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On (α^*, ψ)-contractive multi-valued mappings

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Abstract

In this paper, we generalize the contractive condition for multi-valued mappings given by Asl, Rezapour and Shahzad in 2012. We establish some fixed point theorems for multi-valued mappings from a complete metric space to the space of closed or bounded subsets of the metric space satisfying generalized (α^*, ψ)-contractive condition.

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Keywords: α-admissible; α^*-ψ-contractive mapping; generalized (α^*, ψ)-contractive mapping

1 Introduction

Samet *et al.* [1] introduced the notion of α-ψ-contractive self-mappings of a metric space. Recently, Asl *et al.* [2] introduced the notion of α^*-ψ-contractive mappings to extend the notion α-ψ-contractive mappings. In this paper, we generalize the notion of α^*-ψ-contractive mappings and prove some fixed point theorems for such mappings.

Let Ψ be a family of nondecreasing functions, $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, where ψ^n is the nth iterate of ψ. It is known that for each $\psi \in \Psi$, we have $\psi(t) < t$ for all $t > 0$ and $\psi(0) = 0$ for $t = 0$ [1]. Let (X,d) be a metric space. A mapping $G : X \to X$ is called α-ψ-contractive if there exist two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that $\alpha(x,y)d(Gx,Gy) \le \psi(d(x,y))$ for each $x,y \in X$. A mapping $G : X \to X$ is called α-admissible [1] if $\alpha(x,y) \ge 1 \Rightarrow \alpha(Gx,Gy) \ge 1$. We denote by $N(X)$ the space of all nonempty subsets of X, by $B(X)$ the space of all nonempty bounded subsets of X and by $CL(X)$ the space of all nonempty closed subsets of X. For $A \in N(X)$ and $x \in X$, $d(x,A) = \inf\{d(x,a) : a \in A\}$. For every $A,B \in B(X)$, $\delta(A,B) = \sup\{d(a,b) : a \in A, b \in B\}$. When $A = \{x\}$, we denote $\delta(A,B)$ by $\delta(x,B)$. For every $A,B \in CL(X)$, let

$$H(A,B) = \begin{cases} \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\} & \text{if the maximum exists;} \\ \infty & \text{otherwise.} \end{cases}$$

Such a map H is called generalized Hausdorff metric induced by d. Let $(X,\preceq,d)$ be an ordered metric space and $A,B \subseteq X$. We say that $A \prec_r B$ if for each $a \in A$ and $b \in B$, we have $a \preceq b$. We give a few definitions and the result due to Asl *et al.* [2] for convenience.

Definition 1.1 [2] Let (X,d) be a metric space and let $\alpha : X \times X \to [0,\infty)$ be a mapping. A mapping $G : X \to CL(X)$ is α^*-admissible if $\alpha(x,y) \geq 1 \Rightarrow \alpha^*(Gx,Gy) \geq 1$, where $\alpha^*(Gx,Gy) = \inf\{\alpha(a,b) : a \in Gx, b \in Gy\}$.

Definition 1.2 [2] Let (X,d) be a metric space. A mapping $G : X \to CL(X)$ is called α^*-ψ-contractive if there exist two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that

$$\alpha^*(Gx,Gy)H(Gx,Gy) \leq \psi\big(d(x,y)\big) \tag{1.1}$$

for all $x, y \in X$.

Theorem 1.3 [2] *Let (X,d) be a complete metric space, let $\alpha : X \times X \to [0,\infty)$ be a function, let $\psi \in \Psi$ be a strictly increasing map and T be a closed-valued, α^*-admissible and α^*-ψ-contractive multi-function on X. Suppose that there exist $x_0 \in X$ and $x_1 \in Gx_0$ such that $\alpha(x_0,x_1) \geq 1$. Assume that if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all n and $x_n \to x$, then $\alpha(x_n,x) \geq 1$ for all n. Then G has a fixed point.*

2 Main results

We begin this section by introducing the following definition.

Definition 2.1 Let (X,d) be a metric space and let $G : X \to CL(X)$ be a mapping. We say that G is generalized (α^*,ψ)-contractive if there exists $\psi \in \Psi$ such that

$$\alpha^*(Gx,Gy)d(y,Gy) \leq \psi\big(d(x,y)\big) \tag{2.1}$$

for each $x \in X$ and $y \in Gx$, where $\alpha^*(Gx,Gy) = \inf\{\alpha(a,b) : a \in Gx, b \in Gy\}$.

Note that an α^*-ψ-contractive mapping is generalized (α^*,ψ)-contractive. In case when $\psi \in \Psi$ is strictly increasing, generalized (α^*,ψ)-contractive is called strictly generalized (α^*,ψ)-contractive. The following lemma is inspired by [3, Lemma 2.2].

Lemma 2.2 *Let (X,d) be a metric space and $B \in CL(X)$. Then, for each $x \in X$ with $d(x,B) > 0$ and $q > 1$, there exists an element $b \in B$ such that*

$$d(x,b) < qd(x,B). \tag{2.2}$$

Proof It is given that $d(x,B) > 0$. Choose

$$\epsilon = (q-1)d(x,B).$$

Then, by using the definition of $d(x,B)$, it follows that there exists $b \in B$ such that

$$d(x,b) < d(x,B) + \epsilon = qd(x,B). \qquad \square$$

Lemma 2.3 *Let (X,d) be a metric space and $G : X \to CL(X)$. Assume that there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} d(x_n,Gx_n) = 0$ and $x_n \to x \in X$. Then x is a fixed point of G if and only if the function $f(\xi) = d(\xi,G\xi)$ is lower semi-continuous at x.*

Proof Suppose $f(\xi) = d(\xi, G\xi)$ is lower semi-continuous at x, then

$$d(x, Gx) \leq \liminf_n f(x_n) = \liminf_n d(x_n, Gx_n) = 0.$$

By the closedness of G it follows that $x \in Gx$. Conversely, suppose that x is a fixed point of G, then $f(x) = 0 \leq \liminf_n f(x_n)$. □

Theorem 2.4 *Let (X, d) be a complete metric space and let $G : X \to CL(X)$ be an α^*-admissible strictly generalized (α^*, ψ)-contractive mapping. Assume that there exist $x_0 \in X$ and $x_1 \in Gx_0$ such that $\alpha(x_0, x_1) \geq 1$. Then x is a fixed point of G if and only if $f(\xi) = d(\xi, G\xi)$ is lower semi-continuous at x.*

Proof By the hypothesis, there exist $x_0 \in X$ and $x_1 \in Gx_0$ such that $\alpha(x_0, x_1) \geq 1$. If $x_0 = x_1$, then we have nothing to prove. Let $x_0 \neq x_1$. If $x_1 \in Gx_1$, then x_1 is a fixed point. Let $x_1 \notin Gx_1$. Since G is α^*-admissible, so $\alpha^*(Gx_0, Gx_1) \geq 1$, we have

$$0 < d(x_1, Gx_1) \leq \alpha^*(Gx_0, Gx_1)d(x_1, Gx_1). \tag{2.3}$$

For given $q > 1$ by Lemma 2.2, there exists $x_2 \in Gx_1$ such that

$$0 < d(x_1, x_2) < qd(x_1, Gx_1). \tag{2.4}$$

It follows from (2.3), (2.4) and (2.1) that

$$0 < d(x_1, x_2) < q\psi\bigl(d(x_0, x_1)\bigr). \tag{2.5}$$

It is clear that $x_1 \neq x_2$ and $\alpha(x_1, x_2) \geq 1$. Thus $\alpha^*(Gx_1, Gx_2) \geq 1$. Since ψ is strictly increasing, by (2.5), we have

$$\psi\bigl(d(x_1, x_2)\bigr) < \psi\bigl(q\psi\bigl(d(x_0, x_1)\bigr)\bigr).$$

Put $q_1 = \frac{\psi(q\psi(d(x_0,x_1)))}{\psi(d(x_1,x_2))}$, then $q_1 > 1$. If $x_2 \in Gx_2$, then x_2 is a fixed point. Let $x_2 \notin Gx_2$, then by Lemma 2.2, there exists $x_3 \in Gx_2$ such that

$$\begin{aligned} 0 &< d(x_2, x_3) < q_1 d(x_2, Gx_2) \leq q_1\alpha^*(Gx_1, Gx_2)d(x_2, Gx_2) \\ &\leq q_1\psi\bigl(d(x_1, x_2)\bigr) = \psi\bigl(q\psi\bigl(d(x_0, x_1)\bigr)\bigr). \end{aligned}$$

It is clear that $x_2 \neq x_3$, $\alpha(x_2, x_3) \geq 1$ and $\psi(d(x_2, x_3)) < \psi^2(q\psi(d(x_0, x_1)))$. Now put $q_2 = \frac{\psi^2(q\psi(d(x_0,x_1)))}{\psi(d(x_2,x_3))}$. Then $q_2 > 1$. If $x_3 \in Gx_3$, then x_3 is a fixed point. Let $x_3 \notin Gx_3$. Then by Lemma 2.2 there exists $x_4 \in Gx_3$ such that

$$\begin{aligned} 0 &< d(x_3, x_4) < q_2 d(x_3, Gx_3) \leq q_2\alpha^*(Gx_2, Gx_3)d(x_3, Gx_3) \\ &\leq q_2\psi\bigl(d(x_2, x_3)\bigr) = \psi^2\bigl(q\psi\bigl(d(x_0, x_1)\bigr)\bigr). \end{aligned}$$

By continuing the same process, we get a sequence $\{x_n\}$ in X such that $x_{n+1} \in Gx_n$. Also, $x_n \neq x_{n+1}$, $\alpha(x_n, x_{n+1}) \geq 1$ and $0 < d(x_n, x_{n+1}) < \psi^{n-1}(q\psi(d(x_0, x_1)))$ or

$$0 < d(x_n, Gx_n) < \psi^{n-1}\bigl(q\psi\bigl(d(x_0, x_1)\bigr)\bigr). \tag{2.6}$$

For each $m > n$, we have

$$d(x_n, x_m) \le \sum_{i=n}^{m-1} d(x_i, x_{i+1}) < \sum_{i=n}^{m-1} \psi^{i-1}\bigl(q\psi\bigl(d(x_0, x_1)\bigr)\bigr).$$

Since $\psi \in \Psi$, it follows that $\{x_n\}$ is a Cauchy sequence in X. Thus there is $x \in X$ such that $x_n \to x$. Letting $n \to \infty$ in (2.6), we have

$$\lim_{n\to\infty} d(x_n, Gx_n) = 0. \tag{2.7}$$

The rest of the proof follows from Lemma 2.3. □

Example 2.5 Let $X = \mathbb{R}$ be endowed with the usual metric d. Define $G : X \to CL(X)$ and $\alpha : X \times X \to [0, \infty)$ by

$$Gx = \begin{cases} [x, \infty) & \text{if } x \ge 0, \\ (-\infty, -x^2] & \text{if } x < 0 \end{cases} \tag{2.8}$$

and

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \ge 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

Let $\psi(t) = \frac{t}{2}$ for all $t \ge 0$. For each $x \in X$ and $y \in Gx$, we have

$$\alpha^*(Gx, Gy)d(y, Gy) = 0 \le \frac{1}{2}d(x, y).$$

Hence G is a strictly generalized (α^*, ψ)-contractive mapping. Clearly, G is α^*-admissible. Also, we have $x_0 = 1$ and $x_1 = 1 \in Gx_0$ such that $\alpha(x_0, x_1) = 1$. Therefore, all conditions of Theorem 2.4 are satisfied and G has infinitely many fixed points. Note that Theorem 1.3 in Section 1 is not applicable here. For example, take $x = 1$ and $y = -1$.

Corollary 2.6 *Let $(X, \preceq, d)$ be a complete ordered metric space, $\psi \in \Psi$ be a strictly increasing map and $G : X \to CL(X)$ be a mapping such that for each $x \in X$ and $y \in Gx$ with $x \preceq y$, we have*

$$d(y, Gy) \le \psi\bigl(d(x, y)\bigr). \tag{2.10}$$

Also, assume that

(i) *there exist $x_0 \in X$ and $x_1 \in Gx_0$ such that $x_0 \preceq x_1$,*
(ii) *if $x \preceq y$, then $Gx \prec_r Gy$.*

Then x is a fixed point of G if and only if $f(\xi) = d(\xi, G\xi)$ is lower semi-continuous at x.

Proof Define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \preceq y, \\ 0 & \text{otherwise.} \end{cases}$$

By using condition (i) and the definition of α, we have $\alpha(x_0, x_1) = 1$. Also, from condition (ii), we have $x \preceq y$ implies $Gx \prec_r Gy$; by using the definitions of α and $\prec_r$, we have $\alpha(x, y) = 1$ implies $\alpha^*(Gx, Gy) = 1$. Moreover, it is easy to check that G is a strictly generalized (α^*, ψ)-contractive mapping. Therefore, by Theorem 2.4, x is a fixed point of G if and only if $f(\xi) = d(\xi, G\xi)$ is lower semi-continuous at x. □

Definition 2.7 Let (X, d) be a metric space and $G : X \to B(X)$ be a mapping. We say that G is a generalized (α^*, ψ, δ)-contractive mapping if there exists $\psi \in \Psi$ such that

$$\alpha^*(Gx, Gy)\delta(y, Gy) \leq \psi\big(d(x, y)\big) \tag{2.11}$$

for each $x \in X$ and $y \in Gx$, where $\alpha^*(Gx, Gy) = \inf\{\alpha(a, b) : a \in Gx, b \in Gy\}$.

Lemma 2.8 *Let (X, d) be a metric space and $G : X \to B(X)$. Assume that there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} \delta(x_n, Gx_n) = 0$ and $x_n \to x \in X$. Then $\{x\} = Gx$ if and only if the function $f(\xi) = \delta(\xi, G\xi)$ is lower semi-continuous at x.*

Proof Suppose that $f(\xi) = \delta(\xi, G\xi)$ is lower semi-continuous at x, then

$$\delta(x, Gx) \leq \liminf_n f(x_n) = \liminf_n \delta(x_n, Gx_n) = 0.$$

Hence, $\{x\} = Gx$ because $\delta(A, B) = 0$ implies $A = B = \{a\}$. Conversely, suppose that $\{x\} = Gx$. Then $f(x) = 0 \leq \liminf_n f(x_n)$. □

Theorem 2.9 *Let (X, d) be a complete metric space and let $G : X \to B(X)$ be an α^*-admissible generalized (α^*, ψ, δ)-contractive mapping. Assume that there exist $x_0 \in X$ and $x_1 \in Gx_0$ such that $\alpha(x_0, x_1) \geq 1$. Then there exists $x \in X$ such that $\{x\} = Gx$ if and only if $f(\xi) = \delta(\xi, G\xi)$ is lower semi-continuous at x.*

Proof By the hypothesis of the theorem, there exist $x_0 \in X$ and $x_1 \in Gx_0$ such that $\alpha(x_0, x_1) \geq 1$. Assume that $x_0 \neq x_1$, for otherwise, x_0 is a fixed point. Let $x_1 \notin Gx_1$. As G is α^*-admissible, we have $\alpha^*(Gx_0, Gx_1) \geq 1$. Then

$$\delta(x_1, Gx_1) \leq \alpha^*(Gx_0, Gx_1)\delta(x_1, Gx_1) \leq \psi\big(d(x_0, x_1)\big). \tag{2.12}$$

Since $Gx_1 \neq \emptyset$, there is $x_2 \in Gx_1$. Then

$$0 < d(x_1, x_2) \leq \delta(x_1, Gx_1). \tag{2.13}$$

From (2.12) and (2.13), we have

$$0 < d(x_1, x_2) \leq \psi\big(d(x_0, x_1)\big). \tag{2.14}$$

Since ψ is nondecreasing, we have

$$\psi\big(d(x_1, x_2)\big) \leq \psi^2\big(d(x_0, x_1)\big). \tag{2.15}$$

As $x_2 \in Gx_1$, we have $\alpha(x_1, x_2) \geq 1$. Since $Gx_2 \neq \emptyset$, there is $x_3 \in Gx_2$. Assume that $x_2 \neq x_3$, for otherwise, x_2 is a fixed point of G. Then

$$\begin{aligned} 0 < d(x_2, x_3) &\leq \delta(x_2, Gx_2) \leq \alpha^*(Gx_1, Gx_2)\delta(x_2, Gx_2) \\ &\leq \psi\bigl(d(x_1, x_2)\bigr) \leq \psi^2\bigl(d(x_0, x_1)\bigr). \end{aligned} \tag{2.16}$$

Since ψ is nondecreasing, we have

$$\psi\bigl(d(x_2, x_3)\bigr) \leq \psi^3\bigl(d(x_0, x_1)\bigr). \tag{2.17}$$

By continuing in this way, we get a sequence $\{x_n\}$ in X such that $x_{n+1} \in Gx_n$ and $x_n \neq x_{n+1}$ for $n = 0, 1, 2, 3, \ldots$. Further we have

$$0 < d(x_n, x_{n+1}) \leq \delta(x_n, Gx_n) \leq \psi^n\bigl(d(x_0, x_1)\bigr). \tag{2.18}$$

For each $m > n$, we have

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \psi^i\bigl(d(x_0, x_1)\bigr).$$

Since $\psi \in \Psi$, it follows that $\{x_n\}$ is a Cauchy sequence in X. As X is complete, there exists $x \in X$ such that $x_n \to x$. Letting $n \to \infty$ in (2.18), we have

$$\lim_{n \to \infty} \delta(x_n, Gx_n) = 0. \tag{2.19}$$

The rest of the proof follows from Lemma 2.8. □

Example 2.10 Let $X = \{0, 2, 4, 6, 8, 10, \ldots\}$ be endowed with the usual metric d. Define $G : X \to B(X)$ and $\alpha : X \times X \to [0, \infty)$ by

$$Gx = \begin{cases} \{(x-2), x\} & \text{if } x \neq 0, \\ \{0\} & \text{if } x = 0 \end{cases}$$

and

$$\alpha(x, y) = \begin{cases} 0 & \text{if } x = y \neq 0, \\ 1 & \text{if } x = y = 0, \\ \frac{1}{4} & \text{otherwise.} \end{cases}$$

Let $\psi(t) = \frac{t}{2}$ for all $t \geq 0$. For each $x \in X$ and $y \in Gx$, we have

$$\alpha^*(Gx, Gy)\delta(y, Gy) \leq \frac{1}{2}\bigl(d(x, y)\bigr).$$

Hence G is a generalized (α^*, ψ, δ)-contractive mapping. Clearly, G is α^*-admissible. Also, we have $x_0 = 0 \in X$ and $x_1 = 0 \in G0$ such that $\alpha(x_0, x_1) = 1$. Therefore, all conditions of Theorem 2.9 are satisfied and G has infinitely many fixed points.

Corollary 2.11 *Let $(X, \preceq, d)$ be a complete ordered metric space, $\psi \in \Psi$ and $G : X \to B(X)$ be a mapping such that for each $x \in X$ and $y \in Gx$ with $x \preceq y$, we have*

$$\delta(y, Gy) \leq \psi\bigl(d(x, y)\bigr). \tag{2.20}$$

Also, assume that

(i) *there exists $x_0 \in X$ such that $\{x_0\} \prec_1 Gx_0$, i.e., there exists $x_1 \in Gx_0$ such that $x_0 \preceq x_1$,*

(ii) *if $x \preceq y$, then $Gx \prec_r Gy$.*

Then there exists $x \in X$ such that $\{x\} = Gx$ if and only if $f(\xi) = \delta(\xi, G\xi)$ is lower semi-continuous at x.

Proof Define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \preceq y, \\ 0 & \text{otherwise.} \end{cases}$$

By using condition (i) and the definition of α, we have $\alpha(x_0, x_1) = 1$. Also, from condition (ii), we have $x \preceq y$ implies $Gx \prec_r Gy$, by using the definitions of α and $\prec_r$, we have $\alpha(x, y) = 1$ implies $\alpha^*(Gx, Gy) = 1$. Moreover, it is easy to check that G is a generalized (α^*, ψ, δ)-contractive mapping. Therefore, by Theorem 2.9, there exists $x \in X$ such that $\{x\} = Gx$ if and only if $f(\xi) = \delta(\xi, G\xi)$ is lower semi-continuous at x. □

Competing interests
The authors declare that they have no competing interests.

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