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A general iterative algorithm for an infinite family of nonexpansive operators in Hilbert spaces

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Abstract

In this paper, we introduce a new general iterative algorithm for an infinite family of nonexpansive operators in Hilbert spaces. Under suitable assumptions, we prove that the sequence generated by the iterative algorithm converges strongly to a common point of the sets of fixed points, which solves a variational inequality. Our results improve and extend the corresponding results announced by many others. As applications, at the end of the paper, we apply our results to the split common fixed point problem.

Keywords: an infinite family of nonexpansive operators; strong convergence; *k*-Lipschitizian; η-strongly monotone; split common fixed point problem

1 Introduction

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let T be a nonexpansive operator. The set of fixed points of T is denoted by $\operatorname{Fix}(T)$. In 2000, Moudafi [1] introduced the viscosity approximation method for a nonexpansive operator and considered the sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n f x_n + (1-\alpha_n)Tx_n, \tag{1.1}$$

where f is a contraction on H and $\{\alpha_n\}$ is a sequence in $(0,1)$. In 2004, Xu [2] proved that under some conditions on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (1.1) strongly converges to x^* in $\operatorname{Fix}(T)$ which is the unique solution of the variational inequality

$$\langle (I-f)x^*, x-x^*\rangle \ge 0, \quad \forall x \in \operatorname{Fix}(T).$$

It is well known that iterative methods for nonexpansive operators have been used to solve convex minimization problems; see, *e.g.*, [3, 4]. A typical problem is to minimize a quadratic function over the set of fixed points of a nonexpansive operator T on a real Hilbert space H:

$$\min_{x\in \operatorname{Fix}(T)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1.2}$$

where b is a given point in H and A is a strongly positive bounded linear operator. In [3], Xu proved that the sequence $\{x_n\}$ defined by the following iterative method:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n b, \tag{1.3}$$

converges strongly to the unique solution of the minimization problem (1.2). In [5], Marino and Xu combined the iterative method (1.3) and the viscosity method (1.1) and considered the following general iterative method:

$$x_{n+1} = \alpha_n \gamma f x_n + (I - \alpha_n A)Tx_n. \tag{1.4}$$

They proved that the sequence $\{x_n\}$ generated by (1.4) converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \operatorname{Fix}(T),$$

which is the optimality condition for the minimization problem

$$\min_{x \in \operatorname{Fix}(T)} \frac{1}{2} \langle Ax, x \rangle - h(x),$$

where h is a potential function for γf (*i.e.*, $h'(x) = \gamma f(x)$ for $x \in H$).

On the other hand, Yamada [4] in 2001 introduced the following hybrid iterative method:

$$x_{n+1} = Tx_n - \mu \lambda_n FTx_n, \quad n \geq 0, \tag{1.5}$$

where F is a k-Lipschitzian and η-strongly monotone operator with $k > 0$, $\eta > 0$ and $0 < \mu < 2\eta/k^2$. Under some appropriate conditions, he proved that the sequence $\{x_n\}$ generated by (1.5) converges strongly to the unique solution of the variational inequality

$$\langle F\tilde{x}, x - \tilde{x} \rangle \geq 0, \quad \forall x \in \operatorname{Fix}(T).$$

Recently, combining (1.4) and (1.5), Tian [6] considered the following general iterative method:

$$x_{n+1} = \alpha_n \gamma f x_n + (I - \mu \alpha_n F)Tx_n. \tag{1.6}$$

Improving and extending the corresponding results given by Marino, Xu and Yamada, he proved that the sequence $\{x_n\}$ generated by (1.6) converges strongly to the unique solution $x^* \in \operatorname{Fix}(T)$ of the variational inequality

$$\langle (\gamma f - \mu F)\tilde{x}, x - \tilde{x} \rangle \leq 0, \quad \forall x \in \operatorname{Fix}(T).$$

Based on the above results of Marino, Xu, Yamada and Tian, much generalization work has been made by the corresponding authors; for instance, [7–23]. The problem of finding an element in the intersection of the fixed point sets of an infinite family of nonexpansive operators has attracted much attention because of its extraordinary utility and broad applicability in many branches of mathematical science and engineering. For example, if the nonexpansive operators are projection onto some closed convex sets C_i ($i \in \mathbb{N}$) in a real Hilbert space H, then such a fixed point problem becomes the convex feasibility problem of finding a point in $\bigcap_{i\in\mathbb{N}} C_i$. Many previous results [24–31] and many results not cited here considered the common fixed point about an infinite family of nonexpansive operators by W_n-mappings.

Motivated and inspired by the above results, we consider the following iterative algorithm without W_n-mappings:

$$\begin{cases} y_n = \beta_n x_n + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)T_i x_n, \\ x_{n+1} = \alpha_n \gamma V x_n + (I - \mu\alpha_n F)y_n, \end{cases} \tag{1.7}$$

where $\{\alpha_n\}$ is a sequence in $(0,1]$ and $\{\beta_n\}$ is a strictly decreasing sequence in $(0,1]$. Under some appropriate conditions, we proved the sequence $\{x_n\}$ generated by (1.7) converges strongly to the unique solution of the variational inequality:

$$\big\langle(\mu F - \gamma V)\tilde{x}, z - \tilde{x}\big\rangle \geq 0, \quad \forall z \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i).$$

Our results improve and extend the corresponding results announced by many others. As applications, at the end of the paper, we apply our results to the split common fixed point problem.

2 Preliminaries

Throughout this paper, we write $x_n \rightharpoonup x$ and $x_n \to x$ to indicate that $\{x_n\}$ converges weakly to x and converges strongly to x, respectively.

An operator $T : H \to H$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. It is well known that $\mathrm{Fix}(T)$ is closed and convex. It is known that A is called strongly positive if there exists a constant $\gamma > 0$ such that $\langle Ax, x\rangle \geq \gamma\|x\|^2$ for all $x \in H$. The operator F is called η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle x - y, Fx - Fy\rangle \geq \eta\|x - y\|^2$$

for all $x, y \in H$.

In order to prove our main results, we collect the following lemmas in this section.

Lemma 2.1 (Demiclosedness principle [32]) *Let H be a Hilbert space, C be a closed convex subset of H, and $T : C \to C$ be a nonexpansive operator with $\mathrm{Fix}(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to $x \in C$ and $\{(I - T)x_n\}$ converges strongly to $y \in C$, then $(I - T)x = y$. In particular, if $y = 0$, then $x \in \mathrm{Fix}(T)$.*

Lemma 2.2 [2] *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,
(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3 [33] *Let H be a real Hilbert space, let $V : H \to H$ be an L-Lipschitzian operator with $L > 0$, and let $F : H \to H$ be a k-Lipschitzian continuous operator and η-strongly monotone operator with $k > 0$, $\eta > 0$. Then, for $0 < \gamma < \frac{\mu\eta}{L}$, $\mu F - \gamma V$ is strongly monotone with coefficient $\mu\eta - \gamma L$.*

Lemma 2.4 [34] *Let C be a closed convex subset of a real Hilbert space H, given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if the following inequality holds*:

$$\langle x - y, z - y\rangle \leq 0$$

for every $z \in C$.

3 Main results

Lemma 3.1 *Let $\{T_n\} : H \to H$ be an infinite family of nonexpansive operators, let $F : H \to H$ be a k-Lipschitzian and η-strongly monotone operator with $k > 0$ and $\eta > 0$, and let $V : H \to H$ be an L-Lipschitzian operator. Let $0 < \mu < \frac{2\eta}{k^2}$ and $0 < \gamma < \frac{\mu(\eta - \frac{1}{2}\mu k^2)}{L} = \frac{\tau}{L}$. Assume that $S_n = \beta_n I + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)T_i$, where $\{\beta_n\}$ is a strictly decreasing sequence with $\beta_0 = 1$ and $\beta_n \in (0,1]$. Consider the following mapping G_n on H defined by*

$$G_n x = \alpha_n \gamma V x + (I - \mu\alpha_n F)S_n x,$$

where $\{\alpha_n\}$ is a sequence in $(0,1]$. Then G_n is a contraction.

Proof Observe that

$$\begin{aligned}
\|G_n x - G_n y\| &\leq \alpha_n \gamma \|Vx - Vy\| + (1 - \alpha_n\tau)\|S_n x - S_n y\| \\
&\leq \alpha_n\gamma L\|x - y\| + (1 - \alpha_n\tau)\left\|\beta_n(x - y) + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)(T_i x - T_i y)\right\| \\
&\leq \alpha_n\gamma L\|x - y\| + (1 - \alpha_n\tau)\left(\beta_n\|x - y\| + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)\|x - y\|\right) \\
&= \alpha_n\gamma L\|x - y\| + (1 - \alpha_n\tau)\|x - y\| \\
&= \bigl(1 - \alpha_n(\tau - \gamma L)\bigr)\|x - y\|.
\end{aligned}$$

Since $0 < 1 - \alpha_n(\tau - \gamma L) < 1$, G_n is a contraction. This completes the proof. □

Since G_n is a contraction, using the Banach contraction principle, G_n has a unique fixed point $x_n^V \in H$ such that

$$x_n^V = \alpha_n\gamma V x_n^V + (I - \mu\alpha_n F)S_n x_n^V.$$

For simplicity, we denote x_n for x_n^V without confusion.

Now we state and prove our main results in this paper.

Theorem 3.2 *Let $\{T_n\}$ be an infinite family of nonexpansive self-mappings of a real Hilbert space H, let F be a k-Lipschitzian and η-strongly monotone operator on H with $k > 0$ and $\eta > 0$, and let V be an L-Lipschitzian operator. Suppose that $\Omega = \bigcap_{i=1}^{\infty} \operatorname{Fix}(T_i)$ is nonempty. Suppose that $\{x_n\}$ is generated by the following algorithm*:

$$\begin{cases} y_n = \beta_n x_n + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)T_i x_n, \\ x_n = \alpha_n \gamma V x_n + (I - \mu\alpha_n F)y_n, \end{cases} \tag{3.1}$$

where $0 < \mu < \frac{2\eta}{k^2}$ and $0 < \gamma < \frac{\tau}{L}$ with $\tau = \mu(\eta - \frac{1}{2}\mu k^2)$. If the following conditions are satisfied:

(i) *$\{\alpha_n\}$ is a sequence in $(0,1]$ and $\lim_{n\to\infty} \alpha_n = 0$;*
(ii) *$\{\beta_n\}$ is a strictly decreasing sequence in $(0,1]$ and $\beta_0 = 1$.*

Then $\{x_n\}$ converges strongly to $\tilde{x} \in \Omega$, which solves the variational inequality:

$$\langle (\mu F - \gamma V)\tilde{x}, z - \tilde{x} \rangle \geq 0, \quad \forall z \in \Omega. \tag{3.2}$$

Equivalently, we have $P_\Omega(I - \mu F + \gamma V)\tilde{x} = \tilde{x}$.

Proof We proceed with the following steps:

Step 1: First we show that $\{x_n\}$ is bounded.

In fact, let $p \in \Omega$, then for every $i \in \mathbb{N}$, $T_i p = p$. Observe that

$$\|y_n - p\| \leq \beta_n \|x_n - p\| + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)\|T_i x_n - T_i p\| \leq \|x_n - p\|.$$

Thus it follows that

$$\begin{aligned} \|x_n - p\| &= \|\alpha_n \gamma V x_n + (I - \mu\alpha_n F)y_n - p\| \\ &= \|\alpha_n(\gamma V x_n - \mu F p) + (I - \mu\alpha_n F)y_n - (I - \mu\alpha_n F)p\| \\ &\leq (1 - \alpha_n\tau)\|y_n - p\| + \alpha_n\big(\|\gamma V x_n - \gamma V p\| + \|\gamma V p - \mu F p\|\big) \\ &\leq (1 - \alpha_n\tau)\|y_n - p\| + \alpha_n\gamma L\|x_n - p\| + \alpha_n\|\gamma V p - \mu F p\| \\ &\leq \big(1 - \alpha_n(\tau - \gamma L)\big)\|x_n - p\| + \alpha_n\|\gamma V p - \mu F p\|. \end{aligned}$$

Then we have

$$\|x_n - p\| \leq \frac{1}{\tau - \gamma L}\|\gamma V p - \mu F p\|,$$

which implies that $\{x_n\}$ is bounded. Hence we can obtain $\{y_n\}$, $\{T_i x_n\}$, $\{F y_n\}$ and $\{V x_n\}$ are bounded. Note that

$$\|x_n - y_n\| = \|\alpha_n \gamma V x_n + (I - \mu\alpha_n F)y_n - y_n\| = \alpha_n\|\gamma V x_n - \mu F y_n\|,$$

we immediately obtain that

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{3.3}$$

Step 2: We show $\lim_{n\to\infty}\|x_n - T_ix_n\| = 0$.

Since $p \in \Omega$, we note that

$$\begin{aligned}\|x_n - p\|^2 \geq \|T_ix_n - p\|^2 &= \|T_ix_n - x_n + x_n - p\|^2 \\ &= \|T_ix_n - x_n\|^2 + \|x_n - p\|^2 + 2\langle T_ix_n - x_n, x_n - p\rangle,\end{aligned}$$

which implies that

$$\frac{1}{2}\|T_ix_n - x_n\|^2 \leq \langle x_n - T_ix_n, x_n - p\rangle.$$

Thus

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^{n}(\beta_{i-1} - \beta_i)\|T_ix_n - x_n\|^2 &\leq \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)\langle x_n - T_ix_n, x_n - p\rangle \\ &= \left\langle (1 - \beta_n)x_n - \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)T_ix_n, x_n - p\right\rangle \\ &= \big\langle (1 - \beta_n)x_n - y_n + \beta_nx_n, x_n - p\big\rangle \\ &= \langle x_n - y_n, x_n - p\rangle \\ &\leq \|x_n - y_n\|\|x_n - p\|.\end{aligned}$$

Then we immediately obtain $\lim_{n\to\infty}\sum_{i=1}^{n}(\beta_{i-1} - \beta_i)\|T_ix_n - x_n\|^2 = 0$. Since $\{\beta_n\}$ is strictly decreasing, it follows that

$$\lim_{n\to\infty}\|T_ix_n - x_n\| = 0 \tag{3.4}$$

for every $i \in \mathbb{N}$. Since $S_n = \beta_nI + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)T_i$, thus

$$\|x_n - S_nx_n\|^2 \leq \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)\|T_ix_n - x_n\|^2.$$

It shows that

$$\lim_{n\to\infty}\|x_n - S_nx_n\| = 0. \tag{3.5}$$

Step 3: We show that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to \tilde{x}$.

Since $\{x_n\}$ is bounded, there exist a point $\tilde{x} \in H$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \tilde{x}$. By Lemma 2.1 and (3.4), we obtain $\tilde{x} \in \operatorname{Fix}(T_i)$ for any $i \in \mathbb{N}$. This shows that $\tilde{x} \in \Omega$. On the other hand, we note that

$$\begin{aligned}x_n - \tilde{x} &= \alpha_n\gamma Vx_n + (I - \mu\alpha_nF)y_n - \tilde{x} \\ &= \big(I - \alpha_n(\mu F - \gamma V)\big)y_n - \big(I - \alpha_n(\mu F - \gamma V)\big)\tilde{x} \\ &\quad - \alpha_n(\mu F - \gamma V)\tilde{x} + \alpha_n(\gamma Vx_n - \gamma Vy_n).\end{aligned}$$

Hence we obtain

$$\begin{aligned}\|x_n-\tilde{x}\|^2 &= \big\langle \big(I-\alpha_n(\mu F-\gamma V)\big)y_n-\big(I-\alpha_n(\mu F-\gamma V)\big)\tilde{x}, x_n-\tilde{x}\big\rangle \\ &\quad -\alpha_n\big\langle(\mu F-\gamma V)\tilde{x}, x_n-\tilde{x}\big\rangle+\alpha_n\langle\gamma Vx_n-\gamma Vy_n, x_n-\tilde{x}\rangle \\ &\le \big\|\big(I-\alpha_n(\mu F-\gamma V)\big)y_n-\big(I-\alpha_n(\mu F-\gamma V)\big)\tilde{x}\big\|\|x_n-\tilde{x}\| \\ &\quad -\alpha_n\big\langle(\mu F-\gamma V)\tilde{x}, x_n-\tilde{x}\big\rangle+\alpha_n\gamma L\|x_n-y_n\|\|x_n-\tilde{x}\| \\ &\le \big(1-\alpha_n(\tau-\gamma L)\big)\|x_n-\tilde{x}\|^2-\alpha_n\big\langle(\mu F-\gamma V)\tilde{x}, x_n-\tilde{x}\big\rangle \\ &\quad +\alpha_n\gamma L\|x_n-y_n\|\|x_n-\tilde{x}\|.\end{aligned}$$

Then it follows that

$$\|x_n-\tilde{x}\|^2\le\frac{\gamma L}{\tau-\gamma L}\|x_n-y_n\|\|x_n-\tilde{x}\|-\frac{1}{\tau-\gamma L}\big\langle(\mu F-\gamma V)\tilde{x}, x_n-\tilde{x}\big\rangle.$$

In particular,

$$\|x_{n_k}-\tilde{x}\|^2\le\frac{\gamma L}{\tau-\gamma L}\|x_{n_k}-y_{n_k}\|\|x_{n_k}-\tilde{x}\|-\frac{1}{\tau-\gamma L}\big\langle(\mu F-\gamma V)\tilde{x}, x_{n_k}-\tilde{x}\big\rangle.$$

From $x_{n_k}\rightharpoonup\tilde{x}$ and (3.3), it follows that $x_{n_k}\to\tilde{x}$.

Step 4: We show that $\tilde{x}$ solves the variational inequality (3.2).

Observe that

$$x_n=\alpha_n\gamma Vx_n+(I-\mu\alpha_nF)S_nx_n.$$

Hence, we conclude that

$$\begin{aligned}(\mu F-\gamma V)x_n &= (\mu F-\gamma V)(x_n-S_nx_n)+\mu FS_nx_n-\gamma VS_nx_n \\ &= (\mu F-\gamma V)(x_n-S_nx_n)+(\gamma Vx_n-\gamma VS_nx_n)-\gamma Vx_n+\mu FS_nx_n \\ &= (\mu F-\gamma V)(x_n-S_nx_n)+(\gamma Vx_n-\gamma VS_nx_n)-\frac{1}{\alpha_n}(I-S_n)x_n.\end{aligned}$$

Since S_n is nonexpansive, we have that $I-S_n$ is monotone. Note that for any $z\in\Omega$, $S_nz=z$. Then we deduce

$$\begin{aligned}\big\langle(\mu F-\gamma V)x_n, x_n-z\big\rangle &= -\frac{1}{\alpha_n}\big\langle(I-S_n)x_n-(I-S_n)z, x_n-z\big\rangle \\ &\quad +\big\langle(\mu F-\gamma V)(I-S_n)x_n, x_n-z\big\rangle+\langle\gamma Vx_n-\gamma VS_nx_n, x_n-z\rangle \\ &\le \big\langle(\mu F-\gamma V)(I-S_n)x_n, x_n-z\big\rangle+\gamma L\|x_n-S_nx_n\|\|x_n-z\|.\end{aligned}$$

Now, replacing n with n_k in the above inequality, and letting $k\to\infty$, by (3.5) we have

$$\begin{aligned}\big\langle(\mu F-\gamma V)\tilde{x}, \tilde{x}-z\big\rangle &= \lim_{k\to 0}\big\langle(\mu F-\gamma V)x_{n_k}, x_{n_k}-z\big\rangle \\ &\le \lim_{k\to 0}\big(\big\langle(\mu F-\gamma V)(x_{n_k}-S_{n_k}x_{n_k}), x_{n_k}-z\big\rangle\end{aligned}$$

$$+ \gamma L\|x_{n_k} - S_{n_k}x_{n_k}\|\|x_{n_k} - z\|\big)$$
$$= 0.$$

That is, $\langle(\mu F - \gamma V)\tilde{x}, z - \tilde{x}\rangle \geq 0$ for every $z \in \Omega$. It follows that $\tilde{x}$ is a solution of the variational inequality (3.2). Since $\mu F - \gamma V$ is $(\mu\eta - \gamma L)$-strongly monotone and $(\mu k - \gamma L)$-Lipschitzian, the variational inequality (3.2) has a unique solution. So, we conclude that $x_n \to \tilde{x}$ as $n \to \infty$. The variational inequality (3.2) can be written as

$$\big\langle (I - \mu F + \gamma V)\tilde{x} - \tilde{x}, z - \tilde{x}\big\rangle \leq 0, \quad \forall z \in \Omega.$$

By Lemma 2.4, we have $P_\Omega(I - \mu F + \gamma V)\tilde{x} = \tilde{x}$. □

Theorem 3.3 *Let $\{T_n\}$ be an infinite family of nonexpansive self-mappings of a real Hilbert space H, let F be a k-Lipschitzian and η-strongly monotone operator on H with $k > 0$ and $\eta > 0$, and let V be an L-Lipschitzian operator. Suppose that $\Omega = \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i)$ is nonempty. Suppose that $x_1 \in H$, $0 < \mu < \frac{2\eta}{k^2}$ and $0 < \gamma < \frac{\tau}{L}$ with $\tau = \mu(\eta - \frac{1}{2}\mu k^2)$. Let $\beta_0 = 1$, $\{\alpha_n\}$ be a sequence in $(0,1]$, and let $\{\beta_n\}$ be a strictly decreasing sequence in $(0,1]$. If the following conditions are satisfied*:

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(iv) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then $\{x_n\}$ generated by (1.7) converges strongly to $\tilde{x} \in \Omega$, which solves the variational inequality

$$\big\langle (\mu F - \gamma V)\tilde{x}, z - \tilde{x}\big\rangle \geq 0, \quad \forall z \in \Omega. \tag{3.6}$$

Equivalently, we have $P_\Omega(I - \mu F + \gamma V)\tilde{x} = \tilde{x}$.

Proof We proceed with the following steps:

Step 1: First show that there exists $\tilde{x} \in \Omega$ such that $\tilde{x} = P_\Omega(I - \mu F + \gamma V)\tilde{x}$.

In fact, by Lemma 2.3, $\mu F - \gamma V$ is strongly monotone. So, the variational inequality (3.6) has only one solution. We set $\tilde{x} \in \Omega$ to indicate the unique solution of (3.6). The variational inequality (3.6) can be written as

$$\big\langle (I - \mu F + \gamma V)\tilde{x} - \tilde{x}, z - \tilde{x}\big\rangle \leq 0, \quad \forall z \in \Omega.$$

So, by Lemma 2.4, it is equivalent to the fixed point equation

$$P_\Omega(I - \mu F + \gamma V)\tilde{x} = \tilde{x}.$$

Step 2: Now we show that $\{x_n\}$ is bounded.

Let $p \in \Omega$, then for every $i \in \mathbb{N}$, $T_i p = p$. Observe that

$$\begin{aligned}\|y_n - p\| &\leq \beta_n\|x_n - p\| + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)\|T_i x_n - p\| \\ &\leq \beta_n\|x_n - p\| + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)\|x_n - p\| = \|x_n - p\|.\end{aligned}$$

Thus it follows that

$$\begin{aligned}\|x_{n+1}-p\| &= \big\|\alpha_n\gamma Vx_n + (I-\mu\alpha_nF)y_n - p\big\| \\ &= \big\|\alpha_n(\gamma Vx_n - \mu Fp) + (I-\mu\alpha_nF)y_n - (I-\mu\alpha_nF)p\big\| \\ &\le (1-\alpha_n\tau)\|y_n-p\| + \alpha_n\big(\|\gamma Vx_n - \gamma Vp\| + \|\gamma Vp - \mu Fp\|\big) \\ &\le (1-\alpha_n\tau)\|y_n-p\| + \alpha_n\gamma L\|x_n-p\| + \alpha_n\|\gamma Vp-\mu Fp\| \\ &\le \big[1-\alpha_n(\tau-\gamma L)\big]\|x_n-p\| + \alpha_n(\tau-\gamma L)\frac{\|\gamma Vp-\mu Fp\|}{\tau-\gamma L} \\ &\le \max\left\{\|x_n-p\|, \frac{\|\gamma Vp-\mu Fp\|}{\tau-\gamma L}\right\} \\ &\le \cdots \le \max\left\{\|x_1-p\|, \frac{\|\gamma Vp-\mu Fp\|}{\tau-\gamma L}\right\}.\end{aligned}$$

Therefore, $\{x_n\}$ is bounded. Hence we can obtain that $\{y_n\}$, $\{T_ix_n\}$, $\{Fy_n\}$ and $\{Vx_n\}$ are bounded.

Step 3: we show $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$.

We observe that

$$\begin{aligned}x_{n+1}-x_n &= \alpha_n\gamma Vx_n + (I-\mu\alpha_nF)y_n - \alpha_{n-1}\gamma Vx_{n-1} - (I-\mu\alpha_{n-1}F)y_{n-1} \\ &= \alpha_n\gamma(Vx_n - Vx_{n-1}) + (\alpha_n-\alpha_{n-1})\gamma Vx_{n-1} \\ &\quad + \big((I-\mu\alpha_nF)y_n - (I-\mu\alpha_nF)y_{n-1}\big) + (\alpha_{n-1}-\alpha_n)\mu Fy_{n-1}.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-x_n\| &\le \alpha_n\gamma\|Vx_n - Vx_{n-1}\| + |\alpha_n-\alpha_{n-1}|\gamma\|Vx_{n-1}\| \\ &\quad + (1-\alpha_n\tau)\|y_n-y_{n-1}\| + |\alpha_{n-1}-\alpha_n|\mu\|Fy_{n-1}\|.\end{aligned} \tag{3.7}$$

We have

$$\begin{aligned}\|y_n-y_{n-1}\| &= \left\|\beta_nx_n + \sum_{i=1}^{n}(\beta_{i-1}-\beta_i)T_ix_n - \beta_{n-1}x_{n-1} - \sum_{i=1}^{n-1}(\beta_{i-1}-\beta_i)T_ix_{n-1}\right\| \\ &\le \beta_n\|x_n-x_{n-1}\| + |\beta_n-\beta_{n-1}|\|x_{n-1}\| \\ &\quad + \sum_{i=1}^{n}(\beta_{i-1}-\beta_i)\|T_ix_n - T_ix_{n-1}\| + |\beta_n-\beta_{n-1}|\|T_nx_{n-1}\| \\ &\le \|x_n-x_{n-1}\| + |\beta_n-\beta_{n-1}|\big(\|x_{n-1}\| + \|T_nx_{n-1}\|\big).\end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8), we obtain that

$$\begin{aligned}\|x_{n+1}-x_n\| &\le \alpha_n\gamma L\|x_n-x_{n-1}\| + |\alpha_n-\alpha_{n-1}|\big(\gamma\|Vx_{n-1}\| + \mu\|Fy_{n-1}\|\big) \\ &\quad + (1-\alpha_n\tau)\|x_n-x_{n-1}\| + |\beta_n-\beta_{n-1}|(1-\alpha_n\tau)\big(\|x_{n-1}\| + \|T_nx_{n-1}\|\big) \\ &= \big(1-\alpha_n(\tau-\gamma L)\big)\|x_n-x_{n-1}\| + |\alpha_n-\alpha_{n-1}|\big(\gamma\|Vx_{n-1}\| + \mu\|Fy_{n-1}\|\big) \\ &\quad + |\beta_n-\beta_{n-1}|(1-\alpha_n\tau)\big(\|x_{n-1}\| + \|T_nx_{n-1}\|\big).\end{aligned}$$

Using (ii), (iii), (iv) and Lemma 2.2, we have $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$.

Step 4: We show $\lim_{n\to\infty}\|T_i x_n - x_n\| = 0$ for all $i \in \mathbb{N}$.

Since $p \in \Omega$, we note that

$$\begin{aligned}\|x_n - p\|^2 &\geq \|T_i x_n - T_i p\|^2 = \|T_i x_n - x_n + x_n - p\|^2 \\ &= \|T_i x_n - x_n\|^2 + \|x_n - p\|^2 + 2\langle T_i x_n - x_n, x_n - p\rangle,\end{aligned}$$

which implies that

$$\frac{1}{2}\|T_i x_n - x_n\|^2 \leq \langle x_n - T_i x_n, x_n - p\rangle. \tag{3.9}$$

From (1.7) and (3.9), we deduce

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^{n}(\beta_{i-1} - \beta_i)\|T_i x_n - x_n\|^2 &\leq \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)\langle x_n - T_i x_n, x_n - p\rangle \\ &= \left\langle (1 - \beta_n)x_n - \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)T_i x_n, x_n - p\right\rangle \\ &= \langle (1 - \beta_n)x_n - y_n + \beta_n x_n, x_n - p\rangle \\ &= \langle x_n - y_n, x_n - p\rangle \\ &= \langle x_n - x_{n+1}, x_n - p\rangle + \langle x_{n+1} - y_n, x_n - p\rangle.\end{aligned}$$

Using (1.7), we can have

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^{n}(\beta_{i-1} - \beta_i)\|T_i x_n - x_n\|^2 &\leq \langle x_n - x_{n+1}, x_n - p\rangle + \alpha_n\langle \gamma V x_n - \mu F y_n, x_n - p\rangle \\ &\leq \|x_n - x_{n+1}\|\|x_n - p\| + \alpha_n\|\gamma V x_n - \mu F y_n\|\|x_n - p\|.\end{aligned}$$

Noting that $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$ and $\lim_{n\to\infty}\alpha_n = 0$, we immediately obtain

$$\lim_{n\to\infty}\sum_{i=1}^{n}(\beta_{i-1} - \beta_i)\|T_i x_n - x_n\|^2 = 0.$$

Since $\{\beta_n\}$ is strictly decreasing, it follows that for every $i \in \mathbb{N}$,

$$\lim_{n\to\infty}\|T_i x_n - x_n\| = 0. \tag{3.10}$$

Step 5: Show $\limsup_{n\to\infty}\langle(\gamma V - \mu F)\tilde{x}, x_n - \tilde{x}\rangle \leq 0$, where $\tilde{x} = P_\Omega(I - \mu F + \gamma V)\tilde{x}$.

Since $\{x_n\}$ is bounded, there exist a point $v \in H$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma V - \mu F)\tilde{x}, x_n - \tilde{x}\rangle = \lim_{k\to\infty}\langle(\gamma V - \mu F)\tilde{x}, x_{n_k} - \tilde{x}\rangle$$

and $x_{n_k} \rightharpoonup v$. Now, applying (3.10) and Lemma 2.1, we conclude that $v \in \mathrm{Fix}(T_i)$ for every $i \in \mathbb{N}$. Hence, $v \in \Omega$. Since Ω is closed and convex, by Lemma 2.4, we get

$$\begin{aligned}\limsup_{n\to\infty}\langle(\gamma V - \mu F)\tilde{x}, x_n - \tilde{x}\rangle &= \lim_{k\to\infty}\langle(\gamma V - \mu F)\tilde{x}, x_{n_k} - \tilde{x}\rangle \\ &= \langle(\gamma V - \mu F)\tilde{x}, v - \tilde{x}\rangle \leq 0.\end{aligned} \tag{3.11}$$

Step 6: Show $x_n \to \tilde{x} = P_\Omega(I - \mu F + \gamma V)(\tilde{x})$.
Since $\tilde{x} \in \Omega$, we have $T_i\tilde{x} = \tilde{x}$ for every $i \in \mathbb{N}$. Using (1.7), we have

$$\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &= \big\|\alpha_n\gamma Vx_n + (I - \mu\alpha_n F)y_n - \tilde{x}\big\|^2 \\
&= \big\|(I - \mu\alpha_n F)y_n - (I - \mu\alpha_n F)\tilde{x} + \alpha_n(\gamma Vx_n - \mu F\tilde{x})\big\|^2 \\
&\le \big\|(I - \mu\alpha_n F)y_n - (I - \mu\alpha_n F)\tilde{x}\big\|^2 + 2\alpha_n\langle\gamma Vx_n - \mu F\tilde{x}, x_{n+1} - \tilde{x}\rangle \\
&\le (1 - \alpha_n\tau)^2\|y_n - \tilde{x}\|^2 + 2\alpha_n\gamma\langle Vx_n - V\tilde{x}, x_{n+1} - \tilde{x}\rangle \\
&\quad + 2\alpha_n\langle\gamma V\tilde{x} - \mu F\tilde{x}, x_{n+1} - \tilde{x}\rangle \\
&\le (1 - \alpha_n\tau)^2\|x_n - \tilde{x}\|^2 + \alpha_n L\gamma\left(\|x_n - \tilde{x}\|^2 + \|x_{n+1} - \tilde{x}\|^2\right) \\
&\quad + 2\alpha_n\langle\gamma V\tilde{x} - \mu F\tilde{x}, x_{n+1} - \tilde{x}\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &\le \frac{(1 - \alpha_n\tau)^2 + \alpha_n\gamma L}{1 - \alpha_n\gamma L}\|x_n - \tilde{x}\|^2 + \frac{2\alpha_n}{1 - \alpha_n\gamma L}\langle\gamma V\tilde{x} - \mu F\tilde{x}, x_{n+1} - \tilde{x}\rangle \\
&\le \left(1 - \frac{2\alpha_n(\tau - \gamma L)}{1 - \alpha_n\gamma L}\right)\|x_n - \tilde{x}\|^2 \\
&\quad + \frac{2\alpha_n(\tau - \gamma L)}{1 - \alpha_n\gamma L}\left[\frac{1}{\tau - \gamma L}\langle\gamma V\tilde{x} - \mu F\tilde{x}, x_{n+1} - \tilde{x}\rangle\right].
\end{aligned}$$

Consequently, according to (3.11) and Lemma 2.2, we deduce that $\{x_n\}$ converges strongly to $\tilde{x} = P_\Omega(I - \mu F + \gamma V)\tilde{x}$. This completes the proof. □

Corollary 3.4 *Let T be a nonexpansive self-mapping of a real Hilbert space H, let F be a k-Lipschitzian and η-strongly monotone operator on H with $k > 0$ and $\eta > 0$, and let V be an L-Lipschitzian operator. Suppose that $\Omega = \mathrm{Fix}(T)$ is nonempty. Suppose that $x_1 \in H$ and that $\{x_n\}$ is generated by the following algorithm*:

$$x_{n+1} = \alpha_n\gamma V(x_n) + (I - \mu\alpha_n F)Tx_n,$$

where $0 < \mu < \frac{2\eta}{k^2}$ and $0 < \gamma < \frac{\tau}{L}$ with $\tau = \mu(\eta - \frac{1}{2}\mu k^2)$. Let $\{\alpha_n\}$ be in $(0,1]$. If the following conditions are satisfied:

(i) $\lim_{n\to\infty}\alpha_n = 0$;
(ii) $\sum_{n=1}^\infty \alpha_n = \infty$;
(iii) $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$.

Then $\{x_n\}$ converges strongly to $\tilde{x} \in \Omega$, which solves the variational inequality

$$\big\langle(\mu F - \gamma V)\tilde{x}, z - \tilde{x}\big\rangle \ge 0, \quad \forall z \in \Omega.$$

Equivalently, we have $P_\Omega(I - \mu F + \gamma V)\tilde{x} = \tilde{x}$.

Proof Set $\{T_n\}$ to be the sequences of operators defined by $T_n = T$ for all $n \in \mathbb{N}$ in Theorem 3.3. Then by Theorem 3.3, we obtain the desired result. □

4 Application in the split common fixed point problem

Let H_1 and H_2 be Hilbert spaces, let $A : H_1 \to H_2$ be a bounded linear operator. The split common fixed point problem (SCFPP) is to find a point $x^* \in H_1$ satisfied with

$$x^* \in \bigcap_{i=1}^{p} \mathrm{Fix}(U_i), \qquad Ax^* \in \bigcap_{j=1}^{r} \mathrm{Fix}(T_j),$$

where $U_i : H_1 \to H_1$ $(i = 1, 2, \dots, p)$ and $T_j : H_2 \to H_2$ $(j = 1, 2, \dots, r)$ are nonlinear operators. The concept of the SCFPP in finite-dimensional Hilbert spaces was firstly introduced by Censor and Segal in [35]. Now we consider a generalized split common fixed point problem (GSCFPP) which is to find a point

$$x^* \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(U_i), \qquad Ax^* \in \bigcap_{j=1}^{\infty} \mathrm{Fix}(T_j). \tag{4.1}$$

We know that if for all i and j, U_i and T_j are nonexpansive operators, the GSCFPP is equivalent to the following common fixed point problem:

$$x^* \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(U_i), \qquad x^* \in \bigcap_{j=1}^{\infty} \mathrm{Fix}(V_j),$$

where $V_j = I - \gamma A^*(I - T_j)A$ with $0 < \gamma \leq \frac{1}{\|A\|^2}$ for every $j \in \mathbb{N}$ (see [36]). The solution set of GSCFPP (4.1) is denoted by S.

Theorem 4.1 *Let $\{U_n\}$ and $\{T_n\}$ be sequences of nonexpansive operators on real Hilbert spaces H_1 and H_2, respectively. Let F be a k-Lipschitzian and η-strongly monotone operator on H_1 with $k > 0$ and $\eta > 0$. Let V be an L-Lipschitzian operator. Suppose that S is nonempty. Suppose that $x_1 \in H$ and that $\{x_n\}$ is generated by the following algorithm*:

$$\begin{cases} y_n = \beta_n x_n + \sum_{i=1}^{n} (\beta_{i-1} - \beta_i) U_i (I - \gamma A^*(I - T_i)A) x_n, \\ x_{n+1} = \alpha_n \gamma V(x_n) + (I - \mu\alpha_n F) y_n, \end{cases}$$

where $0 < \mu < \frac{2\eta}{k^2}$ and $0 < \gamma < \frac{\tau}{L}$ with $\tau = \mu(\eta - \frac{1}{2}\mu k^2)$. Let $\{\alpha_n\}$ be in $(0, 1]$, $\{\beta_n\}$ be a strictly decreasing sequence in $(0, 1]$ and $\beta_0 = 1$. If the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(iv) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then $\{x_n\}$ converges strongly to $\tilde{x} \in S$, which solves the variational inequality

$$\langle (\mu F - \gamma V)\tilde{x}, z - \tilde{x} \rangle \geq 0, \quad \forall z \in S.$$

Equivalently, we have $P_\Omega(I - \mu F + \gamma V)\tilde{x} = \tilde{x}$.

Proof Set $\{T_n\}$ to be the sequences of operators defined by $T_n := U_n(I - \gamma A^*(I - T_n)A)$ for all $n \in \mathbb{N}$ in Theorem 3.3. By Theorem 3.3, we can obtain

$$\lim_{n\to\infty} \| U_i (I - \gamma A^*(I - T_i)A) x_n - x_n \| = 0$$

in Step 4. But it does not imply that the set of cluster points of the weak topology $\omega_w(x_n)$ is a subset of S. In order to prove this, we only show $\lim_{n\to\infty}\|T_iAx_n - x_n\| = 0$ and $\lim_{n\to\infty}\|U_ix_n - x_n\| = 0$.

Since $p \in \Omega$, $T_iAp = Ap$. Hence, for every $i \in \mathbb{N}$,

$$\begin{aligned}\|Ax_n - Ap\|^2 &\geq \|T_iAx_n - T_iAp\|^2 = \|T_iAx_n - Ap\|^2 \\ &= \|T_iAx_n - Ax_n + Ax_n - Ap\|^2 \\ &= \|T_iAx_n - Ax_n\|^2 + \|Ax_n - Ap\|^2 + 2\langle T_iAx_n - Ax_n, Ax_n - Ap\rangle,\end{aligned}$$

which yields that

$$\langle T_iAx_n - Ax_n, Ax_n - Ap\rangle \leq -\frac{1}{2}\|T_iAx_n - Ax_n\|^2 \tag{4.2}$$

for every $i \in \mathbb{N}$. Using (4.2), we note that

$$\begin{aligned}\|y_n - p\|^2 &= \left\|\beta_n(x_n - p) + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)\bigl(U_i\bigl(I - \gamma A^*(I - T_i)A\bigr)x_n - p\bigr)\right\|^2 \\ &\leq \beta_n\|x_n - p\|^2 + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)\bigl\|x_n + \gamma A^*(T_i - I)Ax_n - p\bigr\|^2 \\ &\leq \beta_n\|x_n - p\|^2 + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)\bigl(\|x_n - p\|^2 + \gamma^2\|A\|^2\|T_iAx_n - Ax_n\|^2 \\ &\quad + 2\gamma\langle Ax_n - Ap, T_iAx_n - Ax_n\rangle\bigr) \\ &\leq \|x_n - p\|^2 + \sum_{i=1}^{n}(\beta_{i-1} - \beta_i)\bigl(\gamma^2\|A\|^2\|T_iAx_n - Ax_n\|^2 \\ &\quad - \gamma\|T_iAx_n - Ax_n\|^2\bigr) \\ &= \|x_n - p\|^2 + \gamma\bigl(\gamma\|A\|^2 - 1\bigr)\sum_{i=1}^{n}(\beta_{i-1} - \beta_i)\|T_iAx_n - Ax_n\|^2.\end{aligned}$$

Thus

$$\begin{aligned}&\gamma\bigl(1 - \gamma\|A\|^2\bigr)\sum_{i=1}^{n}(\beta_{i-1} - \beta_i)\|T_iAx_n - Ax_n\|^2 \\ &\quad = \|x_n - p\|^2 - \|y_n - p\|^2 \\ &\quad = \bigl(\|x_n - p\| - \|y_n - p\|\bigr)\bigl(\|x_n - p\| + \|y_n - p\|\bigr) \\ &\quad \leq \|x_n - y_n\|\bigl(\|x_n - p\| + \|y_n - p\|\bigr).\end{aligned}$$

It follows that $\lim_{n\to\infty}\|T_iAx_n - Ax_n\| = 0$. Now we show that $\lim_{n\to\infty}\|U_ix_n - x_n\| = 0$. Note that

$$\begin{aligned}\|U_ix_n - x_n\| &\leq \bigl\|U_ix_n - U_i\bigl(I - \gamma A^*(I - T_i)A\bigr)x_n\bigr\| + \bigl\|U_i\bigl(I - \gamma A^*(I - T_i)A\bigr)x_n - x_n\bigr\| \\ &\leq \bigl\|x_n - \bigl(I - \gamma A^*(I - T_i)A\bigr)x_n\bigr\| + \bigl\|U_i\bigl(I - \gamma A^*(I - T_i)A\bigr)x_n - x_n\bigr\| \\ &\leq \gamma\|A\|\bigl\|(T_i - I)Ax_n\bigr\| + \bigl\|U_i\bigl(I - \gamma A^*(I - T_i)A\bigr)x_n - x_n\bigr\|.\end{aligned}$$

Then we have $\lim_{n\to\infty} \|U_i x_n - x_n\| = 0$ for every $i \in \mathbb{N}$. Then we can have $\omega_w(x_n) \subset S$. Hence, by Theorem 3.3, we obtain the desired result. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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