of [23, Section 1.2, p.17] (see also [36]), the function g is called *totally convex* at a point $x \in \operatorname{int}\operatorname{dom} g$ if its *modulus of total convexity* at x, that is, the function $\nu_g : \operatorname{int}\operatorname{dom} g \times [0, +\infty) \to [0, +\infty)$, defined by

$$\nu_g(x, t) := \inf\big\{D_g(y, x) : y \in \operatorname{int}\operatorname{dom} g, \|y - x\| = t\big\},$$

is positive whenever $t > 0$. The function g is called *totally convex* when it is *totally convex* at every point $x \in \operatorname{int}\operatorname{dom} g$. Moreover, the function f is called *totally convex on bounded subsets of* E if $\nu_g(x, t) > 0$ for any bounded subset X of E and for any $t > 0$, where the *modulus of total convexity of the function* g on the set X is the function $\nu_g : \operatorname{int}\operatorname{dom} g \times [0, +\infty) \to [0, +\infty)$ defined by

$$\nu_g(X, t) := \inf\big\{\nu_g(x, t) : x \in X \cap \operatorname{int}\operatorname{dom} g\big\}.$$

It is well known that any uniformly convex function is totally convex, but the converse is not true in general (see [23, Section 1.3, p.30]).

It is also well known that g is totally convex on bounded subsets if and only if g is uniformly convex on bounded subsets (see [37, Theorem 2.10, p.9]).

Examples of totally convex functions can be found, for instance, in [23, 37].

1.5 Some facts about resolvent

Let E be a reflexive Banach space with the dual space E^* and let $g: E \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function. Let A be a maximal monotone operator from E to E^*. For any $r > 0$, let the mapping $\mathrm{Res}_{rA}^g : E \to \operatorname{dom} A$ be defined by

$$\mathrm{Res}_{rA}^g = (\nabla g + rA)^{-1}\nabla g.$$

The mapping Res_{rA}^g is called the *g-resolvent* of A (see [38]). It is well known that $A^{-1}(0) = F(\mathrm{Res}_{rA}^g)$ for each $r > 0$ (for more details, see, for example, [14]).

Examples and some important properties of such operators are discussed in [39].

1.6 Some facts about Bregman quasi-nonexpansive mappings

Let C be a nonempty, closed and convex subset of a reflexive Banach space E. Let $g: E \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function. Recall that a mapping $T: C \to C$ is said to be *Bregman quasi-nonexpansive* [40] if $F(T) \neq \varnothing$ and

$$D_g(p, Tx) \le D_g(p, x), \quad \forall x \in C, p \in F(T).$$

A mapping $T: C \to C$ is said to be *Bregman relatively nonexpansive* [40] if the following conditions are satisfied:

(1) $F(T)$ is nonempty;
(2) $D_g(p, Tv) \le D_g(p, v)$, $\forall p \in F(T)$, $v \in C$;
(3) $\hat{F}(T) = F(T)$.

Now, we are in a position to introduce the following new class of Bregman quasi-nonexpansive type mappings. A mapping $T: C \to C$ is said to be *Bregman weak relatively nonexpansive* if the following conditions are satisfied:

(1) $F(T)$ is nonempty;
(2) $D_g(p, Tv) \le D_g(p, v)$, $\forall p \in F(T)$, $v \in C$;
(3) $\tilde{F}(T) = F(T)$.

It is clear that any Bregman relatively nonexpansive mapping is a Bregman quasi-nonexpansive mapping. It is also obvious that every Bregman relatively nonexpansive mapping is a Bregman weak relatively nonexpansive mapping, but the converse in not true in general. Indeed, for any mapping $T: C \to C$, we have $F(T) \subset \tilde{F}(T) \subset \hat{F}(T)$. If T is Bregman relatively nonexpansive, then $F(T) = \tilde{F}(T) = \hat{F}(T)$. Below we show that there exists a Bregman weak relatively nonexpansive mapping which is not a Bregman relatively nonexpansive mapping.

Example 1.1 Let $E = l^2$, where

$$l^2 = \left\{\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots) : \sum_{n=1}^{\infty} \|\sigma_n\|^2 < \infty\right\}, \qquad \|\sigma\| = \left(\sum_{n=1}^{\infty} \|\sigma_n\|^2\right)^{\frac{1}{2}}, \quad \forall \sigma \in l^2,$$

$$\langle \sigma, \eta \rangle = \sum_{n=1}^{\infty} \sigma_n \eta_n, \quad \forall \delta = (\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots), \eta = (\eta_1, \eta_2, \ldots, \eta_n, \ldots) \in l^2.$$

Let $\{x_n\}_{n\in\mathbb{N}\cup\{0\}} \subset E$ be a sequence defined by

$$x_0 = (1,0,0,0,\ldots),$$
$$x_1 = (1,1,0,0,0,\ldots),$$
$$x_2 = (1,0,1,0,0,0,\ldots),$$
$$x_3 = (1,0,0,1,0,0,0,\ldots),$$
$$\cdots,$$
$$x_n = (\sigma_{n,1},\sigma_{n,2},\ldots,\sigma_{n,k},\ldots),$$
$$\cdots,$$

where

$$\sigma_{n,k} = \begin{cases} 1 & \text{if } k = 1, n+1, \\ 0 & \text{if } k \neq 1, k \neq n+1 \end{cases}$$

for all $n \in \mathbb{N}$. It is clear that the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to x_0. Indeed, for any $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots) \in l^2 = (l^2)^*$, we have

$$\Lambda(x_n - x_0) = \langle x_n - x_0, \Lambda\rangle = \sum_{k=2}^{\infty} \lambda_k \sigma_{n,k} \to 0$$

as $n \to \infty$. It is also obvious that $\|x_n - x_m\| = \sqrt{2}$ for any $n \neq m$ with n, m sufficiently large. Thus, $\{x_n\}_{n\in\mathbb{N}}$ is not a Cauchy sequence. Let k be an even number in $\mathbb{N}$ and let $g : E \to \mathbb{R}$ be defined by

$$g(x) = \frac{1}{k}\|x\|^k, \quad x \in E.$$

It is easy to show that $\nabla g(x) = J_k(x)$ for all $x \in E$, where

$$J_k(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\| \|x^*\|, \|x^*\| = \|x\|^{k-1}\}.$$

It is also obvious that

$$J_k(\lambda x) = \lambda^{k-1} J_k(x), \quad \forall x \in E, \lambda \in \mathbb{R}.$$

Now, we define a mapping $T : E \to E$ by

$$T(x) = \begin{cases} \frac{n}{n+1}x & \text{if } x = x_n; \\ -x & \text{if } x \neq x_n. \end{cases}$$

It is clear that $F(T) = \{0\}$ and for any $n \in \mathbb{N}$,

$$\begin{aligned} D_g(0, Tx_n) &= g(0) - g(Tx_n) - \langle 0 - Tx_n, \nabla g(Tx_n)\rangle \\ &= -\frac{n^k}{(n+1)^k} g(x_n) + \frac{n^k}{(n+1)^k} \langle x_n, \nabla g(x_n)\rangle \end{aligned}$$

$$
\begin{aligned}
&= \frac{n^k}{(n+1)^k}\bigl[-g(x_n) + \langle x, \nabla g(x_n)\rangle\bigr] \\
&= \frac{n^k}{(n+1)^k} D_g(0, x_n) \\
&\leq D_g(0, x_n).
\end{aligned}
$$

If $x \neq x_n$, then we have

$$
\begin{aligned}
D_g(0, Tx) &= g(0) - g(Tx) - \langle 0 - Tx, \nabla g(Tx)\rangle \\
&= -g(x) - \langle x, -\nabla g(x)\rangle \\
&= -g(x) - \langle -x, \nabla g(x)\rangle \\
&= D_g(0, x).
\end{aligned}
$$

Therefore, T is a Bregman quasi-nonexpansive mapping. Next, we claim that T is a Bregman weak relatively nonexpansive mapping. Indeed, for any sequence $\{z_n\}_{n\in\mathbb{N}} \subset E$ such that $z_n \to z_0$ and $\|z_n - Tz_n\| \to 0$ as $n \to \infty$, since $\{x_n\}_{n\in\mathbb{N}}$ is not a Cauchy sequence, there exists a sufficiently large number $N \in \mathbb{N}$ such that $z_n \neq x_m$ for any $n, m > N$. If we suppose that there exists $m \leq N$ such that $z_n = x_m$ for infinitely many $n \in \mathbb{N}$, then a subsequence $\{x_{n_i}\}_{i\in\mathbb{N}}$ would satisfy $z_{n_i} = x_m$, so $z_0 = \lim_{i\to\infty} z_{n_i} = x_m$ and $z_0 = \lim_{i\to\infty} Tz_{n_i} = Tx_m = \frac{m}{m+1}x_m$, which is impossible. This implies that $Tz_n = -z_n$ for all $n > N$. It follows from $\|z_n - Tz_n\| \to 0$ that $2z_n \to 0$ and hence $z_n \to z_0 = 0$. Since $z_0 \in F(T)$, we conclude that T is a Bregman weak relatively nonexpansive mapping.

Finally, we show that T is not Bregman relatively nonexpansive. In fact, though $x_n \rightharpoonup x_0$ and

$$
\|x_n - Tx_n\| = \left\| x_n - \frac{n}{n+1}x_n \right\| = \frac{1}{n+1}\|x_n\| \to 0
$$

as $n \to \infty$, but $x_0 \notin F(T)$. Thus we have $\hat{F}(T) \neq F(T)$.

Let us give an example of a Bregman quasi-nonexpansive mapping which is neither a Bregman relatively nonexpansive mapping nor a Bregman weak relatively nonexpansive mapping (see also [41]).

Example 1.2 Let E be a smooth Banach space, let k be an even number in $\mathbb{N}$ and let $g : E \to \mathbb{R}$ be defined by

$$
g(x) = \frac{1}{k}\|x\|^k, \quad x \in E.
$$

Let $x_0 \neq 0$ be any element of E. We define a mapping $T : E \to E$ by

$$
T(x) = \begin{cases} (\frac{1}{2} + \frac{1}{2^{n+1}})x_0 & \text{if } x = (\frac{1}{2} + \frac{1}{2^n})x_0; \\ -x & \text{if } x \neq (\frac{1}{2} + \frac{1}{2^n})x_0 \end{cases}
$$

for all $n \geq 0$. It could easily be seen that T is neither a Bregman weak relatively nonexpansive mapping nor a Bregman relatively nonexpansive mapping. To this end, we set

$$x_n = \left(\frac{1}{2} + \frac{1}{2^n}\right)x_0, \quad \forall n \in \mathbb{N}.$$

Though $x_n \to \frac{1}{2}x_0$ ($x_n \rightharpoonup \frac{1}{2}x_0$) as $n \to \infty$ and

$$\|x_n - Tx_n\| = \left\|\left(\frac{1}{2} + \frac{1}{2^n}\right)x_0 - \left(\frac{1}{2} + \frac{1}{2^{n+1}}\right)x_0\right\| = \frac{1}{2^{n-1}}\|x_0\| \to 0$$

as $n \to \infty$, but $\frac{1}{2}x_0 \notin F(T)$. Therefore, $\hat{F}(T) \neq F(T)$ and $\tilde{F}(T) \neq F(T)$.

In [42], Bauschke and Combettes introduced an iterative method to construct the Bregman projection of a point onto a countable intersection of closed and convex sets in reflexive Banach spaces. They proved a strong convergence theorem of the sequence produced by their method; for more detail, see [42, Theorem 4.7].

In [40], Reich and Sabach introduced a proximal method for finding common zeros of finitely many maximal monotone operators in a reflexive Banach space. More precisely, they proved the following strong convergence theorem.

Theorem 1.3 *Let E be a reflexive Banach space and let $A_i : E \to 2^{E^*}$, $i = 1, 2, \dots, N$, be N maximal monotone operators such that $Z := \bigcap_{i=1}^N A_i^{-1}(0^*) \neq \emptyset$. Let $g : E \to \mathbb{R}$ be a Legendre function that is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence defined by the following iterative algorithm*:

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ y_n^i = \operatorname{Res}^g_{\lambda_n^i A_i}(x_n + e_n^i), \\ C_n^i = \{z \in E : D_g(z, y_n^i) \leq D_g(z, x_n + e_n^i)\}, \\ C_n := \bigcap_{i=1}^N C_n^i, \\ Q_n = \{z \in E : \langle \nabla g(x_0) - \nabla g(x_n), z - x_n\rangle \leq 0\}, \\ x_{n+1} = \operatorname{proj}^g_{C_n \cap Q_n} x_0 \quad \textit{and} \quad n \in \mathbb{N} \cup \{0\}. \end{cases} \tag{1.10}$$

If, for each $i = 1, 2, \dots, N$, $\liminf_{n\to\infty} \lambda_n^i > 0$ and the sequences of errors $\{e_n^i\}_{n\in\mathbb{N}} \subset E$ satisfy $\liminf_{n\to\infty} e_n^i = 0$, then each such sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\operatorname{proj}_Z^g(x_0)$ as $n \to \infty$.

Let C be a nonempty, closed and convex subset of a reflexive Banach space E. Let $g : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function. Recall that a mapping $T : C \to C$ is said to be *Bregman firmly nonexpansive* (for short, *BFNE*) if

$$D_g(Tx, Ty) + D_g(Ty, Tx) + D_g(Tx, x) + D_g(Ty, y) \leq D_g(Tx, y) + D_g(Ty, x)$$

for all $x, y \in C$. The mapping T is called *quasi-Bregman firmly nonexpansive* (for short, *QBFNE*) [43], if $F(T) \neq \emptyset$ and

$$D_g(p, Tx) + D_g(Tx, x) \leq D_g(p, x)$$

for all $x \in C$ and $p \in F(T)$. It is clear that any quasi-Bregman firmly nonexpansive mapping is Bregman quasi-nonexpansive. For more information on Bregman firmly nonexpansive mappings, we refer the readers to [38, 44]. In [44], Reich and Sabach proved that for any BFNE operator T, $\hat{F}(T) = F(T)$.

In [43], Reich and Sabach introduced a Mann-type process to approximate fixed points of quasi-Bregman firmly nonexpansive mappings defined on a nonempty, closed and convex subset C of a reflexive Banach space E. More precisely, they proved the following theorem.

Theorem 1.4 *Let E be a reflexive Banach space and let $T_i : E \to E$, $i = 1, 2, \dots, N$, be N QBFNE operators which satisfy $F(T_i) = \hat{F}(T_i)$ for each $1 \le i \le N$ and $F := \bigcap_{i=1}^N F(T_i) \neq \varnothing$. Let $g : E \to \mathbb{R}$ be a Legendre function that is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence defined by the following iterative algorithm*:

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ Q_0^i = E, \quad i = 1, 2, \dots, N, \\ y_n^i = T_i(x_n + e_n^i), \\ Q_{n+1}^i = \{z \in Q_n^i : \langle \nabla g(x_n + e_n^i) - \nabla g(y_n^i), z - y_n^i \rangle \le 0\}, \\ Q_{n+1} := \bigcap_{i=1}^N Q_{n+1}^i, \\ x_{n+1} = \operatorname{proj}_{Q_{n+1}}^g x_0 \quad \textit{and} \quad n \in \mathbb{N} \cup \{0\}. \end{cases} \tag{1.11}$$

If, for each $i = 1, 2, \dots, N$, the sequences of errors $\{e_n^i\}_{n\in\mathbb{N}} \subset E$ satisfy $\liminf_{n\to\infty} e_n^i = 0$, then each such sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\operatorname{proj}_F^g(x_0)$ as $n \to \infty$.

Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Let C be a nonempty, closed and convex subset of E. Recall that a mapping $T : C \to C$ is said to be *(quasi-)Bregman strongly firmly nonexpansive* (for short, *BSNE*) with respect to a nonempty $\hat{F}(T)$ if $F(T) \neq \varnothing$ and

$$D_g(p, Tx) \le D_g(p, x)$$

for all $x \in C$ and $p \in \hat{F}(T)$, and if whenever $\{x_n\}_{n\in\mathbb{N}} \subset C$ is bounded and $p \in \hat{F(T)}$, then we have

$$\lim_{n\to\infty} \bigl(D_g(p, x_n) - D_g(p, Tx_n)\bigr) = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} D_g(Tx_n, x_n) = 0.$$

The class of (quasi-)Bregman strongly nonexpansive mappings was first introduced in [21, 45] (for more details, see also [46]). We know that the notion of a strongly nonexpansive operator (with respect to the norm) was first introduced and studied in [47, 48].

In [46], Reich and Sabach introduced iterative algorithms for finding common fixed points of finitely many Bregman strongly nonexpansive operators in a reflexive Banach space. They established the following strong convergence theorem in a reflexive Banach space.

Theorem 1.5 *Let E be a reflexive Banach space and let $T_i : E \to E$, $i = 1, 2, \dots, N$, be N BSNE operators which satisfy $F(T_i) = \hat{F}(T_i)$ for each $1 \le i \le N$ and $F := \bigcap_{i=1}^N F(T_i) \neq \varnothing$.*

Let $g : E \to \mathbb{R}$ be a Legendre function that is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence defined by the following iterative algorithm:

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ y_n^i = T_i(x_n + e_n^i), \\ C_n^i = \{z \in E : D_g(z, y_n^i) \le D_g(z, x_n + e_n^i)\}, \\ C_n := \bigcap_{i=1}^N C_n^i, \\ Q_n = \{z \in E : \langle \nabla g(x_0) - \nabla g(x_n), z - x_n \rangle \le 0\}, \\ x_{n+1} = \operatorname{proj}_{C_n \cap Q_n}^g x_0 \quad \textit{and} \quad n \in \mathbb{N} \cup \{0\}. \end{cases} \tag{1.12}$$

If, for each $i = 1, 2, \ldots, N$, the sequences of errors $\{e_n^i\}_{n\in\mathbb{N}} \subset E$ satisfy $\liminf_{n\to\infty} e_n^i = 0$, then each such sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $\operatorname{proj}_F^g(x_0)$ as $n \to \infty$.

But it is worth mentioning that, in all the above results for Bregman nonexpansive-type mappings, the assumption $\hat{F}(T) = F(T)$ is imposed on the map T.

Remark 1.2 Though the iteration processes (1.10) and (1.12), as introduced by the authors mentioned above, worked, it is easy to see that these processes seem cumbersome and complicated in the sense that at each stage of iteration, two different sets C_n and Q_n are computed and the next iterate taken as the Bregman projection of x_0 on the intersection of C_n and Q_n. This seems difficult to do in application. It is important to state clearly that the iteration process (1.11) involves computation of only one set Q_n at each stage of iteration. In [49], Sabach proposed an excellent modification of algorithm (1.10) for finding common zeros of finitely many maximal monotone operators in reflexive Banach spaces.

Our concern now is the following:

> Is it possible to obtain strong convergence of modified Mann-type schemes (1.10)-(1.12) to a fixed point of a Bregman quasi-nonexpansive type mapping T without imposing the assumption $\hat{F}(T) = F(T)$ on T?

In this paper, using Bregman functions, we introduce new hybrid iterative algorithms for finding common fixed points of an infinite family of Bregman weak relatively nonexpansive mappings in Banach spaces. We prove strong convergence theorems for the sequences produced by the methods. Furthermore, we apply our method to prove strong convergence theorems of iterative algorithms for finding common fixed points of finitely many Bregman weak relatively nonexpansive mappings in reflexive Banach spaces. These algorithms take into account possible computational errors. We also apply our main results to solve equilibrium problems in reflexive Banach spaces. Finally, we study hybrid iterative schemes for finding common solutions of an equilibrium problem, fixed points of an infinite family of Bregman weak relatively nonexpansive mappings and null spaces of a γ-inverse strongly monotone mapping in 2-uniformly convex Banach spaces. Some application of our results to the solution of equations of Hammerstein type is presented. No assumption $\hat{F}(T) = F(T)$ is imposed on the mapping T. Consequently, the above concern is answered in the affirmative in reflexive Banach space setting. Our results improve and generalize many known results in the current literature; see, for example, [4, 7, 8, 11, 22, 40, 42–44, 46, 50–52].

2 Preliminaries

In this section, we begin by recalling some preliminaries and lemmas which will be used in the sequel.

The following definition is slightly different from that in Butnariu and Iusem [23].

Definition 2.1 [24] Let E be a Banach space. The function $g : E \to \mathbb{R}$ is said to be a Bregman function if the following conditions are satisfied:

(1) g is continuous, strictly convex and Gâteaux differentiable;
(2) the set $\{y \in E : D_g(x,y) \le r\}$ is bounded for all $x \in E$ and $r > 0$.

The following lemma follows from Butnariu and Iusem [23] and Zălinscu [35].

Lemma 2.1 *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function. Then*

(1) $\nabla g : E \to E^*$ *is one-to-one, onto and norm-to-weak** *continuous*;
(2) $\langle x - y, \nabla g(x) - \nabla g(y)\rangle = 0$ *if and only if* $x = y$;
(3) $\{x \in E : D_g(x,y) \le r\}$ *is bounded for all* $y \in E$ *and* $r > 0$;
(4) $\operatorname{dom} g^* = E^*$, g^* *is Gâteaux differentiable and* $\nabla g^* = (\nabla g)^{-1}$.

Now, we are ready to prove the following key lemma.

Lemma 2.2 *Let E be a Banach space, let $r > 0$ be a constant and let $g : E \to \mathbb{R}$ be a convex function which is uniformly convex on bounded subsets of E. Then*

$$g\left(\sum_{k=0}^{n} \alpha_k x_k\right) \le \sum_{k=0}^{n} \alpha_k g(x_k) - \alpha_i \alpha_j \rho_r\big(\|x_i - x_j\|\big)$$

for all $i, j \in \{0, 1, 2, \dots, n\}$, $x_k \in B_r$, $\alpha_k \in (0,1)$ and $k = 0, 1, 2, \dots, n$ with $\sum_{k=0}^{n} \alpha_k = 1$, where ρ_r is the gauge of uniform convexity of g.

Proof Without loss of generality, we may assume that $i = 0$ and $j = 1$. By induction on n, for $n = 1$, in view of Remark 1.1 we get the desired result. Now suppose that it is true for $n = k$, *i.e.*,

$$g\left(\sum_{m=0}^{k} \alpha_m x_m\right) \le \sum_{m=0}^{k} \alpha_m g(x_m) - \alpha_0 \alpha_1 \rho_r\big(\|x_0 - x_1\|\big).$$

Now, we prove that the conclusion holds for $n = k + 1$. Put $x = \frac{\sum_{m=0}^{k} \alpha_m x_m}{1 - \alpha_{k+1}}$ and observe that $x \in B_r$. Since g is convex, given assumption, we conclude that

$$\begin{aligned} g\left(\sum_{m=0}^{k+1} \alpha_m x_m\right) &= g\left((1 - \alpha_{k+1}) \sum_{m=0}^{k} \frac{\alpha_m x_m}{1 - \alpha_{k+1}} + \alpha_{k+1} x_{k+1}\right) \\ &\le (1 - \alpha_{k+1}) g\left(\sum_{m=0}^{k} \frac{\alpha_m x_m}{1 - \alpha_{k+1}}\right) + \alpha_{k+1} g(x_{k+1}) \end{aligned}$$

$$\begin{aligned}&\le \sum_{m=0}^{k} \alpha_m g(x_m) - \alpha_0\alpha_1\rho_r\big(\|x_0 - x_1\|\big) + \alpha_{k+1}g(x_{k+1})\\&= \sum_{m=0}^{k+1} \alpha_m g(x_m) - \alpha_0\alpha_1\rho_r\big(\|x_0 - x_1\|\big).\end{aligned}$$

This completes the proof. □

Lemma 2.3 *Let E be a Banach space, let $r > 0$ be a constant and let $g : E \to \mathbb{R}$ be a continuous and convex function which is uniformly convex on bounded subsets of E. Then*

$$g\left(\sum_{k=0}^{\infty} \alpha_k x_k\right) \le \sum_{k=0}^{\infty} \alpha_k g(x_k) - \alpha_i\alpha_j\rho_r\big(\|x_i - x_j\|\big)$$

for all $i, j \in \mathbb{N} \cup \{0\}$, $x_k \in B_r$, $\alpha_k \in (0,1)$ and $k \in \mathbb{N} \cup \{0\}$ with $\sum_{k=0}^{\infty} \alpha_k = 1$, where ρ_r is the gauge of uniform convexity of g.

Proof Let $i, j \in \mathbb{N} \cup \{0\}$ and $k > i, j$. Put $v_k = \frac{\alpha_0 x_0}{\sum_{m=0}^{k}\alpha_m} + \frac{\alpha_1 x_1}{\sum_{m=0}^{k}\alpha_m} + \cdots + \frac{\alpha_k x_k}{\sum_{m=0}^{k}\alpha_m}$ and observe that $v_k \in B_r$ for all $k \in \mathbb{N}$. In view of Lemma 2.2, we obtain that

$$\begin{aligned}g(v_k) &= g\left(\frac{\alpha_0 x_0}{\sum_{m=0}^{k}\alpha_m} + \frac{\alpha_1 x_1}{\sum_{m=0}^{k}\alpha_m} + \cdots + \frac{\alpha_k x_k}{\sum_{m=0}^{k}\alpha_m}\right)\\&\le \frac{1}{\sum_{m=0}^{k}\alpha_m} \sum_{m=0}^{k} \alpha_m g(x_m) - \alpha_i\alpha_j\rho_r\big(\|x_i - x_j\|\big).\end{aligned} \tag{2.1}$$

Since g is continuous and $v_k \to \sum_{m=0}^{\infty} \alpha_m x_m$ as $k \to \infty$, we have

$$\lim_{k\to\infty} g(v_k) = g\left(\sum_{m=0}^{\infty} \alpha_m x_m\right).$$

Letting $k \to \infty$ in (2.1), we conclude that

$$g\left(\sum_{m=0}^{\infty} \alpha_m x_m\right) \le \sum_{m=0}^{\infty} \alpha_m g(x_m) - \alpha_i\alpha_j\rho_r\big(\|x_i - x_j\|\big),$$

which completes the proof. □

We know the following two results; see [35, Proposition 3.6.4].

Theorem 2.1 *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex function which is bounded on bounded subsets of E. Then the following assertions are equivalent*:

(1) *g is strongly coercive and uniformly convex on bounded subsets of E;*
(2) $\operatorname{dom} g^* = E^*$, *$g^*$ is bounded on bounded subsets and uniformly smooth on bounded subsets of E^*;*
(3) $\operatorname{dom} g^* = E^*$, *$g^*$ is Fréchet differentiable and ∇g^* is uniformly norm-to-norm continuous on bounded subsets of E^*.*

Theorem 2.2 *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a continuous convex function which is strongly coercive. Then the following assertions are equivalent*:

(1) *g is bounded on bounded subsets and uniformly smooth on bounded subsets of E*;
(2) *g^* is Fréchet differentiable and ∇g^* is uniformly norm-to-norm continuous on bounded subsets of E^**;
(3) $\operatorname{dom} g^* = E^*$, *$g^*$ is strongly coercive and uniformly convex on bounded subsets of E^**.

Let E be a Banach space and let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. Then the Bregman distance [32, 33] satisfies the *three point identity* that is

$$D_g(x,z) = D_g(x,y) + D_g(y,z) + \langle x - y, \nabla g(y) - \nabla g(z)\rangle, \quad \forall x, y, z \in E. \tag{2.2}$$

In particular, it can be easily seen that

$$D_g(x,y) = -D_g(y,x) + \langle y - x, \nabla g(y) - \nabla g(x)\rangle, \quad \forall x, y \in E. \tag{2.3}$$

Indeed, by letting $z = x$ in (2.2) and taking into account that $D_g(x,x) = 0$, we get the desired result.

Lemma 2.4 *Let E be a Banach space and let $g : E \to \mathbb{R}$ be a Gâteaux differentiable function which is uniformly convex on bounded subsets of E. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be bounded sequences in E. Then the following assertions are equivalent*:

(1) $\lim_{n\to\infty} D_g(x_n, y_n) = 0$;
(2) $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Proof The implication (1) $\Longrightarrow$ (2) was proved in [23] (see also [24]). For the converse implication, we assume that $\lim_{n\to\infty} \|x_n - y_n\| = 0$. Then, in view of (2.3), we have

$$\begin{aligned} D_g(x_n, y_n) &= -D_g(y_n, x_n) + \langle x_n - y_n, \nabla g(x_n) - \nabla g(y_n)\rangle \\ &\leq \|x_n - y_n\| \|\nabla g(x_n) - \nabla g(y_n)\|, \quad \forall n \in \mathbb{N}. \end{aligned} \tag{2.4}$$

The function g is bounded on bounded subsets of E and therefore ∇g is also bounded on bounded subsets of E^* (see, for example, [23, Proposition 1.1.11] for more details). This, together with (2.3)-(2.4), implies that $\lim_{n\to\infty} D_g(x_n, y_n) = 0$, which completes the proof. □

The following result was first proved in [37] (see also [24]).

Lemma 2.5 *Let E be a reflexive Banach space, let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function and let V be the function defined by*

$$V(x, x^*) = g(x) - \langle x, x^*\rangle + g^*(x^*), \quad x \in E, x^* \in E^*.$$

Then the following assertions hold:

(1) *$D_g(x, \nabla g^*(x^*)) = V(x, x^*)$ for all $x \in E$ and $x^* \in E^*$.*
(2) *$V(x, x^*) + \langle \nabla g^*(x^*) - x, y^*\rangle \leq V(x, x^* + y^*)$ for all $x \in E$ and $x^*, y^* \in E^*$.*

Corollary 2.1 [35] *Let E be a Banach space, let $g : E \to (-\infty, \infty]$ be a proper, lower semicontinuous and convex function and let $p, q \in \mathbb{R}$ with $1 \le p \le 2 \le q$ and $p^{-1} + q^{-1} = 1$. Then the following statements are equivalent.*

(1) *There exists $c_1 > 0$ such that g is ρ-convex with $\rho(t) := \frac{c_1}{q} t^q$ for all $t \ge 0$.*
(2) *There exists $c_2 > 0$ such that for all $(x, x^*), (y, y^*) \in G(\partial g)$; $\|x^* - y^*\| \ge \frac{2c_2}{q} \|x - y\|^{q-1}$.*

3 Strong convergence theorems without computational errors

In this section, we prove strong convergence theorems without computational errors in a reflexive Banach space. We start with the following simple lemma whose proof will be omitted since it can be proved by a similar argument as that in [44, Lemma 15.5].

Lemma 3.1 *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of E. Let $T : C \to C$ be a Bregman weak relatively nonexpansive mapping. Then $F(T)$ is closed and convex.*

Using ideas in [22], we can prove the following result.

Theorem 3.1 *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function which is bounded on bounded subsets and uniformly convex and uniformly smooth on bounded subsets of E. Let C be a nonempty, closed and convex subset of E and let $\{T_j\}_{j\in\mathbb{N}}$ be an infinite family of Bregman weak relatively nonexpansive mappings from C into itself such that $F := \bigcap_{j=1}^{\infty} F(T_j) \neq \emptyset$. Suppose in addition that $T_j^0 = T_0 = I$ for all $j \in \mathbb{N}$, where I is the identity mapping on E. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by*

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily}, \\ C_0 = C, \\ z_n = \nabla g^*[\alpha_{n,0} \nabla g(x_n) + \sum_{j=1}^{\infty} \alpha_{n,j} \nabla g(T_j x_n)], \\ y_n = \nabla g^*[\beta_n \nabla g(x_n) + (1 - \beta_n) \nabla g(z_n)], \\ C_{n+1} = \{z \in C_n : D_g(z, y_n) \le D_g(z, x_n)\}, \\ x_{n+1} = \operatorname{proj}^g_{C_{n+1}} x \quad \text{and} \quad n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{3.1}$$

where ∇g is the right-hand derivative of g. Let $\{\alpha_{n,j} : j, n \in \mathbb{N} \cup \{0\}\}$ and $\{\beta_n\}_{n\in\mathbb{N}\cup\{0\}}$ be sequences in $[0, 1)$ satisfying the following control conditions:

(1) $\sum_{j=0}^{\infty} \alpha_{n,j} = 1$, $\forall n \in \mathbb{N} \cup \{0\}$;
(2) *There exists $i \in \mathbb{N}$ such that* $\liminf_{n\to\infty} \alpha_{n,i}\alpha_{n,j} > 0$, $\forall j \in \mathbb{N} \cup \{0\}$;
(3) $0 \le \beta_n < 1$ *for all $n \in \mathbb{N} \cup \{0\}$ and* $\limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined in (3.1) *converges strongly to* $\operatorname{proj}^g_F x$ *as $n \to \infty$.*

Proof We divide the proof into several steps.

Step 1. We show that C_n is closed and convex for each $n \in \mathbb{N} \cup \{0\}$.

It is clear that $C_0 = C$ is closed and convex. Let C_m be closed and convex for some $m \in \mathbb{N}$. For $z \in C_m$, we see that

$$D_g(z, y_m) \le D_g(z, x_m)$$

is equivalent to

$$\langle z, \nabla g(x_m) - \nabla g(y_m)\rangle \le g(y_m) - g(x_m) + \langle x_m, \nabla g(x_m)\rangle - \langle y_m, \nabla g(y_m)\rangle.$$

An easy argument shows that C_{m+1} is closed and convex. Hence C_n is closed and convex for each $n \in \mathbb{N} \cup \{0\}$.

Step 2. We claim that $F \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$.

It is obvious that $F \subset C_0 = C$. Assume now that $F \subset C_m$ for some $m \in \mathbb{N}$. Employing Lemma 2.5, for any $w \in F \subset C_m$, we obtain

$$\begin{aligned}
D_g(w, z_m) &= D_g\left(w, \nabla g^*\left[\alpha_{m,0}\nabla g(x_m) + \sum_{j=1}^{\infty}\alpha_{m,j}\nabla g(T_j x_m)\right]\right) \\
&= V\left(w, \alpha_{m,0}\nabla g(x_m) + \sum_{j=1}^{\infty}\alpha_{m,j}\nabla g\big(T_j^m x_m\big)\right) \\
&= g(w) - \left\langle w, \alpha_{m,0}\nabla g(x_m) + \sum_{j=1}^{\infty}\alpha_{m,j}\nabla g(T_j x_m)\right\rangle \\
&\quad + g^*\left(\alpha_{m,0}\nabla g(x_m) + \sum_{j=1}^{\infty}\alpha_{m,j}\nabla g(T_j x_m)\right) \\
&\le \alpha_{m,0} g(w) + \sum_{j=1}^{\infty}\alpha_{m,j} g(w) + \alpha_{m,0} g^*\big(\nabla g(x_m)\big) + \sum_{j=1}^{\infty}\alpha_{m,j} g^*\big(\nabla g(T_j x_m)\big) \\
&= \alpha_{m,0} V\big(w, \nabla g(x_m)\big) + \sum_{j=1}^{\infty}\alpha_{m,j} V\big(w, \nabla g(T_j x_m)\big) \\
&= \alpha_{m,0} D_g(w, x_m) + \sum_{j=1}^{\infty}\alpha_{m,j} D_g(w, T_j x_m) \\
&\le \alpha_{m,0} D_g(w, x_m) + \sum_{j=1}^{\infty}\alpha_{m,j} D_g(w, x_m) \\
&= D_g(w, x_m).
\end{aligned}$$

This implies that

$$\begin{aligned}
D_g(w, y_m) &= D_g\big(w, \nabla g^*\big[\beta_m \nabla g(x_m) + (1-\beta_m)\nabla g(z_m)\big]\big) \\
&= V\big(w, \beta_m \nabla g(x_m) + (1-\beta_m)\nabla g(z_m)\big) \\
&\le \beta_m V\big(w, \nabla g(x_m)\big) + (1-\beta_m) V\big(w, \nabla(z_m)\big) \\
&= \beta_m D_g(w, x_m) + (1-\beta_m) D_g(w, z_m) \\
&\le \beta_m D_g(w, x_m) + (1-\beta_m) D_g(w, x_m) \\
&= D_g(w, x_m).
\end{aligned} \tag{3.2}$$

This proves that $w \in C_{m+1}$. Thus, we have $F \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$.

Step 3. We prove that $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$, $\{z_n\}_{n\in\mathbb{N}}$ and $\{T_jx_n : j, n \in \mathbb{N} \cup \{0\}\}$ are bounded sequences in C.

In view of (1.9), we conclude that

$$\begin{aligned} D_g(x_n, x) &= D_g\big(\operatorname{proj}^g_{C_n} x, x\big) \le D_g(w, x) - D_g(w, x_n) \\ &\le D_g(w, x), \quad \forall w \in F \subset C_n, n \in \mathbb{N} \cup \{0\}. \end{aligned}$$

This implies that the sequence $\{D(x_n, x)\}_{n\in\mathbb{N}}$ is bounded and hence there exists $M > 0$ such that

$$D_g(x_n, x) \le M, \quad \forall n \in \mathbb{N}.$$

In view of Lemma 2.1(3), we conclude that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded. Since $\{T_j\}_{j\in\mathbb{N}}$ is an infinite family of Bregman weak relatively nonexpansive mappings from C into itself, we have for any $q \in F$ that

$$D_g(q, T_jx_n) \le D_g(q, x_n), \quad \forall j, n \in \mathbb{N}.$$

This, together with Definition 2.1 and the boundedness of $\{x_n\}_{n\in\mathbb{N}}$, implies that the sequence $\{T_jx_n : j, n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Step 4. We show that $x_n \to u$ for some $u \in F$, where $u = \operatorname{proj}^g_F x$.

By Step 3, we have that $\{x_n\}_{n\in\mathbb{N}}$ is bounded. By the construction of C_n, we conclude that $C_m \subset C_n$ and $x_m = \operatorname{proj}^g_{C_m} x \in C_m \subset C_n$ for any positive integer $m \ge n$. This, together with (1.9), implies that

$$\begin{aligned} D_g(x_m, x_n) &= D_g\big(x_m, \operatorname{proj}^g_{C_n} x\big) \le D_g(x_m, x) - D_g\big(\operatorname{proj}^g_{C_n} x, x\big) \\ &= D_g(x_m, x) - D_g(x_n, x). \end{aligned} \tag{3.3}$$

In view of (1.9), we conclude that

$$\begin{aligned} D_g(x_n, x) &= D_g\big(\operatorname{proj}^g_{C_n} x, x\big) \le D_g(w, x) - D_g(w, x_n) \\ &\le D_g(w, x), \quad \forall w \in F \subset C_n, n \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{3.4}$$

It follows from (3.4) that the sequence $\{D_g(x_n, x)\}_{n\in\mathbb{N}}$ is bounded and hence there exists $M > 0$ such that

$$D_g(x_n, x) \le M, \quad \forall n \in \mathbb{N}. \tag{3.5}$$

In view of (3.3), we conclude that

$$D_g(x_n, x) \le D_g(x_n, x) + D_g(x_m, x_n) \le D_g(x_m, x), \quad \forall m \ge n.$$

This proves that $\{D_g(x_n, x)\}_{n\in\mathbb{N}}$ is an increasing sequence in $\mathbb{R}$ and hence by (3.5) the limit $\lim_{n\to\infty} D_g(x_n, x)$ exists. Letting $m, n \to \infty$ in (3.3), we deduce that $D_g(x_m, x_n) \to 0$. In view of Lemma 2.4, we get that $\|x_m - x_n\| \to 0$ as $m, n \to \infty$. This means that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since E is a Banach space and C is closed and convex, we conclude that there exists $u \in C$ such that

$$\lim_{n\to\infty} \|x_n - u\| = 0. \tag{3.6}$$

Now, we show that $u \in F$. In view of (3.3), we obtain

$$\lim_{n\to\infty} D_g(x_{n+1}, x_n) = 0. \tag{3.7}$$

Since $x_{n+1} \in C_{n+1}$, we conclude that

$$D_g(x_{n+1}, y_n) \le D_g(x_{n+1}, x_n).$$

This, together with (3.7), implies that

$$\lim_{n\to\infty} D_g(x_{n+1}, y_n) = 0. \tag{3.8}$$

Employing Lemma 2.4 and (3.7)-(3.8), we deduce that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|x_{n+1} - y_n\| = 0.$$

In view of (3.6), we get

$$\lim_{n\to\infty} \|y_n - u\| = 0. \tag{3.9}$$

From (3.6) and (3.9), it follows that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Since ∇g is uniformly norm-to-norm continuous on any bounded subset of E, we obtain

$$\lim_{n\to\infty} \big\|\nabla g(x_n) - \nabla g(y_n)\big\| = 0. \tag{3.10}$$

In view of (3.1), we have

$$\nabla g(y_n) - \nabla g(x_n) = (1 - \beta_n)\big(\nabla g(z_n) - \nabla g(x_n)\big). \tag{3.11}$$

It follows from (3.10)-(3.11) that

$$\lim_{n\to\infty} \big\|\nabla g(z_n) - \nabla g(x_n)\big\| = 0. \tag{3.12}$$

Since ∇g is uniformly norm-to-norm continuous on any bounded subset of E, we obtain

$$\lim_{n\to\infty} \|z_n - x_n\| = 0.$$

Applying Lemma 2.4, we derive that

$$\lim_{n\to\infty} D_g(z_n, x_n) = 0.$$

It follows from the three point identity (see (2.2)) that

$$\begin{aligned}
\left|D_g(w,x_n) - D_g(w,z_n)\right| &= \left|D_g(w,z_n) + D_g(z_n,x_n)\right.\\
&\quad \left.+ \langle w - z_n, \nabla g(z_n) - \nabla g(x_n)\rangle - D_g(w,z_n)\right|\\
&= \left|D_g(z_n,x_n) - \langle w - z_n, \nabla g(z_n) - \nabla g(x_n)\rangle\right|\\
&\le D_g(z_n,x_n) + \|w - z_n\| \left\|\nabla g(z_n) - \nabla g(x_n)\right\|\\
&\to 0 \qquad (3.13)
\end{aligned}$$

as $n \to \infty$.

The function g is bounded on bounded subsets of E and thus ∇g is also bounded on bounded subsets of E^* (see, for example, [23, Proposition 1.1.11] for more details). This implies that the sequences $\{\nabla g(x_n)\}_{n\in\mathbb{N}}$, $\{\nabla g(y_n)\}_{n\in\mathbb{N}}$, $\{\nabla g(z_n)\}_{n\in\mathbb{N}}$ and $\{\nabla g(T_j^n x_n) : n, j \in \mathbb{N} \cup \{0\}\}$ are bounded in E^*.

In view of Theorem 2.2(3), we know that $\operatorname{dom} g^* = E^*$ and g^* is strongly coercive and uniformly convex on bounded subsets. Let $s = \sup\{\|\nabla g(T_j^n x_n)\| : j \in \mathbb{N} \cup \{0\}, n \in \mathbb{N} \cup \{0\}\}$ and $\rho_s^* : E^* \to \mathbb{R}$ be the gauge of uniform convexity of the conjugate function g^*. Now, we fix $i \in \mathbb{N}$ satisfying condition (2). We prove that for any $w \in F$ and $j \in \mathbb{N} \cup \{0\}$

$$D_g(w,z_n) \le D_g(w,x_n) - \alpha_{n,i}\alpha_{n,j}\rho_s^*\left(\left\|\nabla g(T_i x_n) - \nabla g(T_j x_n)\right\|\right). \qquad (3.14)$$

Let us show (3.14). For any given $w \in F(T)$ and $j \in \mathbb{N}$, in view of the definition of the Bregman distance (see (1.7)), (1.6), Lemmas 2.3 and 2.5, we obtain

$$\begin{aligned}
D_g(w,z_n) &= D_g\left(w, \nabla g^*\left[\alpha_{n,0}\nabla g(x_n) + \sum_{j=1}^{\infty} \alpha_{n,j}\nabla g(T_j x_n)\right]\right)\\
&= V\left(w, \alpha_{n,0}\nabla g(x_n) + \sum_{j=1}^{\infty} \alpha_{n,j}\nabla g(T_j x_n)\right)\\
&= g(w) - \left\langle w, \alpha_{n,0}\nabla g(x_n) + \sum_{j=1}^{\infty} \alpha_{n,j}\nabla g(T_j x_n)\right\rangle\\
&\quad + g^*\left(\alpha_{n,0}\nabla g(x_n) + \sum_{j=1}^{\infty} \alpha_{n,j}\nabla g(T_j x_n)\right)\\
&\le \alpha_{n,0} g(w) + \sum_{j=1}^{\infty} \alpha_{n,j} g(w)\\
&\quad - \alpha_{n,0}\langle w, \nabla g(x_n)\rangle - \sum_{j=1}^{\infty} \alpha_{n,j}\langle w, \nabla g(T_j x_n)\rangle\\
&\quad + \alpha_{n,0} g^*\left(\nabla g(x_n)\right) + \sum_{j=1}^{\infty} \alpha_{n,j} g^*\left(\nabla g(T_j x_n)\right)\\
&\quad - \alpha_{n,i}\alpha_{n,j}\rho_s^*\left(\left\|\nabla g(T_i x_n) - \nabla g(T_j x_n)\right\|\right)\\
&= \alpha_{n,0} V\left(w, \nabla g(x_n)\right) + \sum_{j=1}^{\infty} \alpha_{n,j} V\left(w, \nabla g(T_j x_n)\right)
\end{aligned}$$

$$
\begin{aligned}
&\quad - \alpha_{n,i}\alpha_{n,j}\rho_s^*\big(\big\|\nabla g(T_i x_n) - \nabla g(T_j x_n)\big\|\big) \\
&= \alpha_{n,0} D_g(w, x_n) + \sum_{j=1}^{\infty} \alpha_{n,j} D_g(w, T_j x_n) - \alpha_{n,i}\alpha_{n,j}\rho_s^*\big(\big\|\nabla g(T_i x_n) - \nabla g(T_j x_n)\big\|\big) \\
&\leq \alpha_{n,0} D_g(w, x_n) + \sum_{j=1}^{\infty} \alpha_{n,j} D_g(w, x_n) - \alpha_{n,i}\alpha_{n,j}\rho_s^*\big(\big\|\nabla g(T_i x_n) - \nabla g(T_j x_n)\big\|\big) \\
&= D_g(w, x_n) - \alpha_{n,i}\alpha_{n,j}\rho_s^*\big(\big\|\nabla g(T_i x_n) - \nabla g(T_j x_n)\big\|\big).
\end{aligned}
$$

In view of (3.13), we obtain

$$
D_g(w, x_n) - D_g(w, z_n) \to 0 \quad \text{as } n \to \infty. \tag{3.15}
$$

In view of (3.14) and (3.15), we conclude that

$$
\alpha_{n,i}\alpha_{n,j}\rho_s^*\big(\big\|\nabla g(T_i x_n) - \nabla g(T_j x_n)\big\|\big) \leq D_g(w, x_n) - D_g(w, z_n) \to 0
$$

as $n \to \infty$. From the assumption $\liminf_{n\to\infty} \alpha_{n,i}\alpha_{n,j} > 0$, $\forall j \in \mathbb{N} \cup \{0\}$, we have

$$
\lim_{n\to\infty} \rho_s^*\big(\big\|\nabla g(T_i x_n) - \nabla g(T_j x_n)\big\|\big) = 0, \quad \forall j \in \mathbb{N} \cup \{0\}.
$$

Therefore, from the property of ρ_s^*, we deduce that

$$
\lim_{n\to\infty} \big\|\nabla g(T_i x_n) - \nabla g(T_j x_n)\big\| = 0, \quad \forall j \in \mathbb{N} \cup \{0\}.
$$

Since ∇g^* is uniformly norm-to-norm continuous on bounded subsets of E^*, we arrive at

$$
\lim_{n\to\infty} \|T_i x_n - T_j x_n\| = 0, \quad \forall j \in \mathbb{N} \cup \{0\}. \tag{3.16}
$$

In particular, for $j = 0$, we have

$$
\lim_{n\to\infty} \|T_i x_n - x_n\| = 0.
$$

This, together with (3.16), implies that

$$
\lim_{n\to\infty} \|T_j x_n - x_n\| = 0, \quad \forall j \in \mathbb{N} \cup \{0\}. \tag{3.17}
$$

Since $\{T_j\}_{j\in\mathbb{N}}$ is an infinite family of Bregman weak relatively nonexpansive mappings, from (3.6) and (3.17), we conclude that $T_j u = u$, $\forall j \in \mathbb{N} \cup \{0\}$. Thus, we have $u \in F$.

Finally, we show that $u = \operatorname{proj}_F^g x$. From $x_n = \operatorname{proj}_{C_n}^g x$, we conclude that

$$
\big\langle z - x_n, \nabla g(x_n) - \nabla g(x)\big\rangle \geq 0, \quad \forall z \in C_n.
$$

Since $F \subset C_n$ for each $n \in \mathbb{N}$, we obtain

$$
\big\langle z - x_n, \nabla g(x_n) - \nabla g(x)\big\rangle \geq 0, \quad \forall z \in F. \tag{3.18}
$$

Letting $n \to \infty$ in (3.18), we deduce that

$$\langle z - u, \nabla g(u) - \nabla g(x) \rangle \geq 0, \quad \forall z \in F.$$

In view of (1.8), we have $u = \operatorname{proj}_F^g x$, which completes the proof. □

Remark 3.1 Theorem 3.1 improves Theorem 1.2 in the following aspects.

(1) For the structure of Banach spaces, we extend the duality mapping to a more general case, that is, a convex, continuous and strongly coercive Bregman function which is bounded on bounded subsets and uniformly convex and uniformly smooth on bounded subsets.

(2) For the mappings, we extend the mapping from a relatively nonexpansive mapping to a countable family of Bregman weak relatively nonexpansive mappings. We remove the assumption $\hat{F}(T) = F(T)$ on the mapping T and extend the result to a countable family of Bregman weak relatively nonexpansive mappings, where $\hat{F}(T)$ is the set of asymptotic fixed points of the mapping T.

(3) For the algorithm, we remove the set W_n in Theorem 1.2.

Lemma 3.2 *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function which is bounded on bounded subsets and uniformly convex and uniformly smooth on bounded subsets of E. Let A be a maximal monotone operator from E to E^* such that $A^{-1}(0) \neq \emptyset$. Let $r > 0$ and $\operatorname{Res}_{rA}^g = (\nabla g + rA)^{-1} \nabla g$ be the g-resolvent of A. Then $\operatorname{Res}_{rA}^g$ is a Bregman weak relatively nonexpansive mapping.*

Proof Let $\{z_n\}_{n \in \mathbb{N}} \subset E$ be a sequence such that $z_n \to z$ and $\lim_{n \to \infty} \|z_n - \operatorname{Res}_{rA}^g z_n\| = 0$. Since ∇g is uniformly norm-to-norm continuous on bounded subsets of E, we obtain

$$\frac{1}{r}\bigl(\nabla g(z_n) - \nabla g\bigl(\operatorname{Res}_{rA}^g z_n\bigr)\bigr) \to 0.$$

It follows from

$$\frac{1}{r}\bigl(\nabla g(z_n) - \nabla g\bigl(\operatorname{Res}_{rA}^g z_n\bigr)\bigr) \in A \operatorname{Res}_{rA}^g z_n$$

and the monotonicity of A that

$$\left\langle w - \operatorname{Res}_{rA}^g z_n, y - \frac{1}{r}\bigl(\nabla g(z_n) - \nabla g\bigl(\operatorname{Res}_{rA}^g z_n\bigr)\bigr) \right\rangle \geq 0$$

for all $w \in \operatorname{dom} A$ and $y \in Aw$. Letting $n \to \infty$ in the above inequality, we have $\langle w - z, y \rangle \geq 0$ for all $w \in \operatorname{dom} A$ and $y \in Aw$. Therefore, from the maximality of A, we conclude that $z \in A^{-1}(0) = F(\operatorname{Res}_{rA}^g)$, that is, $z = \operatorname{Res}_{rA}^g z$. Hence $\operatorname{Res}_{rA}^g$ is Bregman weak relatively nonexpansive, which completes the proof. □

As an application of our main result, we include a concrete example in support of Theorem 3.1. Using Theorem 3.1, we obtain the following strong convergence theorem for maximal monotone operators.

Theorem 3.2 *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function which is bounded on bounded subsets and uniformly convex and uniformly smooth on bounded subsets of E. Let A be a maximal monotone operator from E to E^* such that $A^{-1}(0) \neq \emptyset$. Let $r_n > 0$ such that $\liminf_{n\to\infty} r_n > 0$ and $\mathrm{Res}^g_{r_nA} = (\nabla g + r_nA)^{-1}\nabla g$ be the g-resolvent of A. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by*

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily}, \\ C_0 = C, \\ z_n = \nabla g^*[\alpha_{n,0}\nabla g(x_n) + \sum_{j=1}^{\infty} \alpha_{n,j}\nabla g(\mathrm{Res}^g_{r_jA}\, x_n)], \\ y_n = \nabla g^*[\beta_n \nabla g(x_n) + (1-\beta_n)\nabla g(z_n)], \\ C_{n+1} = \{z \in C_n : D_g(z, y_n) \le D_g(z, x_n)\}, \\ x_{n+1} = \mathrm{proj}^g_{C_{n+1}} x \quad \text{and} \quad n \in \mathbb{N}\cup\{0\}, \end{cases} \tag{3.19}$$

where ∇g is the right-hand derivative of g. Let $\{\alpha_{n,j} : j, n \in \mathbb{N}\cup\{0\}\}$ and $\{\beta_n\}_{n\in\mathbb{N}\cup\{0\}}$ be sequences in $[0,1)$ satisfying the following control conditions:

(1) $\sum_{j=0}^{\infty} \alpha_{n,j} = 1$, $\forall n \in \mathbb{N}\cup\{0\}$;
(2) *There exists $i \in \mathbb{N}$ such that* $\liminf_{n\to\infty} \alpha_{n,i}\alpha_{n,j} > 0$, $\forall j \in \mathbb{N}\cup\{0\}$;
(3) $0 \le \beta_n < 1$ *for all $n \in \mathbb{N}\cup\{0\}$ and* $\liminf_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined in (3.19) *converges strongly to* $\mathrm{proj}^g_{A^{-1}(0)} x$ *as $n \to \infty$.*

Proof Letting $T_j = \mathrm{Res}^g_{r_jA}$, $\forall j \in \mathbb{N}\cup\{0\}$, in Theorem 3.1, from (3.1) we obtain (3.19). We need only to show that T_j satisfies all the conditions in Theorem 3.1 for all $j \in \mathbb{N}\cup\{0\}$. In view of Lemma 3.2, we conclude that T_j is a Bregman relatively nonexpansive mapping for each $j \in \mathbb{N}\cup\{0\}$. Thus, we obtain

$$D_g\big(p, \mathrm{Res}^g_{r_jA}\, v\big) \le D_g(p, v), \quad \forall v \in E, p \in F\big(\mathrm{Res}^g_{r_jA}\big), \forall j \in \mathbb{N}\cup\{0\}$$

and

$$\tilde{F}\big(\mathrm{Res}^g_{r_jA}\big) = F\big(\mathrm{Res}^g_{r_jA}\big) = A^{-1}(0), \quad \forall j \in \mathbb{N}\cup\{0\},$$

where $\tilde{F}(\mathrm{Res}^g_{r_jA})$ is the set of all strong asymptotic fixed points of $\mathrm{Res}^g_{r_jA}$. Therefore, in view of Theorem 3.1, we have the conclusions of Theorem 3.2. This completes the proof. □

4 Strong convergence theorems with computational errors

In this section, we study strong convergence of iterative algorithms to find common fixed points of finitely many Bregman weak relatively nonexpansive mappings in a reflexive Banach space. Our algorithms take into account possible computational errors. We prove the following strong convergence theorem concerning Bregman weak relatively nonexpansive mappings.

Theorem 4.1 *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function which is bounded on bounded subsets and uniformly convex and uniformly smooth on bounded subsets of E. Let $N \in \mathbb{N}$ and $\{T_j\}_{j=1}^N$ be a finite family of Bregman weak relatively nonexpansive mappings from E into int* $\mathrm{dom}\, g$ *such that $F := \bigcap_{j=1}^N F(T_j)$ is* *a nonempty subset of E. Suppose in addition that $T_0 = I$, where I is the identity mapping on E. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by*

$$\begin{cases} x_0 = x \in E \quad \textit{chosen arbitrarily}, \\ C_0 = E, \\ y_n = \nabla g^*[\alpha_{n,0}\nabla g(x_n) + \sum_{j=1}^N \alpha_{n,j}\nabla g(T_j(x_n + e_n^j))], \\ C_{n+1} = \{z \in C_n : D_g(z, y_n) \le D_g(z, x_n) + \sum_{j=1}^N \alpha_{n,j} D_g(x_n, x_n + e_n^j) \\ \qquad + \sum_{j=1}^N \alpha_{n,j}\langle z - x_n, \nabla g(x_n) - \nabla g(x_n + e_n^j)\rangle\}, \\ x_{n+1} = \operatorname{proj}_{C_{n+1}}^g x \quad \textit{and} \quad n \in \mathbb{N}\cup\{0\}, \end{cases} \tag{4.1}$$

where ∇g is the right-hand derivative of g. Let $\{\alpha_{n,j} : n \in \mathbb{N}\cup\{0\}, j \in \{0,1,2,\ldots,N\}\}$ be a sequence in $(0,1)$ satisfying the following control conditions:

(1) $\sum_{j=0}^N \alpha_{n,j} = 1$, $\forall n \in \mathbb{N}\cup\{0\}$;

(2) *There exists $i \in \{1,2,\ldots,N\}$ such that* $\liminf_{n\to\infty} \alpha_{n,i}\alpha_{n,j} > 0$, $\forall j \in \{0,1,2,\ldots,N\}$.

If, for each $j = 0,1,2,\ldots,N$, the sequences of errors $\{e_n^j\}_{n\in\mathbb{N}} \subset E$ satisfy $\liminf_{n\to\infty} e_n^j = 0$, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined in (4.1) *converges strongly to* $\operatorname{proj}_F^g x$ *as $n \to \infty$.*

Proof We divide the proof into several steps.

Step 1. We show that C_n is closed and convex for each $n \in \mathbb{N}\cup\{0\}$.

It is clear that $C_0 = E$ is closed and convex. Let C_m be closed and convex for some $m \in \mathbb{N}$. For $z \in C_m$, we see that

$$\begin{aligned} D_g(z, y_m) \le{}& D_g(z, x_m) + \sum_{j=1}^N \alpha_{m,j} D_g\big(x_m, x_m + e_m^j\big) \\ &+ \sum_{j=1}^N \alpha_{m,j}\big\langle z - x_m, \nabla g(x_m) - \nabla g\big(x_m + e_m^j\big)\big\rangle \end{aligned}$$

is equivalent to

$$\begin{aligned} &\big\langle z, \nabla g(x_m) - \nabla g(y_m)\big\rangle + \sum_{j=1}^N \alpha_{m,j}\big\langle x_m - z, \nabla g(x_m) - \nabla g\big(x_m + e_m^j\big)\big\rangle \\ &\quad\le g(y_m) - g(x_m) + \big\langle x_m, \nabla g(x_m)\big\rangle - \big\langle y_m, \nabla g(y_m)\big\rangle + \sum_{j=1}^N \alpha_{m,j} D_g\big(x_m, x_m + e_m^j\big). \end{aligned}$$

An easy argument shows that C_{m+1} is closed and convex. Hence C_n is closed and convex for all $n \in \mathbb{N}\cup\{0\}$.

Step 2. We claim that $F \subset C_n$ for all $n \in \mathbb{N}\cup\{0\}$.

It is obvious that $F \subset C_0 = E$. Assume now that $F \subset C_m$ for some $m \in \mathbb{N}$. Employing Lemma 2.5, for any $w \in F \subset C_m$, we obtain

$$\begin{aligned} D_g(w, y_m) &= D_g\left(w, \nabla g^*\left[\alpha_{m,0}\nabla g(x_m) + \sum_{j=1}^N \alpha_{m,j}\nabla g\big(T_j\big(x_m + e_m^j\big)\big)\right]\right) \\ &= V\left(w, \alpha_{m,0}\nabla g(x_m) + \sum_{j=1}^N \alpha_{m,j}\nabla g\big(T_j\big(x_m + e_m^j\big)\big)\right) \end{aligned}$$

$$\begin{aligned}
&= g(w) - \left\langle w, \alpha_{m,0}\nabla g(x_m) + \sum_{j=1}^{N} \alpha_{m,j}\nabla g\big(T_j\big(x_m + e_m^j\big)\big)\right\rangle \\
&\quad + g^*\big(\alpha_{m,0}\nabla g(x_m)\big) + \sum_{j=1}^{N} \alpha_{m,j}\nabla g\big(T_j\big(x_m + e_m^j\big)\big) \\
&\le \alpha_{m,0}g(w) + \sum_{j=1}^{N} \alpha_{m,j}g(w) \\
&\quad + \alpha_{m,0}g^*\big(\nabla g(x_m)\big) + \sum_{j=1}^{N} \alpha_{m,j}g^*\big(\nabla g\big(T_j\big(x_m + e_m^j\big)\big)\big) \\
&= \alpha_{m,0}V\big(w, \nabla g(x_m)\big) + \sum_{j=1}^{N} \alpha_{m,j}V\big(w, \nabla g\big(T_j\big(x_m + e_m^j\big)\big)\big) \\
&= \alpha_{m,0}D_g(w, x_m) + \sum_{j=1}^{N} \alpha_{m,j}D_g\big(w, T_j\big(x_m + e_m^j\big)\big) \\
&\le \alpha_{m,0}D_g(w, x_m) + \sum_{j=1}^{N} \alpha_{m,j}D_g\big(w, x_m + e_m^j\big) \\
&= \alpha_{m,0}D_g(w, x_m) + \sum_{j=1}^{N} \alpha_{m,j}D_g(w, x_m) + \sum_{j=1}^{N} \alpha_{m,j}D_g\big(x_m, x_m + e_m^j\big) \\
&\quad + \sum_{j=1}^{N} \alpha_{m,j}\big\langle w - x_m, \nabla g(x_m) - \nabla g\big(x_m + e_m^j\big)\big\rangle \\
&= D_g(w, x_m) + \sum_{j=1}^{N} \alpha_{m,j}D_g\big(x_m, x_m + e_m^j\big) \\
&\quad + \sum_{j=1}^{N} \alpha_{m,j}\big\langle w - x_m, \nabla g(x_m) - \nabla g\big(x_m + e_m^j\big)\big\rangle. \qquad (4.2)
\end{aligned}$$

This proves that $w \in C_{m+1}$. Consequently, we see that $F \subset C_n$ for any $n \in \mathbb{N} \cup \{0\}$.

Step 3. We prove that $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ and $\{T_j(x_n + e_n^j) : n \in \mathbb{N}, j \in \{0, 1, 2, \ldots, N\}\}$ are bounded sequences in E.

In view of (1.9), we conclude that

$$\begin{aligned}
D_g(x_n, x) &= D_g\big(\operatorname{proj}^g_{C_n} x, x\big) \le D_g(w, x) - D_g(w, x_n) \\
&\le D_g(w, x), \quad \forall w \in F \subset C_n, n \in \mathbb{N} \cup \{0\}. \qquad (4.3)
\end{aligned}$$

It follows from (4.3) that the sequence $\{D_g(x_n, x)\}_{n\in\mathbb{N}}$ is bounded and hence there exists $M_0 > 0$ such that

$$D_g(x_n, x) \le M_0, \quad \forall n \in \mathbb{N} \cup \{0\}. \qquad (4.4)$$

In view of Lemma 2.1(3), we conclude that the sequence $\{x_n\}_{n\in\mathbb{N}}$ and hence $\{x_n + e_n^j : n \in \mathbb{N} \cup \{0\}, j \in \{0, 1, 2, \ldots, N\}\}$ is bounded. Since $\{T_j\}_{j=1}^N$ is a finite family of Bregman weak relatively nonexpansive mappings from E into $\operatorname{int}\operatorname{dom} g$, for any $q \in F$, we have

$$D_g\big(q, T_j\big(x_n + e_n^j\big)\big) \le D_g\big(q, x_n + e_n^j\big), \quad \forall n \in \mathbb{N} \text{ and } j \in \{0, 1, 2, \dots, N\}. \tag{4.5}$$

This, together with Definition 2.1 and the boundedness of $\{x_n\}_{n\in\mathbb{N}}$, implies that $\{T_j(x_n + e_n^j) : n \in \mathbb{N} \cup \{0\}, j \in \{0, 1, 2, \dots, N\}\}$ is bounded.

Step 4. We show that $x_n \to u$ for some $u \in F$, where $u = \operatorname{proj}_F^g x$.

By Step 3, we deduce that $\{x_n\}_{n\in\mathbb{N}}$ is bounded. By the construction of C_n, we conclude that $C_m \subset C_n$ and $x_m = \operatorname{proj}_{C_m}^g x \in C_m \subset C_n$ for any positive integer $m \ge n$. This, together with (1.9), implies that

$$\begin{aligned} D_g(x_m, x_n) &= D_g\big(x_m, \operatorname{proj}_{C_n}^g x\big) \le D_g(x_m, x) - D_g\big(\operatorname{proj}_{C_n}^g x, x\big) \\ &= D_g(x_m, x) - D_g(x_n, x). \end{aligned} \tag{4.6}$$

In view of (4.6), we have

$$D_g(x_n, x) \le D_g(x_n, x) + D_g(x_m, x_n) \le D_g(x_m, x), \quad \forall m \ge n.$$

This proves that $\{D_g(x_n, x)\}_{n\in\mathbb{N}}$ is an increasing sequence in $\mathbb{R}$ and hence by (4.4) the limit $\lim_{n\to\infty} D_g(x_n, x)$ exists. Letting $m, n \to \infty$ in (4.6), we deduce that $D_g(x_m, x_n) \to 0$. In view of Lemma 2.4, we obtain that $\|x_m - x_n\| \to 0$ as $m, n \to \infty$. Thus we have $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since E is a Banach space, we conclude that there exists $u \in E$ such that

$$\lim_{n\to\infty} \|x_n - u\| = 0. \tag{4.7}$$

Now, we show that $u \in F$. In view of (4.6), we obtain

$$\lim_{n\to\infty} D_g(x_{n+1}, x_n) = 0. \tag{4.8}$$

Since $\lim_{n\to\infty} e_n^j = 0$, for all $j \in \{0, 1, 2, \dots, N\}$, in view of Lemma 2.4 and (4.8), we obtain that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} D\big(x_n, x_n + e_n^j\big) = 0, \quad j \in \{0, 1, 2, \dots, N\}. \tag{4.9}$$

The function g is bounded on bounded subsets of E and thus ∇g is also bounded on bounded subsets of E^* (see, for example, [23, Proposition 1.1.11] for more details). Since $x_{n+1} \in C_{n+1}$, we get

$$\begin{aligned} D_g(x_{n+1}, y_n) &\le D_g(x_{n+1}, x_n) + \sum_{j=1}^{N} \alpha_{n,j} D_g\big(x_n, x_n + e_n^j\big) \\ &\quad + \sum_{j=1}^{N} \alpha_{n,j} \big\langle x_{n+1} - x_n, \nabla g(x_n) - \nabla g\big(x_n + e_n^j\big)\big\rangle. \end{aligned}$$

This, together with (4.9), implies that

$$\lim_{n\to\infty} D_g(x_{n+1}, y_n) = 0. \tag{4.10}$$

Employing Lemma 2.4 and (4.9)-(4.10), we deduce that

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{4.11}$$

In view of (4.7) and (4.11), we get

$$\lim_{n\to\infty} \|y_n - u\| = 0. \tag{4.12}$$

Thus, $\{y_n\}_{n\in\mathbb{N}}$ is a bounded sequence.

From (4.11) and (4.12), it follows that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Since ∇g is uniformly norm-to-norm continuous on any bounded subset of E, we obtain

$$\lim_{n\to\infty} \big\|\nabla g(x_n) - \nabla g(y_n)\big\| = 0. \tag{4.13}$$

Applying Lemma 2.4, we deduce that

$$\lim_{n\to\infty} D_g(y_n, x_n) = 0. \tag{4.14}$$

It follows from the three point identity (see (2.2)) that

$$\begin{aligned}
\big|D_g(w,x_n) - D_g(w,y_n)\big| &= \big|D_g(w,y_n) + D_g(y_n,x_n) \\
&\quad + \langle w - y_n, \nabla g(y_n) - \nabla g(x_n)\rangle - D_g(w,y_n)\big| \\
&= \big|D_g(y_n,x_n) - \langle w - y_n, \nabla g(y_n) - \nabla g(x_n)\rangle\big| \\
&\le D_g(y_n,x_n) + \|w - y_n\| \big\|\nabla g(y_n) - \nabla g(x_n)\big\| \\
&\to 0
\end{aligned} \tag{4.15}$$

as $n \to \infty$.

The function g is bounded on bounded subsets of E and thus ∇g is also bounded on bounded subsets of E^* (see, for example, [23, Proposition 1.1.11] for more details). This, together with Step 3, implies that the sequences $\{\nabla g(x_n)\}_{n\in\mathbb{N}}$, $\{\nabla g(y_n)\}_{n\in\mathbb{N}}$ and $\{\nabla g(T_j(x_n + e_n^j)) : n \in \mathbb{N}\cup\{0\}, j \in \{0,1,2,\dots,N\}\}$ are bounded in E^*.

In view of Theorem 2.2(3), we know that $\operatorname{dom} g^* = E^*$ and g^* is strongly coercive and uniformly convex on bounded subsets. Let $s = \sup\{\|\nabla g(T_j(x_n + e_n^j))\| : j \in \{0,1,2,\dots,N\}, n \in \mathbb{N}\cup\{0\}\}$ and let $\rho_s^* : E^* \to \mathbb{R}$ be the gauge of uniform convexity of the conjugate function g^*. Suppose that $i \in \mathbb{N}$ satisfies condition (2). We prove that for any $w \in F$ and $j \in \{0,1,2,\dots,N\}$,

$$\begin{aligned}
D_g(w,y_n) \le D_g(w,x_n) &+ \sum_{j=1}^{N} \alpha_{n,j} D_g\big(x_n, x_n + e_n^j\big) \\
&+ \sum_{j=1}^{N} \alpha_{n,j} \langle w - x_n, \nabla g(x_n) - \nabla g\big(x_n + e_n^j\big)\rangle \\
&- \alpha_{n,i}\alpha_{n,j}\rho_s^*\big(\big\|\nabla g\big(T_i\big(x_n + e_n^i\big)\big) - \nabla g\big(T_j\big(x_n + e_n^j\big)\big)\big\|\big).
\end{aligned} \tag{4.16}$$

Let us show (4.16). For any given $w \in F$ and $j \in \{0, 1, 2, \dots, N\}$, in view of the definition of the Bregman distance (see (1.7)), (1.6), Lemmas 2.3 and 2.5, we obtain

$$\begin{aligned}
D_g(w, y_n) &= D_g\left(w, \nabla g^*\left[\alpha_{n,0}\nabla g(x_n) + \sum_{j=1}^{N}\alpha_{n,j}\nabla g\big(T_j\big(x_n + e_n^j\big)\big)\right]\right) \\
&= V\left(w, \alpha_{n,0}\nabla g(x_n) + \sum_{j=1}^{N}\alpha_{n,j}\nabla g\big(T_j\big(x_n + e_n^j\big)\big)\right) \\
&= g(w) - \left\langle w, \alpha_{n,0}\nabla g(x_n) + \sum_{j=1}^{N}\alpha_{n,j}\nabla g\big(T_j\big(x_n + e_n^j\big)\big)\right\rangle \\
&\quad + g^*\big(\alpha_{n,0}\nabla g(x_n)\big) + \sum_{j=1}^{N}\alpha_{n,j}\nabla g\big(T_j\big(x_n + e_n^j\big)\big) \\
&\le \alpha_{n,0}g(w) + \sum_{j=1}^{N}\alpha_{n,j}g(w) - \alpha_{n,0}\big\langle w, \nabla g(x_n)\big\rangle - \sum_{j=1}^{N}\alpha_{n,j}\big\langle w, \nabla g\big(T_j\big(x_n + e_n^j\big)\big)\big\rangle \\
&\quad + \alpha_{n,0}g^*\big(\nabla g(x_n)\big) + \sum_{j=1}^{N}\alpha_{n,j}g^*\big(\nabla g\big(T_j\big(x_n + e_n^j\big)\big)\big) \\
&\quad - \alpha_{n,i}\alpha_{n,j}\rho_s^*\big(\big\|\nabla g\big(T_i\big(x_n + e_n^i\big)\big) - \nabla g\big(T_j\big(x_n + e_n^j\big)\big)\big\|\big) \\
&= \alpha_{n,0}V\big(w, \nabla g(x_n)\big) + \sum_{j=1}^{N}\alpha_{n,j}V\big(w, \nabla g\big(T_j\big(x_n + e_n^j\big)\big)\big) \\
&\quad - \alpha_{n,i}\alpha_{n,j}\rho_s^*\big(\big\|\nabla g\big(T_i\big(x_n + e_n^i\big)\big) - \nabla g\big(T_j\big(x_n + e_n^j\big)\big)\big\|\big) \\
&= \alpha_{n,0}D_g(w, x_n) + \sum_{j=1}^{N}\alpha_{n,j}D_g\big(w, T_j\big(x_n + e_n^j\big)\big) \\
&\quad - \alpha_{n,i}\alpha_{n,j}\rho_s^*\big(\big\|\nabla g\big(T_i\big(x_n + e_n^i\big)\big) - \nabla g\big(T_j\big(x_n + e_n^j\big)\big)\big\|\big) \\
&\le \alpha_{n,0}D_g(w, x_n) + \sum_{j=1}^{N}\alpha_{n,j}D_g\big(w, x_n + e_n^j\big) \\
&\quad - \alpha_{n,i}\alpha_{n,j}\rho_s^*\big(\big\|\nabla g\big(T_i\big(x_n + e_n^i\big)\big) - \nabla g\big(T_j\big(x_n + e_n^j\big)\big)\big\|\big) \\
&= \alpha_{n,0}D_g(w, x_n) + \sum_{j=1}^{N}\alpha_{n,j}D_g(w, x_n) + \sum_{j=1}^{N}\alpha_{n,j}D_g\big(x_n, x_n + e_n^j\big) \\
&\quad + \sum_{j=1}^{N}\alpha_{n,j}\big\langle w - x_n, \nabla g(x_n) - \nabla g\big(x_n + e_n^j\big)\big\rangle \\
&\quad - \alpha_{n,i}\alpha_{n,j}\rho_s^*\big(\big\|\nabla g\big(T_i\big(x_n + e_n^i\big)\big) - \nabla g\big(T_j\big(x_n + e_n^j\big)\big)\big\|\big) \\
&= D_g(w, x_n) + \sum_{j=1}^{N}\alpha_{n,j}D_g\big(x_n, x_n + e_n^j\big) \\
&\quad + \sum_{j=1}^{N}\alpha_{n,j}\big\langle w - x_n, \nabla g(x_n) - \nabla g\big(x_n + e_n^j\big)\big\rangle \\
&\quad - \alpha_{n,i}\alpha_{n,j}\rho_s^*\big(\big\|\nabla g\big(T_i\big(x_n + e_n^i\big)\big) - \nabla g\big(T_j\big(x_n + e_n^j\big)\big)\big\|\big).
\end{aligned}$$

Since $\lim_{n\to\infty}\|x_n-(x_n+e_n^j)\|=0$ for all $j\in\{0,1,2,\ldots,N\}$ and ∇g is uniformly norm-to-norm continuous on any bounded subset of E, we obtain

$$\lim_{n\to\infty}\left\|\nabla g(x_n)-\nabla g\left(x_n+e_n^j\right)\right\|=0,\quad \forall j\in\{0,1,2,\ldots,N\}.$$

This, together with (4.15), implies that

$$\begin{aligned}&D_g(w,x_n)-D_g(w,y_n)+\sum_{j=1}^{N}\alpha_{n,j}D_g\left(x_n,x_n+e_n^j\right)\\&\quad+\sum_{j=1}^{N}\alpha_{n,j}\left\langle w-x_n,\nabla g(x_n)-\nabla g\left(x_n+e_n^j\right)\right\rangle\to 0\quad\text{as } n\to\infty.\end{aligned}\tag{4.17}$$

In view of (4.16) and (4.17), we conclude that

$$\begin{aligned}&\alpha_{n,i}\alpha_{n,j}\rho_s^*\left(\left\|\nabla g\left(T_i\left(x_n+e_n^i\right)\right)-\nabla g\left(T_j\left(x_n+e_n^j\right)\right)\right\|\right)\\&\quad\le D_g(w,x_n)-D_g(w,y_n)+\sum_{j=1}^{N}\alpha_{n,j}D_g\left(x_n,x_n+e_n^j\right)\\&\qquad+\sum_{j=1}^{N}\alpha_{n,j}\left\langle x_{n+1}-x_n,\nabla g(x_n)-\nabla g\left(x_n+e_n^j\right)\right\rangle\\&\quad\to 0\end{aligned}$$

as $n\to\infty$. From the assumption $\liminf_{n\to\infty}\alpha_{n,i}\alpha_{n,j}>0$, $\forall j\in\{0,1,2,\ldots,N\}$, we have

$$\lim_{n\to\infty}\rho_s^*\left(\left\|\nabla g\left(T_i\left(x_n+e_n^i\right)\right)-\nabla g\left(T_j\left(x_n+e_n^j\right)\right)\right\|\right)=0,\quad\forall j\in\{0,1,2,\ldots,N\}.$$

Therefore, from the property of ρ_s^*, we deduce that

$$\lim_{n\to\infty}\left\|\nabla g\left(T_i\left(x_n+e_n^i\right)\right)-\nabla g\left(T_j\left(x_n+e_n^j\right)\right)\right\|=0,\quad\forall j\in\{0,1,2,\ldots,N\}.$$

Since ∇g^* is uniformly norm-to-norm continuous on bounded subsets of E^*, we arrive at

$$\lim_{n\to\infty}\left\|T_i\left(x_n+e_n^i\right)-T_j\left(x_n+e_n^j\right)\right\|=0,\quad\forall j\in\{0,1,2,\ldots,N\}.\tag{4.18}$$

In particular, for $j=0$, we have

$$\lim_{n\to\infty}\left\|T_i\left(x_n+e_n^i\right)-x_n\right\|=\lim_{n\to\infty}\left\|T_i\left(x_n+e_n^i\right)-\left(x_n+e_n^0\right)\right\|=0.$$

This, together with (4.7) and (4.18), implies that

$$\lim_{n\to\infty}\left\|T_j\left(x_n+e_n^j\right)-x_n+e_n^j\right\|=0,\quad\forall j\in\{0,1,2,\ldots,N\}.\tag{4.19}$$

From (4.7), we obtain

$$\lim_{n\to\infty}\left\|x_n+e_n^j-u\right\|=0,\quad\forall j\in\{0,1,2,\ldots,N\}.\tag{4.20}$$

In view of (4.19) and (4.20), we conclude that $T_j u = u$, $\forall j \in \{0, 1, 2, \dots, N\}$. Thus, we have $u \in F$.

Finally, we show that $u = \operatorname{proj}_F^g x$. From $x_n = \operatorname{proj}_{C_n}^g x$, we conclude that

$$\langle z - x_n, \nabla g(x_n) - \nabla g(x) \rangle \geq 0, \quad \forall z \in C_n.$$

Since $F \subset C_n$ for each $n \in \mathbb{N}$, we obtain

$$\langle z - x_n, \nabla g(x_n) - \nabla g(x) \rangle \geq 0, \quad \forall z \in F. \tag{4.21}$$

Letting $n \to \infty$ in (4.21), we deduce that

$$\langle z - u, \nabla g(u) - \nabla g(x) \rangle \geq 0, \quad \forall z \in F.$$

In view of (1.8), we have $u = \operatorname{proj}_F^g x$, which completes the proof. □

Remark 4.1 In Theorem 4.1, we present a strong convergence theorem for Bregman weak relatively nonexpansive mappings with a new algorithm and new control conditions. This is complementary to Reich and Sabach [46, Theorem 2]. It also extends and improves Theorems 1.3, 1.4 and 1.5.

5 Equilibrium problems

Let E be a Banach space and let C be a nonempty, closed and convex subset of a reflexive Banach space E. Let $f : C \times C \to \mathbb{R}$ be a bifunction. Consider the following equilibrium problem: Find $\bar{x} \in C$ such that

$$f(\bar{x}, y) \geq 0, \quad \forall y \in C. \tag{5.1}$$

In order to solve the equilibrium problem, let us assume that $f : C \times C \to \mathbb{R}$ satisfies the following conditions [53]:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, *i.e.*, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A3) f is upper hemi-continuous, *i.e.*, for each $x, y, z \in C$,

$$\limsup_{t \downarrow 0} f\big(tz + (1 - t)x, y\big) \leq f(x, y);$$

(A4) for each $x \in C$, the function $y \longmapsto f(x, y)$ is convex and lower semicontinuous.

The set of solutions of problem (5.1) is denoted by $\mathrm{EP}(f)$.

Let C be a nonempty, closed and convex subset of E and let $g : E \to \mathbb{R}$ be a Legendre function. For $r > 0$, we define a mapping $T_r : E \to C$ as follows:

$$T_r(x) = \left\{ z \in C : f(z, y) + \frac{1}{r} \langle y - z, \nabla g(z) - \nabla g(x) \rangle \geq 0 \text{ for all } y \in C \right\} \tag{5.2}$$

for all $x \in E$.

The following two lemmas were proved in [46].

Lemma 5.1 *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a Legendre function. Let C be a nonempty, closed and convex subset of E and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4). *For $r > 0$, let $T_r : E \to C$ be the mapping defined by* (5.2). *Then* $\operatorname{dom}(T_r) = E$.

Lemma 5.2 *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a convex, continuous and strongly coercive function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of E and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4). *For $r > 0$, let $T_r : E \to C$ be the mapping defined by* (5.2). *Then the following statements hold*:

(1) *T_r is single-valued*;

(2) *T_r is a Bregman firmly nonexpansive mapping* [46], *i.e., for all $x, y \in E$,*

$$\langle T_r x - T_r y, \nabla g(T_r x) - \nabla g(T_r y)\rangle \leq \langle T_r x - T_r y, \nabla g(x) - \nabla g(y)\rangle;$$

(3) $F(T_r) = \mathrm{EP}(f)$;

(4) $\mathrm{EP}(f)$ *is closed and convex*;

(5) *T_r is a Bregman quasi-nonexpansive mapping*;

(6) $D_g(q, T_r x) + D_g(T_r x, x) \leq D_g(q, x)$, $\forall q \in F(T_r)$.

Theorem 5.1 *Let E be a reflexive Banach space and let $g : E \to \mathbb{R}$ be a strongly coercive Bregman function which is bounded on bounded subsets and uniformly convex and uniformly smooth on bounded subsets of E. Let f be a bifunction from $E \times E$ to $\mathbb{R}$ satisfying* (A1)-(A4). *Let $N \in \mathbb{N}$ and let $\{T_j\}_{j=1}^N$ be a finite family of Bregman weak relatively nonexpansive mappings from E into $\operatorname{int}\operatorname{dom} g$ such that $F := \bigcap_{j=1}^N F(T_j)$ is a nonempty subset of E. Suppose in addition that $T_0 = I$, where I is the identity mapping on E. Suppose that $F \cap \mathrm{EP}(f)$ is a nonempty subset of E, where $\mathrm{EP}(f)$ is the set of solutions to the equilibrium problem* (5.1). *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by*

$$\begin{cases} x_0 = x \in E \quad \text{chosen arbitrarily}, \\ C_0 = E, \\ y_n = \nabla g^*[\alpha_{n,0}\nabla g(x_n) + \sum_{j=1}^N \alpha_{n,j}\nabla g(T_j(x_n + e_n^j))], \\ u_n \in E \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, \nabla g(u_n) - \nabla g(y_n)\rangle \geq 0, \forall y \in E, \\ C_{n+1} = \{z \in C_n : D_g(z, u_n) \leq D_g(z, x_n) + \sum_{j=1}^N \alpha_{n,j} D_g(x_n, x_n + e_n^j) \\ \qquad + \sum_{j=1}^N \alpha_{n,j}\langle z - x_n, \nabla g(x_n) - \nabla g(x_n + e_n^j)\rangle\}, \\ x_{n+1} = \operatorname{proj}_{C_{n+1}}^g x \quad \text{and} \quad n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{5.3}$$

where ∇g is the right-hand derivative of g. Let $\{\alpha_{n,j} : n \in \mathbb{N} \cup \{0\}, j \in \{0, 1, 2, \ldots, N\}\}$ be a sequence in $(0, 1)$ *satisfying the following control conditions*:

(1) $\sum_{j=0}^N \alpha_{n,j} = 1$, $\forall n \in \mathbb{N} \cup \{0\}$;

(2) *There exists $i \in \{1, 2, \ldots, N\}$ such that* $\liminf_{n\to\infty} \alpha_{n,i}\alpha_{n,j} > 0$, $\forall j \in \{0, 1, 2, \ldots, N\}$.

If, for each $j = 0, 1, 2, \ldots, N$, the sequences of errors $\{e_n^j\}_{n\in\mathbb{N}} \subset E$ satisfy $\liminf_{n\to\infty} e_n^j = 0$; *then the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined in* (5.3) *converges strongly to* $\operatorname{proj}_{F\cap\mathrm{EP}(f)}^g x$ *as $n \to \infty$.*

Proof By the same argument, as in the proof of Theorem 4.1, we can prove the following:

(i) $\lim_{n\to\infty}\|x_n-u_n\|=0$ and $\lim_{n\to\infty}\|\nabla g(x_n)-\nabla g(u_n)\|=0$.
(ii) For each $w\in F$, $\lim_{n\to\infty}|D_g(w,x_n)-D_g(w,u_n)|=0$.
(iii) There exists $u\in F$ such that $x_n\to u$ as $n\to\infty$.

Since $u_n=T_{r_n}y_n$, for any $w\in F$, we have

$$\begin{aligned} D_g(w,u_n) &= D_g(w,T_{r_n}y_n) \\ &\le D_g(w,y_n) \\ &\le D_g(w,x_n)+\sum_{j=1}^N \alpha_{n,j}D_g\bigl(x_n,x_n+e_n^j\bigr) \\ &\quad+\sum_{j=1}^N \alpha_{n,j}\bigl\langle w-x_n,\nabla g(x_n)-\nabla g\bigl(x_n+e_n^j\bigr)\bigr\rangle. \end{aligned} \tag{5.4}$$

Next, we show that $u\in \mathrm{EP}(f)$. From Lemma 5.2(6), (5.4) and $u_n=T_{r_n}y_n$, we conclude that

$$\begin{aligned} D_g(u_n,y_n) &= D_g(T_{r_n}y_n,y_n) \\ &\le D_g(w,y_n)-D_g(w,T_{r_n}y_n) \\ &\le D_g(w,x_n)-D_g(w,u_n)+\sum_{j=1}^N \alpha_{n,j}D_g\bigl(x_n,x_n+e_n^j\bigr) \\ &\quad+\sum_{j=1}^N \alpha_{n,j}\bigl\langle w-x_n,\nabla g(x_n)-\nabla g\bigl(x_n+e_n^j\bigr)\bigr\rangle \\ &\to 0 \end{aligned} \tag{5.5}$$

as $n\to\infty$. In view of (5.5) and Lemma 2.4, we obtain

$$\lim_{n\to\infty}\|u_n-y_n\|=0. \tag{5.6}$$

Since ∇g is uniformly norm-to-norm continuous on any bounded subset of E, it follows from (5.6) that

$$\lim_{n\to\infty}\bigl\|\nabla g(u_n)-\nabla g(y_n)\bigr\|=0.$$

By the assumption $r_n\ge a$, we have

$$\lim_{n\to\infty}\frac{\|\nabla g(u_n)-\nabla g(y_n)\|}{r_n}=0. \tag{5.7}$$

In view of $u_n=T_{r_n}y_n$, we obtain

$$f(u_n,y)+\frac{1}{r_n}\bigl\langle y-u_n,\nabla g(u_n)-\nabla g(y_n)\bigr\rangle\ge 0,\quad \forall y\in E.$$

From condition (A2), we deduce that

$$\begin{aligned}\|y-u_n\|\frac{\|\nabla g(u_n)-\nabla g(y_n)\|}{r_n} &\geq \frac{1}{r_n}\langle y-u_n, \nabla g(u_n)-\nabla g(y_n)\rangle \\ &\geq -f(u_n,y)\geq f(y,u_n)\geq 0, \quad \forall y\in E.\end{aligned}$$

Letting $n\to\infty$ in the above inequality, we have from (5.7) and (A4) that

$$f(y,u)\leq 0, \quad \forall y\in E.$$

For $t\in(0,1]$ and $y\in E$, let $y_t=ty+(1-t)u$. Then we have $y_t\in E$, which yields that $f(y_t,u)\leq 0$. From (A1), we also have

$$0=f(y_t,y_t)\leq tf(y_t,y)+(1-t)f(y_t,u)\leq tf(y_t,y).$$

Dividing by t, we get

$$f(y_t,y)\geq 0, \quad \forall y\in E.$$

Letting $t\downarrow 0$, from the condition (A3), we obtain that

$$f(u,y)\geq 0, \quad \forall y\in E.$$

This means that $u\in \mathrm{EP}(f)$. Therefore, $u\in F\cap \mathrm{EP}(f)$. □

Theorem 5.2 *Let E be a 2-uniformly convex Banach space and let $g:E\to\mathbb{R}$ be a strongly coercive Bregman function which is bounded on bounded subsets and uniformly convex and uniformly smooth on bounded subsets of E. Assume that there exists $c_1>0$ such that g is ρ-convex with $\rho(t):=\frac{c_1}{2}t^2$ for all $t\geq 0$. Let C be a nonempty, closed and convex subset of E and let f be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying* (A1)-(A4). *Assume that $\{T_j\}_{j\in\mathbb{N}}$ is an infinite family of Bregman weak relatively nonexpansive mappings from C into itself and that $A:C\to E^*$ is a γ-inverse strongly monotone mapping for some $\gamma>0$. Suppose that $F:=\bigcap_{j=1}^{\infty}F(T_j)\cap A^{-1}(0)\cap \mathrm{EP}(f)$ is a nonempty subset of C, where $\mathrm{EP}(f)$ is the set of solutions to the equilibrium problem* (5.1). *Suppose in addition that $T_0=I$, where I is the identity mapping on E. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by*

$$\begin{cases} x_0=x\in C \quad \text{chosen arbitrarily}, \\ C_0=C, \\ y_n=\mathrm{proj}_C^g(\nabla g^*[\nabla g(x_n)-\beta Ax_n]), \\ z_n=\nabla g^*[\alpha_{n,0}\nabla g(x_n)+\sum_{j=1}^{\infty}\alpha_{n,j}\nabla g(T_j y_n)], \\ u_n\in C \quad \text{such that } f(u_n,y)+\frac{1}{r_n}\langle y-u_n,\nabla g(u_n)-\nabla g(y_n)\rangle\geq 0, \forall y\in C, \\ C_{n+1}=\{z\in C_n: D_g(z,u_n)\leq D_g(z,x_n)\}, \\ x_{n+1}=\mathrm{proj}_{C_{n+1}}^g x \quad \text{and} \quad n\in\mathbb{N}\cup\{0\}, \end{cases} \tag{5.8}$$

where ∇g is the right-hand derivative of g. Let β be a constant such that $0<\beta<\frac{c_2^2\gamma}{2}$, where c_2 is the 2-uniformly convex constant of E satisfying Corollary 2.1(2). *Let $\{\alpha_{n,j}:n\in\mathbb{N}\cup\{0\}, j\in\mathbb{N}\cup\{0\}\}$ be a sequence in $(0,1)$ satisfying the following control conditions*:

(1) $\sum_{j=0}^{\infty} \alpha_{n,j} = 1$, $\forall n \in \mathbb{N} \cup \{0\}$;
(2) $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,j} > 0$, $\forall j \in \mathbb{N}$.

Then the sequence $\{x_n\}_{n\in\mathbb{N}}$ *defined in* (5.8) *converges strongly to* $\operatorname{proj}_F^g x$ *as* $n \to \infty$.

Proof We divide the proof into several steps.

Step 1. Following the method of the proof of Theorem 3.1 Step 1, we obtain that C_n is both closed and convex for each $n \in \mathbb{N} \cup \{0\}$.

Step 2. We claim that $F \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$.

It is obvious that $F \subset C_0 = C$. Assume now that $F \subset C_m$ for some $m \in \mathbb{N}$. It follows from Lemma 2.5 that, for each $w \in F \subset C_m$, we have

$$\begin{aligned}
D_g(w, y_m) &= D_g\bigl(w, \operatorname{proj}_C^g\bigl(\nabla g^*\bigl[\nabla g(x_m) - \beta A x_m\bigr]\bigr)\bigr) \\
&\le D_g\bigl(w, \nabla g^*\bigl[\nabla g(x_m) - \beta A x_m\bigr]\bigr) \\
&= V\bigl(w, \nabla g(x_m) - \beta A x_m\bigr) \\
&\le V\bigl(w, \nabla g(x_m) - \beta A x_m + \beta A x_m\bigr) - \bigl\langle \nabla g^*\bigl(\nabla g(x_m) - \beta A x_m\bigr) - w, \beta A x_m\bigr\rangle \\
&= V\bigl(w, \nabla g(x_m)\bigr) - \beta\bigl\langle \nabla g^*\bigl(\nabla g(x_m) - \beta A x_m\bigr) - w, A x_m\bigr\rangle \\
&= D_g(w, x_m) - \beta\langle x_m - w, A x_m\rangle - \beta\bigl\langle \nabla g^*\bigl(\nabla g(x_m) - \beta A x_m\bigr) - x_m, A x_m\bigr\rangle \\
&\le D_g(w, x_m) - \beta\gamma\|A x_m\|^2 + \beta\bigl\|\nabla g^*\bigl(\nabla g(x_m) - \beta A x_m\bigr) - \nabla g^*\nabla g(x_m)\bigr\|\|A x_m\| \\
&\le D_g(w, x_m) - \beta\gamma\|A x_m\|^2 + \frac{4\beta^2}{c_2^2}\|A x_m\|^2 \\
&\le D_g(w, x_m) + \beta\left(\frac{4\beta}{c_2^2} - \gamma\right)\|A x_m\|^2.
\end{aligned} \tag{5.9}$$

This, together with $\frac{4\beta}{c_2^2} - \gamma < 0$, implies that

$$D_g(w, y_m) \le D_g(w, x_m).$$

Since T_j is Bregman weak relatively nonexpansive, for each $j \in \mathbb{N}$, we obtain

$$\begin{aligned}
D_g(w, u_m) &= D_g(w, T_{r_m} z_m) \\
&\le D_g(w, z_m) \\
&= D_g\left(w, \nabla g^*\left[\alpha_{m,0}\nabla g(x_m) + \sum_{j=1}^{\infty}\alpha_{m,j}\nabla g(T_j y_m)\right]\right) \\
&= V\left(w, \alpha_{m,0}\nabla g(x_m) + \sum_{j=1}^{\infty}\alpha_{m,j}\nabla g(T_j y_m)\right) \\
&= g(w) - \left\langle w, \alpha_{m,0}\nabla g(x_m) + \sum_{j=1}^{\infty}\alpha_{m,j}\nabla g(T_j y_m)\right\rangle \\
&\quad + g^*\left(\alpha_{m,0}\nabla g(x_m) + \sum_{j=1}^{\infty}\alpha_{m,j}\nabla g(T_j y_m)\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_{m,0} g(w) + \sum_{j=1}^{\infty} \alpha_{m,j} g(w) \\
&\quad + \alpha_{m,0} g^*\big(\nabla g(x_m)\big) + \sum_{j=1}^{\infty} \alpha_{m,j} g^*\big(\nabla g(T_j y_m)\big) \\
&= \alpha_{m,0} V\big(w, \nabla g(x_m)\big) + \sum_{j=1}^{\infty} \alpha_{m,j} V\big(w, \nabla g(T_j y_m)\big) \\
&= \alpha_{m,0} D_g(w, x_m) + \sum_{j=1}^{\infty} \alpha_{m,j} D_g(w, T_j y_m) \\
&\leq \alpha_{m,0} D_g(w, x_m) + \sum_{j=1}^{\infty} \alpha_{m,j} D_g(w, y_m) \\
&\leq D_g(w, x_m).
\end{aligned}$$

This proves that $w \in C_{m+1}$. Consequently, we see that $F \subset C_n$ for any $n \in \mathbb{N} \cup \{0\}$.

Step 3. By the same manner, as mentioned in the proof of Theorem 3.1, Step 3, we can prove that the sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$, $\{z_n\}_{n\in\mathbb{N}}$, $\{u_n\}_{n\in\mathbb{N}}$ and $\{T_j y_n : j \in \mathbb{N} \cup \{0\}, n \in \mathbb{N} \cup \{0\}\}$ are bounded.

Step 4. We show that $x_n \to u$ for some $u \in F$, where $u = \mathrm{proj}_F^g x$.

A similar argument, as mentioned in Theorem 3.1, Step 4, shows that there exists $u \in C$ such that

$$\lim_{n\to\infty} \|x_n - u\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{5.10}$$

In view of Lemma 2.4, we deduce that

$$\lim_{n\to\infty} D_g(u_n, x_n) = 0.$$

Since ∇g is uniformly norm-to-norm continuous on any bounded subset of E, we obtain

$$\lim_{n\to\infty} \big\|\nabla g(u_n) - \nabla g(x_n)\big\| = 0.$$

It follows from the three point identity (see (2.2)) that

$$\begin{aligned}
\big|D_g(w, x_n) - D_g(w, u_n)\big| &= \big|D_g(w, u_n) + D_g(u_n, x_n) \\
&\quad + \big\langle w - u_n, \nabla g(u_n) - \nabla g(x_n)\big\rangle - D_g(w, u_n)\big| \\
&= \big|D_g(u_n, x_n) - \big\langle w - u_n, \nabla g(u_n) - \nabla g(x_n)\big\rangle\big| \\
&\leq D_g(u_n, x_n) + \|w - u_n\| \big\|\nabla g(u_n) - \nabla g(x_n)\big\| \\
&\to 0
\end{aligned} \tag{5.11}$$

as $n \to \infty$. The function g is bounded on bounded subsets of E and thus ∇g is also bounded on bounded subsets of E^* (see, for example, [23, Proposition 1.1.11] for more details). This, together with Step 3, implies that the sequences $\{\nabla g(x_n)\}_{n\in\mathbb{N}}$, $\{\nabla g(y_n)\}_{n\in\mathbb{N}}$, $\{\nabla g(z_n)\}_{n\in\mathbb{N}}$ and $\{\nabla g(T_j x_n) : j \in \mathbb{N} \cup \{0\}, n \in \mathbb{N} \cup \{0\}\}$ are bounded in E^*.

In view of Theorem 2.2(3), we know that $\operatorname{dom} g^* = E^*$ and g^* is strongly coercive and uniformly convex on bounded subsets. Let $s = \sup\{\|\nabla g(x_n)\|, \|\nabla g(T_j x_n)\| : j \in \mathbb{N} \cup \{0\}, n \in \mathbb{N} \cup \{0\}\}$ and $\rho_s^* : E^* \to \mathbb{R}$ be the gauge of uniform convexity of the conjugate function g^*. For any given $w \in F(T)$ and $j \in \mathbb{N}$, in view of the definition of the Bregman distance (see (1.7)), (1.6), Lemmas 2.3 and 2.5, we obtain

$$\begin{aligned}
D_g(w, u_n) &= D_g(w, T_{r_n} z_n) \\
&\leq D_g(w, z_n) \\
&= D_g\left(w, \nabla g^*\left[\alpha_{n,0} \nabla g(x_n) + \sum_{j=1}^{\infty} \alpha_{n,j} \nabla g(T_j y_n)\right]\right) \\
&= V\left(w, \alpha_{n,0} \nabla g(x_n) + \sum_{j=1}^{\infty} \alpha_{n,j} \nabla g(T_j y_n)\right) \\
&= g(w) - \left\langle w, \alpha_{n,0} \nabla g(x_n) + \sum_{j=1}^{\infty} \alpha_{n,j} \nabla g(T_j y_n)\right\rangle \\
&\quad + g^*\left(\alpha_{n,0} \nabla g(x_n) + \sum_{j=1}^{\infty} \alpha_{n,j} \nabla g(T_j y_n)\right) \\
&\leq \alpha_{n,0} g(w) + \sum_{j=1}^{\infty} \alpha_{n,j} g(w) \\
&\quad - \alpha_{n,0} \langle w, \nabla g(x_n)\rangle - \sum_{j=1}^{\infty} \alpha_{n,j} \langle w, \nabla g(T_j y_n)\rangle \\
&\quad + \alpha_{n,0} g^*\big(\nabla g(x_n)\big) + \sum_{j=1}^{\infty} \alpha_{n,j} g^*\big(\nabla g(T_j y_n)\big) \\
&\quad - \alpha_{n,i} \alpha_{n,j} \rho_s^*\big(\big\|\nabla g(x_n) - \nabla g(T_j y_n)\big\|\big) \\
&= \alpha_{n,0} V\big(w, \nabla g(x_n)\big) + \sum_{j=1}^{\infty} \alpha_{n,j} V\big(w, \nabla g(T_j y_n)\big) \\
&\quad - \alpha_{n,0} \alpha_{n,j} \rho_s^*\big(\big\|\nabla g(x_n) - \nabla g(T_j y_n)\big\|\big) \\
&= \alpha_{n,0} D_g(w, x_n) + \sum_{j=1}^{\infty} \alpha_{n,j} D_g(w, T_j y_n) \\
&\quad - \alpha_{n,0} \alpha_{n,j} \rho_s^*\big(\big\|\nabla g(x_n) - \nabla g(T_j y_n)\big\|\big) \\
&= \alpha_{n,0} D_g(w, x_n) + \sum_{j=1}^{\infty} \alpha_{n,j} D_g(w, y_n) \\
&\quad - \alpha_{n,0} \alpha_{n,j} \rho_s^*\big(\big\|\nabla g(x_n) - \nabla g(T_j y_n)\big\|\big) \\
&\leq \alpha_{n,0} D_g(w, x_n) + \sum_{j=1}^{\infty} \alpha_{n,j} D_g(w, x_n) \\
&\quad - \alpha_{n,0} \alpha_{n,j} \rho_s^*\big(\big\|\nabla g(x_n) - \nabla g(T_j y_n)\big\|\big) \\
&= D_g(w, x_n) - \alpha_{n,0} \alpha_{n,j} \rho_s^*\big(\big\|\nabla g(x_n) - \nabla g(T_j x_n)\big\|\big).
\end{aligned} \tag{5.12}$$

In view of (5.10), (5.11) and (5.12), we conclude that

$$\begin{aligned}&\alpha_{n,0}\alpha_{n,j}\rho_s^*\big(\big\|\nabla g(x_n)-\nabla g(T_jy_n)\big\|\big)\\&\quad\le D_g(w,x_n)-D_g(w,z_n)\\&\quad\to 0\end{aligned}$$

as $n\to\infty$. From the assumption $\liminf_{n\to\infty}\alpha_{n,0}\alpha_{n,j}>0$, $\forall j\in\mathbb{N}$, we have

$$\lim_{n\to\infty}\rho_s^*\big(\big\|\nabla g(x_n)-\nabla g(T_jy_n)\big\|\big)=0,\quad \forall j\in\mathbb{N}.$$

Therefore, from the property of ρ_s^*, we deduce that

$$\lim_{n\to\infty}\big\|\nabla g(x_n)-\nabla g(T_jy_n)\big\|=0,\quad \forall j\in\mathbb{N}.$$

Since ∇g^* is uniformly norm-to-norm continuous on bounded subsets of E^*, we arrive at

$$\lim_{n\to\infty}\|x_n-T_jy_n\|=0,\quad \forall j\in\mathbb{N}. \tag{5.13}$$

Using inequalities (5.9) and (5.12), we obtain

$$\begin{aligned}D_g(w,u_n)&\le\alpha_{n,0}D_g(w,x_n)+\sum_{j=1}^{\infty}\alpha_{n,j}D_g(w,y_n)\\&\le\alpha_{n,0}D_g(w,x_n)+\sum_{j=1}^{\infty}\alpha_{n,j}\biggl[D_g(w,x_n)+\beta\biggl(\frac{4\beta}{c_2^2}-\gamma\biggr)\|Ax_n\|^2\biggr]\\&=D_g(w,x_n)+\beta\sum_{j=1}^{\infty}\alpha_{n,j}\biggl(\frac{4\beta}{c_2^2}-\gamma\biggr)\|Ax_n\|^2.\end{aligned}\tag{5.14}$$

It follows from (5.14) that

$$\beta\sum_{j=1}^{\infty}\alpha_{n,j}\biggl(\gamma-\frac{4\beta}{c_2^2}\biggr)\|Ax_n\|^2\le D_g(w,x_n)-D_g(w,u_n).$$

Since $\frac{4\beta}{c_2^2}-\gamma<0$, we see that

$$\lim_{n\to\infty}\|Ax_n\|=0. \tag{5.15}$$

Furthermore, since $x_n\in C$ for all $n\ge 0$, then using (1.6), Lemma 2.5 and Corollary 2.1, we get

$$\begin{aligned}D_g(x_n,y_n)&=D_g\big(x_n,\operatorname{proj}_C^g\big(\nabla g^*\big[\nabla g(x_n)-\beta Ax_n\big]\big)\big)\\&\le D_g\big(x_n,\nabla g^*\big[\nabla g(x_n)-\beta Ax_n\big]\big)\\&=V\big(x_n,\nabla g(x_n)-\beta Ax_n\big)\\&\le V\big(x_n,\nabla g(x_n)-\beta Ax_n+\beta Ax_n\big)-\big\langle\nabla g^*\big(\nabla g(x_n)-\beta Ax_n\big)-x_n,\beta Ax_n\big\rangle\end{aligned}$$

$$
\begin{aligned}
&= V\big(x_n, \nabla g(x_n)\big) - \beta\big\langle \nabla g^*\big(\nabla g(x_n) - \beta A x_n\big) - w, A x_n\big\rangle \\
&= D_g(x_n, x_n) - \beta\langle x_n - x_n, A x_n\rangle - \beta\big\langle \nabla g^*\big(\nabla g(x_n) - \beta A x_n\big) - x_n, A x_n\big\rangle \\
&\le \beta\big\| \nabla g^*\big(\nabla g(x_n) - \beta A x_n\big) - \nabla g^* \nabla g(x_n)\big\| \|A x_n\| \\
&\le \frac{4\beta^2}{c_2^2}\|A x_n\|^2.
\end{aligned}
$$

It follows from (5.15) that

$$\lim_{n\to\infty} D_g(x_n, y_n) = 0.$$

Lemma 2.2 now implies that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{5.16}$$

Using (5.13) and (5.16), we conclude that

$$\lim_{n\to\infty} \|y_n - u\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|y_n - T_j y_n\| = 0, \quad \forall j \in \mathbb{N}. \tag{5.17}$$

Therefore, $u \in \tilde{F}(T_j) = F(T_j)$, $\forall j \in \mathbb{N}$.

Step 5. We show that $u \in A^{-1}(0)$.

Since A is γ-inverse strongly monotone, it is continuous and hence, using (5.16) and (5.17), we conclude that $Au = \lim_{n\to\infty} A x_n = 0$. Therefore, $u \in A^{-1}(0)$.

Step 6. Finally, we show that $u = \operatorname{proj}_F^g x$.

The proof of this step is similar to that of Theorem 3.1, Step 4 and is omitted here. □

We end this section with the following simple example in order to support Theorem 5.2.

Example 5.1 Let $E = l^2$ and

$$
\begin{aligned}
&x_0 = (1, 0, 0, 0, \ldots), \\
&x_1 = (1, 1, 0, 0, 0, \ldots), \\
&x_2 = (1, 0, 1, 0, 0, 0, \ldots), \\
&x_3 = (1, 0, 0, 1, 0, 0, 0, \ldots), \\
&\cdots, \\
&x_n = (\sigma_{n,1}, \sigma_{n,2}, \ldots, \sigma_{n,k}, \ldots), \\
&\cdots,
\end{aligned}
$$

where

$$\sigma_{n,k} = \begin{cases} 1 & \text{if } k = 1, n+1, \\ 0 & \text{if } k \neq 1, k \neq n+1 \end{cases}$$

for all $n \in \mathbb{N}$. It is easy to see that the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to x_0. Let k be an even number in $\mathbb{N}$ and let $g : E \to \mathbb{R}$ be defined by

$$g(x) = \frac{1}{k}\|x\|^k, \quad x \in E.$$

It is easy to show that $\nabla g(x) = J_k(x)$ for all $x \in E$, where

$$J_k(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|\|x^*\|, \|x^*\| = \|x\|^{k-1}\}.$$

It is also obvious that

$$J_k(\lambda x) = \lambda^{k-1} J_k(x), \quad \forall x \in E, \lambda \in \mathbb{R}.$$

Now, we define a countable family of mappings $T_j : E \to E$ by

$$T_j(x) = \begin{cases} \frac{n}{n+1}x & \text{if } x = x_n; \\ \frac{-x}{j} & \text{if } x \neq x_n \end{cases}$$

for all $j \geq 1$ and $n \geq 0$. It is clear that $F(T_j) = \{0\}$ for all $j \geq 1$. Choose $j \in \mathbb{N}$, then for any $n \in \mathbb{N}$,

$$\begin{aligned} D_g(0, T_j x_n) &= g(0) - g(T_j x_n) - \langle 0 - T_j x_n, \nabla g(T_j x_n)\rangle \\ &= -\frac{n^k}{(n+1)^k} g(x_n) + \frac{n^k}{(n+1)^k}\langle x_n, \nabla g(x_n)\rangle \\ &= \frac{n^k}{(n+1)^k}\left[-g(x_n) + \langle x_n, \nabla g(x_n)\rangle\right] \\ &= \frac{n^k}{(n+1)^k}\left[D_g(0, x_n)\right] \\ &\leq D_g(0, x_n). \end{aligned}$$

If $x \neq x_n$, then we have

$$\begin{aligned} D_g(0, T_j x) &= g(0) - g(T_j x) - \langle 0 - T_j x, \nabla g(T_j x)\rangle \\ &= -\frac{1}{j^k} g(x) - \frac{1}{j^k}\langle x, -\nabla g(x)\rangle \\ &= \frac{1}{j^k}\left[-g(x) - \langle -x, \nabla g(x)\rangle\right] \\ &\leq D_g(0, x). \end{aligned}$$

Therefore, T_j is a Bregman quasi-nonexpansive mapping. Next, we claim that T_j is a Bregman weak relatively nonexpansive mapping. Indeed, for any sequence $\{z_n\}_{n\in\mathbb{N}} \subset E$ such that $z_n \to z_0$ and $\|z_n - T_j z_n\| \to 0$ as $n \to \infty$, there exists a sufficiently large number $N_0 \in \mathbb{N}$ such that $z_n \neq x_m$ for any $n, m > N_0$. This implies that $T_j z_n = -\frac{z_n}{j}$ for all $n > N_0$. It follows from $\|z_n - T_j z_n\| \to 0$ that $\frac{j+1}{j} z_n \to 0$ and hence $z_n \to z_0 = 0$. Since $z_0 \in F(T_j)$, we conclude that T_j is a Bregman weak relatively nonexpansive mapping. It is clear that $\bigcap_{j=1}^{\infty} \tilde{F}(T_j) = \bigcap_{j=1}^{\infty} F(T_j) = \{0\}$. Thus $\{T_j\}_{j\in\mathbb{N}}$ is a countable family of Bregman weak relatively nonexpansive mappings.

Next, we show that $\{T_j\}_{j\in\mathbb{N}}$ is not a countable family of Bregman relatively nonexpansive mappings. In fact, though $x_n \rightharpoonup x_0$ and

$$\|x_n - T_j x_n\| = \left\| x_n - \frac{n}{n+1} x_n \right\| = \frac{1}{n+1} \|x_n\| \to 0$$

as $n \to \infty$, but $x_0 \notin F(T_j)$ for all $j \in \mathbb{N}$. Therefore, $\hat{F}(T_j) \neq F(T_j)$ for all $j \in \mathbb{N}$. This implies that $\bigcap_{j=1}^{\infty} \hat{F}(T_j) \neq \bigcap_{j=1}^{\infty} F(T_j)$.

Finally, it is obvious that the family $\{T_j\}_{j\in\mathbb{N}}$ satisfies all the aspects of the hypothesis of Theorem 5.2.

6 Applications (Hammerstein-type equations)

Let E be a real Banach space with the dual space E^*. The generalized formulation of many boundary value problems for ordinary and partial differential equations leads to operator equations of the type

$$\langle z, Ax \rangle = \langle z, b \rangle, \quad \forall z \in E,$$

which is equivalent to equality of functionals on E. That is, the equality of the form

$$Ax = b, \tag{6.1}$$

where A is a monotone-type operator acting from a Banach space E into E^*. Without loss of generality, we may assume $b = 0$. It is well known that a solution of the equation $Ax = 0$ (*i.e.*, $\langle z, Ax \rangle = 0$, $\forall z \in E$) is a solution of the variational inequality $\langle z - x, Ax \rangle \geq 0$, $\forall z \in E$. Therefore, the theory of monotone operators and its applications to nonlinear partial differential equations and variational inequalities are related and have been involved in a substantial topic in nonlinear functional analysis. One important application of solving (6.1) is finding the zeros of the so-called equation of Hammerstein type (see, *e.g.*, [54]), where a nonlinear integral equation of Hammerstein type is one of the form

$$u(x) + \int_{\Omega} k(x,y) f\big(y, u(y)\big)\, dy = h(x), \tag{6.2}$$

where dy is a σ-finite measure on the measure space Ω; the real kernel k is defined on $\Omega \times \Omega$, f is a real-valued function defined on $\Omega \times \mathbb{R}$ and is, in general, nonlinear and h is a given function on Ω. If we now define an operator K by $Kv(x) = \int_{\Omega} k(x,y) v(y)\, dy$; $x \in \Omega$, and the so-called superposition or Nemytskii operator by $Qu(y) := f(y, u(y))$, then the integral Eq. (6.2) can be put in operator theoretic form as follows:

$$u + KQu = 0, \tag{6.3}$$

where, without loss of generality, we have taken $h = 0$.

Interest in Eq. (6.2) stems mainly from the fact that several problems that arise in differential equations, for instance, elliptic boundary value problems, whose linear parts possess Green's functions, can, as a rule, be transformed into equations of the form (6.2) (see, *e.g.*, [55], Chapter IV). Equations of Hammerstein type play a crucial role in the theory of optimal control systems (see, *e.g.*, [56]). Several existence and uniqueness theorems have been proved for equations of Hammerstein type (see, *e.g.*, [57–62]). Very recently, Ofoedu and Malonza in [63] proposed an iterative solution of the operator Hammerstein Eq. (6.1) in a 2-uniformly convex and uniformly smooth Banach space.

Now, we give an application of Theorem 5.1 to an iterative solution of the operator Hammerstein Eq. (6.1).

Theorem 6.1 *Let E be a real Banach space with a dual space E^* such that $X = E \times E^*$ (with the norm $\|z\|_X^2 = \|u\|_E^2 + \|v\|_{E^*}^2$, $z = (u,v) \in X$) is a 2-uniformly convex and uniformly smooth real Banach space. Let $g : X \to \mathbb{R}$ be a strongly coercive Bregman function which is bounded on bounded subsets and uniformly convex and uniformly smooth on bounded subsets of X. Assume that there exists $c_1 > 0$ such that g is ρ-convex with $\rho(t) := \frac{c_1}{2}t^2$ for all $t \geq 0$. Let $Q : E \to E^*$ and $K : E^* \to E$ with $\operatorname{dom} K = Q(E) = E^*$ be continuous monotone-type operators such that Eq. (6.3) has a solution in E and such that the map $A : X \to X^*$ defined by $Az := A(u,v) = (Qu - v, u + Kv)$ is γ-inverse strongly monotone. Let C be a nonempty, closed and convex subset of X, let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and let $\{T_j\}_{j\in\mathbb{N}}$ be an infinite family of Bregman weak relatively nonexpansive mappings from C into itself. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by*

$$\begin{cases} x_0 = x \in C \quad \textit{chosen arbitrarily}, \\ C_0 = C, \\ y_n = \operatorname{proj}_C^g(\nabla g^*[\nabla g(x_n) - \beta A x_n]), \\ z_n = \nabla g^*[\alpha_{n,0}\nabla g(x_n) + \sum_{j=1}^{\infty} \alpha_{n,j}\nabla g(T_j y_n)], \\ u_n \in C \quad \textit{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, \nabla g(u_n) - \nabla g(y_n)\rangle \geq 0, \forall y \in C, \\ C_{n+1} = \{z \in C_n : D_g(z, u_n) \leq D_g(z, x_n)\}, \\ x_{n+1} = \operatorname{proj}_{C_{n+1}}^g x \quad \textit{and} \quad n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{6.4}$$

where ∇g is the right-hand derivative of g. Let β be a constant such that $0 < \beta < \frac{c_2^2\gamma}{2}$, where c_2 is the 2-uniformly convex constant of E satisfying Corollary 2.1(2). Let $\{\alpha_{n,j} : n \in \mathbb{N} \cup \{0\}, j \in \mathbb{N} \cup \{0\}\}$ be a sequence in $(0,1)$ satisfying the following control conditions:

(1) $\sum_{j=0}^{\infty} \alpha_{n,j} = 1$, $\forall n \in \mathbb{N} \cup \{0\}$;
(2) $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,j} > 0$, $\forall j \in \mathbb{N}$.

Suppose that $F := \bigcap_{j=1}^{\infty} F(T_j) \cap A^{-1}(0) \cap \mathrm{EP}(f) \neq \emptyset$, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by (6.4) converges strongly to $\operatorname{proj}_F^g x$ as $n \to \infty$.

Remark 6.1 Observe that $z_0 \in F$ implies, in particular, that $z_0 \in A^{-1}(0) \Longleftrightarrow Az_0 = 0$. But $z_0 = (u_0, v_0)$ for some $u_0 \in E$ and $v_0 \in E^*$; moreover, $Az_0 = A(u_0, v_0) = (Qu_0 - v_0, u_0 + Kv_0)$. So, $Az_0 = 0$ implies that $(Qu_0 - v_0, u_0 + Kv_0) = (0, 0)$. This is equivalent to $Qu_0 - v_0 = 0$ and $u_0 + Kv_0 = 0$. Thus we have $v_0 = Qu_0$ which in turn implies that $u_0 + Kv_0 = 0$. Therefore, $u_0 \in E$ solves the Hammerstein-type Eq. (6.3).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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Acknowledgements
Dedicated to Professor Wataru Takahashi on the occasion of his seventieth birthday.
The work of Eskandar Naraghirad was conducted with a postdoctoral fellowship at the National Sun Yat-sen University of Kaohsing. This research was partially supported by the Grant NSC 99-2115-M-037-002-MY3.

Received: 25 April 2013 Accepted: 9 May 2013 Published: 30 May 2013

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doi:10.1186/1687-1812-2013-141
Cite this article as: Naraghirad and Yao: **Bregman weak relatively nonexpansive mappings in Banach spaces.** *Fixed Point Theory and Applications* 2013 **2013**:141.