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Fixed point theorems for cyclic contraction mappings in fuzzy metric spaces

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Abstract

In the present paper, an extension of the Edelstein contraction theorem for cyclic contractions in a fuzzy metric space is established, which also can be considered as a generalization of the fuzzy Edelstein contraction theorem introduced by Grabiec. Additionally, we extend a fixed point theorem in G -complete fuzzy metric spaces given by Shen *et al.* to M -complete fuzzy metric spaces. Meantime, two examples are constructed to illustrate the corresponding results, respectively.

1 Introduction

The contraction type mappings in fuzzy metric spaces play a crucial role in fixed point theory. In 1988, Grabiec [1] first defined the Banach contraction in a fuzzy metric space and extended fixed point theorems of Banach and Edelstein to fuzzy metric spaces. Following Grabiec's approach, Mishra *et al.* [2] obtained some common fixed point theorems for asymptotically commuting mappings in fuzzy metric spaces. In the sequel, Vasuki [3] offered a generalization of Grabiec's fuzzy Banach contraction theorem and proved a common fixed point theorem for a sequence of mappings in a fuzzy metric space. Afterwards, Cho [4] presented the concept of compatible mappings of type (α) in fuzzy metric spaces and then studied the fixed point theory. Several years later, Singh and Chauhan [5] introduced the concept of compatible mapping and proved two common fixed point theorems in the fuzzy metric space with the strongest triangular norm. In 2002, Sharma [6] further extended some known results of fixed point theory for compatible mappings in fuzzy metric spaces. In the same year, Gregori and Sapena [7] introduced the notion of fuzzy contractive mapping and presented some fixed point theorems for complete fuzzy metric spaces in the sense of George and Veeramani, and also for Kramosil and Michalek's fuzzy metric spaces which are complete in Grabiec's sense. Soon after, Mihet [8] proposed a fuzzy Banach theorem for (weak) B -contraction in M -complete fuzzy metric spaces. Later, Mihet [9, 10] further studied the fixed point theory for the different contraction mappings in fuzzy metric spaces, and introduced some new contraction mappings, such as Edelstein fuzzy contractive mappings, fuzzy ψ -contraction of (ε, λ) type, *etc.* Based on the definition of fuzzy contractive mapping introduced by Mihet [9], Abbas *et al.* [11] proposed the notion of φ -weak contraction and obtained several results of fixed point in a G -complete metric space. Recently, Shen *et al.* [12] constructed a novel class of φ -contractions and proved a fixed point theorem for this kind of mappings in an M -complete fuzzy metric space.

In 2003, Kirk *et al.* [13] first introduced a novel class of contractive conditions, namely, cyclical contractive conditions, and extended several fixed point theorems including Banach, Edelstein, Caristi and a fixed point theorem for nonexpansive mappings in a Banach space. Motivated by the notion of cyclic contraction, Păcurar and Ioan [14] proposed the concept of cyclic φ -contraction. Based on this type of contractive condition, they constructed a fixed point theorem in a classical complete metric space. Moreover, several problems related to the fixed point are investigated, such as data dependence, well-posedness, limit shadowing property, and so on. Hereafter, Karapinar [15] proposed the concept of cyclic weak ϕ -contraction, and then discussed some main results proposed by Păcurar and Ioan [14] under this kind of contractive conditions. Following these ideas mentioned above, Shen *et al.* [16] extended the notion of cyclic φ -contraction to fuzzy metric spaces, and proposed a fixed point theorem for this type of mappings in G -complete fuzzy metric spaces in the sense of George and Veeramani. Simultaneously, some of the results concerned with the fixed point proposed by Păcurar and Ioan [14] were also generalized to fuzzy metric spaces. Analogously, Gopal *et al.* [17] also introduced the notion of cyclic weak φ -contraction in fuzzy metric spaces and obtained some results on the existence and uniqueness of a fixed point in fuzzy metric spaces in the sense of G -completeness. Furthermore, Murthy *et al.* [18] generalized the concept of cyclic weak φ -contraction to cyclic weak (φ, ψ) -contraction and utilized these contractive mappings to prove a fixed point theorem in a G -complete fuzzy metric space.

In this paper, we present an extension of the Edelstein contraction theorem for cyclic contractions in a fuzzy metric space. Our results can also be viewed as a generalization of the fuzzy Edelstein contraction theorem proposed by Grabiec [1]. Besides, we also generalized a fixed theorem in G -complete fuzzy metric spaces given by Shen *et al.* [16] to M -complete fuzzy metric spaces. In addition, it should be pointed out that the function φ of cyclic φ -contraction proposed by us is different from that of cyclic φ -contraction introduced in [17, 18].

2 Preliminaries

For completeness and clarity, in this section, some related concepts and conclusions are summarized and introduced below. Let \mathbb{N} denote the set of all positive integers.

Definition 2.1 (Schweizer and Sklar [19]) A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a *continuous triangular norm* (in short, continuous t -norm) if it satisfies the following conditions:

- (TN-1) $*$ is commutative and associative;
- (TN-2) $*$ is continuous;
- (TN-3) $a * 1 = a$ for every $a \in [0, 1]$;
- (TN-4) $a * b \leq c * d$ whenever $a \leq c$, $b \leq d$ and $a, b, c, d \in [0, 1]$.

Definition 2.2 (George and Veeramani [20]) A *fuzzy metric space* is an ordered triple $(X, M, *)$ such that X is a nonempty set, $*$ is a continuous t -norm and M is a fuzzy set on $X \times X \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X$, $s, t > 0$:

- (FM-1) $M(x, y, t) > 0$;
- (FM-2) $M(x, y, t) = 1$ if and only if $x = y$;
- (FM-3) $M(x, y, t) = M(y, x, t)$;

$$(FM-4) \quad M(x, y, t) * M(y, z, s) \leq M(x, z, t + s);$$

$$(FM-5) \quad M(x, y, \cdot) : (0, \infty) \rightarrow (0, 1] \text{ is continuous.}$$

Definition 2.3 (George and Veeramani [20]) Let $(X, M, *)$ be a fuzzy metric space. Then:

- (i) A sequence $\{x_n\}$ in X is said to *converge to* x in X , denoted by $\lim_{n \rightarrow \infty} x_n = x$ (or $x_n \rightarrow x$), if and only if $\lim_{n \rightarrow \infty} M(x_n, x, t) = 1$ for all $t > 0$, i.e., for each $r \in (0, 1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - r$ for all $n \geq n_0$;
- (ii) A sequence $\{x_n\}$ is an *M-Cauchy sequence* if and only if for all $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_m, x_n, t) > 1 - \epsilon$ for any $m, n > n_0$ and $t > 0$;
- (iii) A fuzzy metric space $(X, M, *)$ is called *M-complete* if every M-Cauchy sequence is convergent.

Notice that a fuzzy metric space is called *compact* if every sequence contains a convergent subsequence.

Lemma 2.1 (Grabiec [1]) *Let $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} y_n = y$. If $M(x, y, \cdot)$ is continuous, then $\lim_{n \rightarrow \infty} M(x_n, y_n, t) = M(x, y, t)$ for any $t > 0$.*

Lemma 2.2 (Grabiec [1], fuzzy Edelstein contraction theorem) *Let $(X, M, *)$ be a compact fuzzy metric space. $T : X \rightarrow X$ a mapping satisfying the following condition:*

$$M(Tx, Ty, \cdot) > M(x, y, \cdot)$$

for all $x \neq y$. Then T has a unique fixed point.

Definition 2.4 (Păcurar and Rus [14]) Let X be a nonempty set, r be a positive integer and $f : X \rightarrow X$ be a mapping. $X = \bigcup_{i=1}^r X_i$ is a *cyclic representation* of X with respect to f if

- (i) $X_i, i = 1, 2, \dots, r$, are nonempty sets;
- (ii) $f(X_1) \subset X_2, \dots, f(X_{r-1}) \subset X_r, f(X_r) \subset X_1$.

Definition 2.5 A function $\varphi : [0, 1] \rightarrow [0, 1]$ is called a *comparison function* if it satisfies:

- (i) φ is nondecreasing and left continuous;
- (ii) $\varphi(t) > t$ for all $t \in (0, 1)$.

Lemma 2.3 (Vetro [21]) *If φ is a comparison function, then*

- (i) $\varphi(1) = 1$;
- (ii) $\lim_{n \rightarrow +\infty} \varphi^n(t) = 1$ for all $t \in (0, 1)$, where $\varphi^n(t)$ denotes the composition of $\varphi(t)$ with itself n times.

Definition 2.6 (Shen *et al.* [16]) Let $(X, M, *)$ be a fuzzy metric space, r be a positive integer, $A_1, A_2, \dots, A_r \in P_{cl}(X)$, where $P_{cl}(X)$ denotes the collection of nonempty closed subsets of X , $Y = \bigcup_{i=1}^r A_i$ and $f : Y \rightarrow Y$ a mapping. If

- (i) $\bigcup_{i=1}^r A_i$ is cyclic representation of Y with respect to f ;
- (ii) there exists a comparison function $\varphi : [0, 1] \rightarrow [0, 1]$ such that

$$M(f(x), f(y), t) \geq \varphi(M(x, y, t))$$

for any $x \in A_i, y \in A_{i+1}$ and $t > 0$, where $A_{r+1} = A_1$, then f is called *cyclic φ -contraction* in the fuzzy metric space.

3 Main results

In this section, we prove an extended fuzzy Edelstein contraction theorem for cyclic contractive mappings in a fuzzy metric space. Meantime, we also prove that a fixed point theorem given by Shen *et al.* [16] does hold even if G -completeness of the fuzzy metric space is replaced by M -completeness.

Theorem 3.1 (The extended fuzzy Edelstein contraction theorem) *Let $(X, M, *)$ be a fuzzy metric space, r be a positive integer, $A_1, A_2, \dots, A_r \in P_{cl}(X)$, at least one of which is compact. $Y = \bigcup_{i=1}^r A_i, f : Y \rightarrow Y$ a mapping. Assume that*

- (C1) $\bigcup_{i=1}^r A_i$ is a cyclic representation of Y with respect to f ;
- (C2) $M(f(x), f(y), \cdot) > M(x, y, \cdot)$ for any $x \in A_i, y \in A_{i+1} (x \neq y), i = 1, 2, \dots, r$.

Then f has a unique fixed point $x^ \in \bigcap_{i=1}^r A_i$.*

Proof Without loss of generality, we assume that A_1 is compact. Define $d(t) = \sup\{M(x, y, t) : x \in A_1, y \in A_r\}$ for each $t > 0$. According to Definition 2.2, we know that $0 < d(t) \leq 1$. Since A_1 is compact, for an arbitrary but fixed $t > 0$, there exist $x_0 \in A_1$ and a sequence $\{y_n\} \subset A_r$ such that $\lim_{n \rightarrow \infty} M(x_0, y_n, t) = d(t)$. Now, we claim that $d(t) = 1$ for each given $t > 0$. Otherwise, we can assume that $0 < d(t) < 1$. By the condition (FM-2), we know that $x_0 \neq y_n$ for each $n \in \mathbb{N}$. Furthermore, we can obtain that $f^i(x_0) \neq f^i(y_n)$ for $1 \leq i \leq r$. Thus, for every $n \in \mathbb{N}$, we have

$$\begin{aligned} M(f^{r+1}(x_0), f^{r+1}(y_n), t) &> M(f^r(x_0), f^r(y_n), t) \\ &> \dots > M(f(x_0), f(y_n), t) > M(x_0, y_n, t). \end{aligned}$$

Since the sequence $\{f^{r+1}(y_n)\} \subset A_1$ and the compactness of A_1 , there exists a subsequence of $\{f^{r+1}(y_n)\}$ such that it converges to some $z \in A_1$. According to the foregoing inequality and Lemma 2.1, by supposing that $n \rightarrow \infty$, it follows that

$$M(f^{r+1}(x_0), z, t) \geq d(t).$$

According to the arbitrariness of t , if $M(f^{r+1}(x_0), z, t) = 1$, then we have $f^{r+1}(x_0) = z$. Furthermore, we can obtain that $M(f^{2r}(x_0), f^{r-1}(z), t) = 1$. Obviously, this is a contradiction. On the other hand, if $M(f^{r+1}(x_0), z, t) < 1$, then we have $f^{r+1}(x_0) \neq z$. Thus, by the conditions (C1) and (C2), we can obtain

$$\begin{aligned} M(f^{2r}(x_0), f^{r-1}(z), t) &> M(f^{2r-1}(x_0), f^{2r-2}(z), t) \\ &> \dots > M(f^{r+1}(x_0), z, t) \geq d(t). \end{aligned}$$

As $f^{2r}(x_0) \in A_1$ and $f^{r-1}(z) \in A_r$, this implies a contradiction. Therefore, we conclude that $d(t) = 1$ for each $t > 0$, and $A_1 \cap A_r \neq \emptyset$. By (C1), $f(A_1 \cap A_r) \subset A_1$ and $f(A_1 \cap A_r) \subset A_2$, we have $f(A_1 \cap A_r) \subset A_1 \cap A_2$. Obviously, $A_1 \cap A_2 \neq \emptyset$.

Now, we redefine the sets $A'_1 = A_1 \cap A_2, A'_2 = A_2 \cap A_3, \dots, A'_r = A_r \cap A_1$. In view of (C1), these sets are both nonempty and closed. As A'_1 is compact, the conditions (C1) and (C2)

are satisfied by f with respect to the family $\{A'_i\}$, $i = 1, 2, \dots, r$. By repeating the above process, we obtain $A'_1 \cap A'_r \neq \emptyset$, which implies $A_1 \cap A_2 \cap A_3 \neq \emptyset$.

Continuing in this manner, we can obtain

$$A := \bigcap_{i=1}^r A_i \neq \emptyset.$$

Consider the restriction $f|_A$ of f on A . Obviously, the mapping $f|_A : A \rightarrow A$ satisfies the condition (C2) and A is compact. By Lemma 2.2, we conclude that $f|_A$ has a unique fixed point in A . However, by the condition (C1), it can easily be shown that the uniqueness follows from the fact that any fixed point of f necessarily lies in A . \square

Remark 1 Theorem 3.1 can be regarded as an extension of the fuzzy Edelstein contraction theorem (Lemma 2.2) given by Grabiec [1], and it is not related with the compactness and completeness of the fuzzy metric space $(X, M, *)$.

Example 1 Let $X = [0, 4]$ be endowed with the usual metric $d(x, y) = |x - y|$. Define $M(x, y, t) = \frac{t}{t + |x - y|}$ for all $x, y \in X$ and $t > 0$. Clearly, $(X, M, *)$ is a fuzzy metric space with respect to t -norm $a * b = ab$.

Let $f : X \rightarrow X$ be defined as $f(x) = 2 - \frac{x}{2}$. Set $A_1 = [0, 2]$, $A_2 = [1, 4]$. Owing to $f(A_1) = [1, 2] \subset [1, 4] = A_2$, $f(A_2) = [0, \frac{3}{2}] \subset [0, 2] = A_1$, we can obtain that $A_1 \cup A_2$ is a cyclic representation of X with respect to f . In addition, for any $x \in A_1$, $y \in A_2$ and $x \neq y$, we have

$$M(f(x), f(y), t) = \frac{t}{t + \frac{1}{2}|x - y|} > \frac{t}{t + |x - y|} = M(x, y, t).$$

Thus, all the conditions of Theorem 3.1 are satisfied and $x = \frac{4}{3}$ is the unique fixed point of f .

Theorem 3.2 Let $(X, M, *)$ be an M -complete fuzzy metric space, r be a positive integer, $A_1, A_2, \dots, A_r \in P_{cl}(X)$, $Y = \bigcup_{i=1}^r A_i$, $\varphi : [0, 1] \rightarrow [0, 1]$ be a comparison function, and $f : Y \rightarrow Y$ be a mapping. Assume that

- (C1) $\bigcup_{i=1}^r A_i$ is a cyclic representation of Y with respect to f ;
- (C2) f is a cyclic φ -contraction.

Then f has a unique fixed point $x^* \in \bigcap_{i=1}^r A_i$ and the iterative sequence $\{x_n\}_{n \geq 0}$ ($x_n = f(x_{n-1})$, $n \in \mathbb{N}$) converges to x^* for any initial point $x_0 \in Y$.

Proof Let $x_0 \in Y$ be an arbitrary initial point. Define $\tau_n(t) = M(x_n, x_{n+1}, t)$ for all $n \in \mathbb{N} \cup \{0\}$, $t > 0$. Now, we show that f has a fixed point. In particular, if there exists an $n_0 \in \mathbb{N} \cup \{0\}$ such that $x_{n_0+1} = x_{n_0}$, i.e., $f(x_{n_0}) = x_{n_0}$, then it is obvious that x_{n_0} is a fixed point of f . Otherwise, the inequality $x_n \neq x_{n+1}$ holds for each n , which implies that $0 < \tau_n(t) < 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $t > 0$. By (C2), $\{\tau_n(t)\}$ is monotone increasing and bounded from below for every $t > 0$ with respect to n . So, $\lim_{n \rightarrow \infty} \tau_n(t) = \tau(t) \leq 1$. Suppose that $0 < \tau(t) < 1$. Since

$$\tau_{n+1}(t) = M(x_{n+1}, x_{n+2}, t) \geq \varphi(M(x_n, x_{n+1}, t)) = \varphi(\tau_n(t))$$

for each t , letting $n \rightarrow \infty$, we conclude that $\tau(t) \geq \varphi(\tau(t))$. Obviously, which is a contradiction. Thus, we obtain $\lim_{n \rightarrow \infty} \tau_n(t) = 1$.

Next, we show that the iterative sequence $\{x_n\}_{n \geq 0}$ is an M -Cauchy sequence. Suppose that it is not. Then there exist $\epsilon \in (0, 1)$, $t > 0$, and two sequences $\{p(n)\}$ and $\{q(n)\}$ such that for every $n \in \mathbb{N} \cup \{0\}$, so we obtain that

$$p(n) > q(n) \geq n, \quad M(x_{p(n)}, x_{q(n)}, t) \leq 1 - \epsilon.$$

Furthermore, for each n , we can choose the smallest number $p(n)$ greater than $q(n)$ such that

$$M(x_{p(n)-1}, x_{q(n)-1}, t) > 1 - \epsilon, \quad M(x_{p(n)-1}, x_{q(n)}, t) > 1 - \epsilon.$$

For each $n \in \mathbb{N} \cup \{0\}$, we put $s_n(t) = M(x_{p(n)}, x_{q(n)}, t)$. Then we have

$$\begin{aligned} 1 - \epsilon &\geq s_n(t) = M(x_{p(n)}, x_{q(n)}, t) \\ &\geq M(x_{p(n)-1}, x_{p(n)}, t/2) * M(x_{p(n)-1}, x_{q(n)}, t/2) \\ &\geq \tau_{p(n)-1}(t/2) * (1 - \epsilon). \end{aligned}$$

Since $\tau_{p(n)-1}(t/2) \rightarrow 1$ as $n \rightarrow \infty$ for the foregoing t , letting $n \rightarrow \infty$, we conclude that the sequence $\{s_n(t)\}$ converges to $1 - \epsilon$. Moreover, for $p(n) > q(n)$, there exists $0 \leq j \leq r - 1$ such that $p(n) - q(n) + j = 1 \pmod r$ for infinitely many n .

Case I: If $j = 0$, then, for such n , we have

$$\begin{aligned} s_n(t) &= M(x_{p(n)}, x_{q(n)}, t) \\ &\geq M(x_{p(n)}, x_{p(n)+1}, t/3) * M(x_{p(n)+1}, x_{q(n)+1}, t/3) * M(x_{q(n)+1}, x_{q(n)}, t/3) \\ &\geq \tau_{p(n)}(t/3) * \varphi(M(x_{p(n)}, x_{q(n)}, t/3)) * \tau_{q(n)}(t/3) \\ &= \tau_{p(n)}(t/3) * \varphi(s_n(t)) * \tau_{q(n)}(t/3). \end{aligned}$$

Letting $n \rightarrow \infty$, we obtain that

$$1 - \epsilon \geq \varphi(1 - \epsilon) > 1 - \epsilon.$$

Clearly, this leads to a contradiction.

Case II: If $j \neq 0$, similarly, for such n , we can obtain

$$\begin{aligned} s_n(t) &= M(x_{p(n)}, x_{q(n)}, t) \\ &\geq M(x_{p(n)}, x_{p(n)+1}, t/(j+3)) * M(x_{p(n)+1}, x_{p(n)+2}, t/(j+3)) * \cdots \\ &\quad * M(x_{p(n)+j}, x_{p(n)+j+1}, t/(j+3)) * M(x_{p(n)+j+1}, x_{q(n)+1}, t/(j+3)) \\ &\quad * M(x_{q(n)+1}, x_{q(n)}, t/(j+3)) \\ &\geq \tau_{p(n)}(t/(j+3)) * \tau_{p(n)+1}(t/(j+3)) * \cdots * \tau_{p(n)+j}(t/(j+3)) \\ &\quad * \varphi(M(x_{p(n)+j}, x_{q(n)}, t/(j+3))) * \tau_{q(n)}(t/(j+3)). \end{aligned}$$

Supposing that $n \rightarrow \infty$, we can get the same contradiction as Case I.

Moreover, we consider another case. That is, there exists $n_0 \in \mathbb{N} \cup \{0\}$ such that $M(x_m, x_n, t) \leq 1 - \epsilon$ for all $m, n \geq n_0$. Therefore, for any $p \in \mathbb{N}$, it is obvious that $M(x_{n_0+p+1}, x_{n_0+p+2}, t) \leq 1 - \epsilon$. By (C2), we know that the sequence $\{M(x_{n_0+p+1}, x_{n_0+p+2}, t)\}$ is monotone increasing and bounded with respect to p for each t . So, there exists $\alpha(t) \in (0, 1 - \epsilon]$ such that $\lim_{p \rightarrow \infty} M(x_{n_0+p+1}, x_{n_0+p+2}, t) = \alpha(t)$. Thus, we can obtain

$$M(x_{n_0+p+2}, x_{n_0+p+3}, t) \geq \varphi(M(x_{n_0+p+1}, x_{n_0+p+2}, t)).$$

Letting $p \rightarrow \infty$, for the forgoing t , we have $\alpha(t) \geq \varphi(\alpha(t))$, which is also a contradiction.

Based on the above analysis, we obtain that $\{x_n\}_{n \geq 0}$ is an M -Cauchy sequence in the M -complete fuzzy metric subspace Y . Furthermore, we conclude that there exists an $x^* \in Y$ such that $\lim_{n \rightarrow \infty} x_n = x^*$.

Now, we show that x^* is a fixed point of f . By the condition (C1), it follows that the iterative sequence $\{x_n\}_{n \geq 0}$ has an infinite number of terms in each $A_i, i = 1, 2, \dots, r$. Since Y is M -complete, from each $A_i, i = 1, 2, \dots, r$, we can extract a subsequence of $\{x_n\}_{n \geq 0}$ which converges to x^* as well. Because each $A_i, i = 1, 2, \dots, r$, is closed, we conclude that $x^* \in \bigcap_{i=1}^r A_i$ and thus $\bigcap_{i=1}^r A_i \neq \emptyset$.

Set $Z = \bigcap_{i=1}^r A_i$. Clearly, Z is closed and M -complete. Consider the restriction of f on Z , that is, $f|_Z : Z \rightarrow Z$. Next, we show that $f|_Z$ has a unique fixed point in $Z \subset Y$.

For the forgoing $x^* \in Z$, there exists $i_n (n \in \mathbb{N})$ such that $x_n \in A_{i_n}$. Therefore, we can choose $A_{i_{n+1}}$ such that $f|_Z(x_n^*) \in A_{i_{n+1}}$. Accordingly, for any $t > 0$, we can obtain

$$\begin{aligned} M(f|_Z(x_n^*), x_n^*, t) &= M(f(x_n^*), x_n^*, t) \\ &\geq M(f(x_n^*), f(x_n), t/2) * M(x_{n+1}, x_n^*, t/2) \\ &\geq \varphi(M(x_n^*, x_n, t/2)) * M(x_{n+1}, x_n^*, t/2). \end{aligned}$$

By supposing that $n \rightarrow \infty$, we conclude that $M(f|_Z(x_n^*), x_n^*, t) = 1$ for any $t > 0$, i.e., $f|_Z(x_n^*) = x_n^*$. That is to say, x^* is a fixed point of $f|_Z$, which is obtained by iteration from the initial point $x_0 \in Y$.

To show uniqueness, we assume that $y \in \bigcap_{i=1}^r A_i$ is another fixed point of $f|_Z$. As $x^*, y \in A_i$ for every $i \in \{1, 2, \dots, r\}$, we obtain that

$$\begin{aligned} M(x^*, y, t) &= M(f|_Z(x^*), f|_Z(y), t) \\ &= M(f(x^*), f(y), t) \\ &\geq \varphi(M(x^*, y, t)) \\ &> M(x^*, y, t). \end{aligned}$$

This leads to a contradiction. Thus, x^* is a unique fixed point of $f|_Z$.

Finally, we have to prove that the iterative sequence $\{x_n\}_{n \geq 0}$ converges to x^* for any starting point $x_0 \in Y$. Since $x_0 \in Y = \bigcup_{i=1}^m A_i$, there exists $i_0 \in \{1, 2, \dots, r\}$ such that $x_0 \in A_{i_0}$. As $x^* \in \bigcap_{i=1}^m A_i$, we know that $x^* \in A_{i_0+1}$ as well. Then, for any $t > 0$, we have

$$M(f(x), f(x^*), t) \geq \varphi(M(x, x^*, t)).$$

By induction, we can obtain

$$\begin{aligned} M(x_n, x^*, t) &= M(f^n(x_0), x^*, t) \\ &= M(f^n(x_0), f(x^*), t) \\ &= M(f(f^{n-1}(x_0)), f(x^*), t) \\ &\geq \varphi(M(f^{n-1}(x_0), x^*, t)) \\ &\geq \dots \geq \varphi^n(M(x_0, x^*, t)). \end{aligned}$$

According to Lemma 2.3, supposing $x_0 \neq x^*$, it follows immediately that $x_n \rightarrow x^*$ as $n \rightarrow \infty$. Hence, the iterative sequence $\{x_n\}_{n \geq 0}$ converges to the unique fixed point x^* of f for any initial point $x_0 \in Y$. This completes the proof of the theorem. \square

Remark 2 Based on Theorem 3.2, the remaining results concerning fixed point theory given in [16] can be directly obtained in the sense of M -completeness.

Example 2 Let $X = [0, 3]$ be equipped with the ordinary metric $d(x, y) = |x - y|$, $\varphi(\tau) = \sqrt{\tau}$ for all $\tau \in [0, 1]$. Define $M(x, y, t) = e^{-\frac{2|x-y|}{t}}$ for all $x, y \in X$ and $t > 0$. Clearly, $(X, M, *)$ is an M -complete fuzzy metric space with respect to t -norm $a * b = ab$.

Let $f : X \rightarrow X$ be defined as

$$f(x) = \begin{cases} 1, & x \in [0, 1], \\ \frac{3-x}{2}, & x \in (1, 3]. \end{cases}$$

Set $A_1 = [\frac{2}{3}, 3]$, $A_2 = [0, \frac{3}{2}]$, it is obvious that $f(A_1) = [0, 1] \subseteq A_2$, $f(A_2) = [\frac{3}{4}, 1] \subseteq A_1$. Besides, for any $x \in A_1$, $y \in A_2$, we have

$$d(f(x), f(y)) = |f(x) - f(y)| \leq \frac{1}{2}|x - y| \leq |x - y| = d(x, y),$$

it follows that $M(f(x), f(y), t) = e^{-\frac{2|f(x)-f(y)|}{t}} \geq e^{-\frac{|x-y|}{t}} = \varphi(M(x, y, t))$ for every $t > 0$. Thus, f is a cyclic φ -contraction in the fuzzy metric space $(X, M, *)$. Now, all the conditions of Theorem 3.2 are satisfied and then f has a unique fixed point, that is $x = 1$.

4 Conclusion

In this paper, we have proposed an extended version of the Edelstein contraction theorem for cyclic contraction mappings in a fuzzy metric space, which can be considered as a generalization of the fuzzy Edelstein contraction theorem introduced by Grabiec [1]. It should be noted that the existence of a unique fixed point, in our results, does not depend on the compactness and completeness of a fuzzy metric space. Moreover, we have also generalized a fixed point theorem given by Shen *et al.* [16] to an M -complete fuzzy metric space. The conclusion has shown that the original proposition and the corresponding conclusions are true even if G -completeness is replaced by M -completeness.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

YS carried out the main proofs and drafted the manuscript. WC provided some valuable suggestions and helped to modify the manuscript. All authors read and approved the final manuscript.

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