Zhai *Fixed Point Theory and Applications* 2013, **2013**:119
http://www.fixedpointtheoryandapplications.com/content/2013/1/119

Fixed Point Theory and Applications
a SpringerOpen Journal

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Fixed point theorems for a class of mixed monotone operators with convexity

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Abstract

In this paper, we use partial order theory to study a class of mixed monotone operators with convexity, and we get the existence and uniqueness of fixed points without assuming the operator to be compact or continuous. Our results compliment the theory of mixed monotone operators in ordered Banach spaces.

MSC: 47H10; 47H07

Keywords: fixed point; mixed monotone operator; normal cone; convexity; α-convex operator

1 Introduction and preliminaries

It is well known that mixed monotone operators were introduced by Guo and Lakshmikantham [1] in 1987. Thereafter many authors have investigated these kinds of operators in Banach spaces and obtained a lot of interesting and important results (see [1–18] and the references therein). Their study has not only important theoretical meaning but also wide applications in engineering, nuclear physics, biological chemistry technology, *etc.* That is, they are used extensively in nonlinear differential and integral equations. For a small sample of such work, we refer the reader to works [13, 15–21]. However, the mixed monotone operators considered in most of these papers are concave; see [2–4, 7–11, 15, 17, 18], for instance. To our knowledge, the fixed point results on mixed monotone operators with convexity are still very few. So it is worthwhile to investigate this operator. The purpose of this paper is to establish the existence and uniqueness of fixed points for a class of mixed monotone operators with convexity. We will use partial order theory to study the mixed monotone operator with convexity and get the existence and uniqueness of fixed points without assuming the operator to be compact or continuous. Our results compliment the theory of mixed monotone operators in ordered Banach spaces.

For the discussion of the following section, we state here some definitions and notations. For convenience of readers, we suggest that one refers to [1, 2, 17, 21–23] for details.

Suppose that $(E, \|\cdot\|)$ is a real Banach space which is partially ordered by a cone $P \subset E$, *i.e.*, $x \leq y$ if and only if $y - x \in P$. If $x \leq y$ and $x \neq y$, then we denote $x < y$ or $y > x$. By θ we denote the zero element of E. Recall that a non-empty closed convex set $P \subset E$ is a cone if it satisfies (i) $x \in P$, $\lambda \geq 0 \Rightarrow \lambda x \in P$; (ii) $x \in P$, $-x \in P \Rightarrow x = \theta$.

Further, P is called normal if there exists a constant $N > 0$ such that, for all $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$; in this case N is called the normality constant of P. If $x_1, x_2 \in E$, the set $[x_1, x_2] = \{x \in E | x_1 \leq x \leq x_2\}$ is called the order interval between x_1 and x_2. We say that an operator $A : E \to E$ is increasing if $x \leq y$ implies $Ax \leq Ay$.

Definition 1.1 (See [1, 2]) $A : P \times P \to P$ is said to be a mixed monotone operator if $A(x,y)$ is increasing in x and decreasing in y, *i.e.*, u_i, v_i $(i = 1,2) \in P$, $u_1 \leq u_2$, $v_1 \geq v_2$ imply $A(u_1,v_1) \leq A(u_2,v_2)$. The element $x \in P$ is called a fixed point of A if $A(x,x) = x$.

2 Main results

In this section we consider the existence and uniqueness of fixed points for a class of mixed monotone operators in ordered Banach spaces.

Theorem 2.1 *Let E be a real Banach space and let P be a normal cone in E. $A : P \times P \to P$ is a mixed monotone operator which satisfies the following*:

(H_1) *for $t \in (0,1)$, $x,y \in P$, there exists $\alpha(t,x,y) \in (1,+\infty)$ such that*

$$A(tx,y) \leq t^{\alpha(t,x,y)}A(x,y); \tag{2.1}$$

(H_2) *there exist $u_0,v_0 \in P$, $r \in (0,1)$ such that*

$$u_0 \leq rv_0, \qquad A(u_0,v_0) \geq u_0, \qquad A(v_0,u_0) \leq v_0. \tag{2.2}$$

Then A has a unique fixed point u^* in $[u_0,rv_0]$. Moreover, constructing successively the sequences

$$x_n = A(x_{n-1},y_{n-1}), \qquad y_n = A(y_{n-1},x_{n-1}), \quad n = 1,2,\ldots,$$

for any initial values $x_0,y_0 \in [u_0,rv_0]$, we have $\|x_n - u^*\| \to 0$, $\|y_n - u^*\| \to 0$ as $n \to \infty$.

Remark 2.1 (i) The operator A which satisfies (H_1) can be called a mixed monotone operator with convexity; (ii) from (H_1), we can get $A(\theta,\theta) = \theta$. In fact, for $t_0 \in (0,1)$, $A(\theta,\theta) = A(t_0\theta,\theta) \leq t_0^{\alpha(t_0,\theta,\theta)}A(\theta,\theta)$, and thus $[1 - t_0^{\alpha(t_0,\theta,\theta)}]A(\theta,\theta) \leq \theta$. It follows from $A(\theta,\theta) \geq \theta$ that $A(\theta,\theta) = \theta$; (iii) from (2.1), we can get

$$A\left(\frac{1}{t}x,y\right) \geq \frac{1}{t^{\alpha(t,\frac{1}{t}x,y)}}A(x,y), \quad x,y \in P, t \in (0,1). \tag{2.3}$$

Proof of Theorem 2.1 Let $w_0 = rv_0$, $\varepsilon = r^{\alpha(r,v_0,u_0)-1}$. Then $w_0 \geq u_0$, $\varepsilon \in (0,1)$, and

$$\begin{aligned} A(w_0,u_0) &= A(rv_0,u_0) \leq r^{\alpha(r,v_0,u_0)}A(v_0,u_0) \leq r^{\alpha(r,v_0,u_0)}v_0 = r^{\alpha(r,v_0,u_0)-1} \cdot rv_0 \\ &= \varepsilon w_0 \leq w_0, \end{aligned} \tag{2.4}$$

$$A(u_0,w_0) = A(u_0,rv_0) \geq A(u_0,v_0) \geq u_0. \tag{2.5}$$

Construct successively the sequences

$$\begin{aligned} &u_n = A(u_{n-1},w_{n-1}), \qquad w_n = A(w_{n-1},u_{n-1}), \\ &w_n' = \frac{1}{\varepsilon}A(w_{n-1}',u_{n-1}), \qquad w_0' = w_0, \quad n = 1,2,\ldots. \end{aligned}$$

From (2.4), (2.5) and the mixed monotonicity of A, we have

$$u_0 \le u_1 \le u_2 \le \cdots \le u_n \le \cdots \le w_n \le \cdots \le w_1 \le w_0. \tag{2.6}$$

Next we prove that

$$u_0 \le w_n' \le w_0, \quad n = 1, 2, \ldots. \tag{2.7}$$

From (2.4),

$$\begin{aligned}
w_1' &= \frac{1}{\varepsilon} A(w_0, u_0) \le \frac{1}{\varepsilon} \varepsilon w_0 = w_0, \\
w_1' &= \frac{1}{\varepsilon} A(w_0, u_0) \ge \frac{1}{\varepsilon} A(u_0, w_0) \ge \frac{1}{\varepsilon} u_0 \ge u_0, \\
w_2' &= \frac{1}{\varepsilon} A\left(w_1', u_1\right) \le \frac{1}{\varepsilon} A(w_0, u_0) \le \frac{1}{\varepsilon} \varepsilon w_0 = w_0, \\
w_2' &= \frac{1}{\varepsilon} A\left(w_1', u_1\right) \ge \frac{1}{\varepsilon} A(u_0, v_0) \ge \frac{1}{\varepsilon} u_0 \ge u_0.
\end{aligned}$$

Suppose that when $n = k$, we have $u_0 \le w_k' \le w_0$. Then, when $n = k + 1$, we obtain

$$\begin{aligned}
w_{k+1}' &= \frac{1}{\varepsilon} A\left(w_k', u_k\right) \le \frac{1}{\varepsilon} A(w_0, u_0) \le \frac{1}{\varepsilon} \varepsilon w_0 = w_0, \\
w_{k+1}' &= \frac{1}{\varepsilon} A\left(w_k', u_k\right) \ge \frac{1}{\varepsilon} A(u_0, v_0) \ge \frac{1}{\varepsilon} u_0 \ge u_0.
\end{aligned}$$

By the induction method, we know that (2.7) holds. On the other hand, from (2.1),

$$\begin{aligned}
w_1 &= A(w_0, u_0) = \varepsilon \frac{1}{\varepsilon} A(w_0, u_0) = \varepsilon w_1', \\
w_2 &= A(w_1, u_1) = A\left(\varepsilon w_1', u_1\right) \le \varepsilon^{\alpha(\varepsilon, w_1', u_1)} A\left(w_1', u_1\right) = \varepsilon^{\alpha(\varepsilon, w_1', u_1)+1} \frac{1}{\varepsilon} A\left(w_1', u_1\right) \le \varepsilon^2 w_2'.
\end{aligned}$$

Suppose that when $n = k$, we have $w_k \le \varepsilon^k w_k'$. Then, when $n = k + 1$, we obtain

$$\begin{aligned}
w_{k+1} &= A(w_k, u_k) \le A\left(\varepsilon^k w_k', u_k\right) \le \left(\varepsilon^k\right)^{\alpha(\varepsilon^k, w_k', u_k)} A\left(w_k', u_k\right) \\
&= \varepsilon^{k\alpha(\varepsilon^k, w_k', u_k)+1} \frac{1}{\varepsilon} A\left(w_k', u_k\right) \le \varepsilon^{k+1} w_{k+1}'.
\end{aligned}$$

By the induction method, we have

$$w_n \le \varepsilon^n w_n', \quad n = 1, 2, \ldots. \tag{2.8}$$

By (2.6)-(2.8), we get

$$\begin{aligned}
&\theta \le w_n - u_n \le \varepsilon^n w_n' - u_n \le \varepsilon^n w_n' - \varepsilon^n u_n = \varepsilon^n \left(w_n' - u_n\right) \le \varepsilon^n (w_0 - u_0), \\
&\theta \le u_{n+p} - u_n \le w_n - u_n, \qquad \theta \le w_n - w_{n+p} \le w_n - u_n.
\end{aligned}$$

Since P is normal, we have

$$\|w_n - u_n\| \le N\varepsilon^n \|w_0 - u_0\| \to 0 \quad (\text{as } n \to \infty).$$

Further,

$$\|u_{n+p} - u_n\| \le N\|w_n - u_n\| \to 0 \quad (\text{as } n \to \infty),$$
$$\|w_n - w_{n+p}\| \le N\|w_n - u_n\| \to 0 \quad (\text{as } n \to \infty).$$

Here N is the normality constant.

So, we can claim that $\{u_n\}$ and $\{w_n\}$ are Cauchy sequences. Because E is complete, there exist $u^*, w^* \in P$ such that

$$u_n \to u^*, \qquad w_n \to w^*, \quad \text{as } n \to \infty.$$

By (2.6), we know that $u_0 \le u_n \le u^* \le w^* \le w_n \le w_0$, and then

$$\theta \le w^* - u^* \le w_n - u_n \le \varepsilon^n (w_0 - u_0).$$

Further, $\|w^* - u^*\| \le N\varepsilon^n \|w_0 - u_0\| \to 0$ (as $n \to \infty$), and thus $w^* = u^*$. Then we obtain

$$u_{n+1} = A(u_n, w_n) \le A\bigl(u^*, u^*\bigr) \le A(w_n, u_n) = w_{n+1}.$$

Let $n \to \infty$, then we get $A(u^*, u^*) = u^*$. That is, u^* is a fixed point of A in $[u_0, w_0] = [u_0, rv_0]$.

In the following, we prove that u^* is the unique fixed point of A in $[u_0, w_0]$. Suppose that there is $x^* \in [u_0, w_0]$ such that $A(x^*, x^*) = x^*$. Then we have $u_0 \le x^* \le w_0$. By the induction method and the mixed monotonicity of A, we have

$$u_{n+1} = A(u_n, w_n) \le x^* = A\bigl(x^*, x^*\bigr) \le A(w_n, u_n) = w_{n+1}, \quad n = 0, 1, 2, \ldots.$$

Then from the normality of P, we have $x^* = u^*$.

Moreover, constructing successively the sequences

$$x_n = A(x_{n-1}, y_{n-1}), \qquad y_n = A(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots,$$

for any initial values $x_0, y_0 \in [u_0, w_0]$, we have $u_n \le x_n, y_n \le w_n$, $n = 1, 2, \ldots$. Letting $n \to \infty$ yields $x_n \to u^*$, $y_n \to u^*$ as $n \to \infty$. □

Remark 2.2 Let $\alpha(t, x, y)$ be a constant $\alpha \in (1, +\infty)$, then Theorem 2.1 also holds.

Corollary 2.2 *Let E be a real Banach space and let P be a normal cone in E. $A : P \times P \to P$ is a mixed monotone operator which satisfies* (H_2) *and, for $t \in (0, 1)$, $x, y \in P$, there exists a constant $\alpha \in (1, +\infty)$ such that $A(tx, y) \le t^\alpha A(x, y)$. Then A has a unique fixed point u^* in $[u_0, rv_0]$. Moreover, constructing successively the sequences*

$$x_n = A(x_{n-1}, y_{n-1}), \qquad y_n = A(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots,$$

for any initial values $x_0, y_0 \in [u_0, rv_0]$, we have $\|x_n - u^*\| \to 0$, $\|y_n - u^*\| \to 0$ as $n \to \infty$.

From the proof of Theorem 2.1, we can easily obtain the following conclusion.

Corollary 2.3 *Let E be a real Banach space and let P be a normal cone in E. $A : P \to P$ is an increasing operator which satisfies the following*:

(H_3) *for $t \in (0,1)$, $x \in P$, there exists a constant $\alpha \in (1,+\infty)$ such that $A(tx) \leq t^{\alpha}Ax$*;
(H_4) *there exist $u_0, v_0 \in P$, $r \in (0,1)$ such that $u_0 \leq rv_0$, $Au_0 \geq u_0$, $Av_0 \leq v_0$.*

Then A has a unique fixed point u^* in $[u_0, rv_0]$. Moreover, constructing successively the sequence

$$x_n = Ax_{n-1}, \quad n = 1,2,\ldots,$$

for any initial value $x_0 \in [u_0, rv_0]$, we have $\|x_n - u^*\| \to 0$ as $n \to \infty$.

Remark 2.3 The operator A which satisfies (H_3) is called α-convex. In 1977, A.J.B. Potter introduced the definition of an α-convex operator and showed that for $\alpha \geq 0$, decreasing $(-\alpha)$-convex mappings have contraction ratios less than or equal to α and gave the existence of solutions to the nonlinear eigenvalue problem $Ax = \lambda x$. The method is based upon Hilbert's projective metric. In [22] Guo studied the existence and uniqueness of fixed points for $(-\alpha)$-convex operators. In [23], we obtained the existence and uniqueness of positive fixed points for α-convex operators by means of the properties of cone, concave operators and the monotonicity of set-valued maps. Here, the result on α-convex operators is new and the method is also new and different from previous ones.

Remark 2.4 If $A : P \to P$ is an α-convex operator, then A is not a decreasing and constant operator. In fact, suppose that A is decreasing, then we have $Ax \leq A(tx) \leq t^{\alpha}Ax$, $x \in P$, $t \in (0,1)$. Hence, $t^{\alpha} \geq 1$, this is a contradiction. Suppose that A is a constant operator, that is, $Ax = u_0 \in P$. Then $u_0 = A(tx) \leq t^{\alpha}Ax = t^{\alpha}u_0$, and thus $t^{\alpha} \geq 1$, this is also a contradiction.

Theorem 2.4 *Let E be a real Banach space and let P be a normal cone in E. $A : P \times P \to P$ is a mixed monotone operator which satisfies* (2.1) *and*

(H_5) *there exist $u_0, v_0 \in P$, $R \in (1,+\infty)$ such that*

$$A(u_0, v_0) \geq u_0, \qquad A(v_0, u_0) \leq v_0, \qquad v_0 \geq Ru_0.$$

Then the operator equation $A(u,u) = bu$ has a unique solution u^* in $[Ru_0, v_0]$, where $b = R^{\alpha(\frac{1}{R}, Ru_0, v_0)-1}$. Moreover, constructing successively the sequences

$$x_n = b^{-1}A(x_{n-1}, y_{n-1}), \qquad y_n = b^{-1}A(y_{n-1}, x_{n-1}), \quad n = 1,2,\ldots,$$

for any initial values $x_0, y_0 \in [Ru_0, v_0]$, we have $\|x_n - u^*\| \to 0$, $\|y_n - u^*\| \to 0$ as $n \to \infty$.

Proof Let $w_0 = Ru_0$. Then $w_0 \leq v_0$. Note that $b > 1$ and from (2.3),

$$\begin{aligned}
A(w_0, v_0) = A(Ru_0, v_0) &\geq \frac{1}{(\frac{1}{R})^{\alpha(\frac{1}{R}, Ru_0, v_0)}} A(u_0, v_0) \\
&\geq R^{\alpha(\frac{1}{R}, Ru_0, v_0)} u_0 = R^{\alpha(\frac{1}{R}, Ru_0, v_0)-1} \cdot Ru_0 = bw_0 \geq w_0, \\
A(v_0, w_0) = A(v_0, Ru_0) &\leq A(v_0, u_0) \leq v_0.
\end{aligned}$$

Set $B(x,y) = b^{-1}A(x,y)$, $x, y \in P$. Then from the above inequalities, we have

$$\begin{aligned}
&B(w_0, v_0) = b^{-1}A(w_0, v_0) \geq b^{-1}bw_0 = w_0, \\
&B(v_0, w_0) = b^{-1}A(v_0, w_0) \leq b^{-1}v_0 \leq v_0.
\end{aligned} \tag{2.9}$$

Also, construct successively the sequences

$$\begin{aligned}
&w_n = B(w_{n-1}, v_{n-1}), \qquad v_n = B(v_{n-1}, w_{n-1}), \\
&v'_n = bB(v'_{n-1}, w_{n-1}), \qquad v'_0 = v_0, \quad n = 1, 2, \ldots.
\end{aligned}$$

From (2.9) and the mixed monotonicity of A, we have

$$w_0 \leq w_1 \leq w_2 \leq \cdots \leq w_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0.$$

Similar to the proof of Theorem 2.1, we can prove that

$$w_0 \leq v'_n \leq v_0, \qquad v_n \leq \left(\frac{1}{b}\right)^n v'_n, \quad n = 1, 2, \ldots.$$

Further, by using the same method with the proof of Theorem 2.1, we can get the following conclusions: (i) B has a unique fixed point u^* in $[w_0, v_0]$; (ii) for any initial values $x_0, y_0 \in [w_0, v_0]$, constructing successively the sequences

$$x_n = B(x_{n-1}, y_{n-1}), \qquad y_n = B(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots,$$

we have $\|x_n - u^*\| \to 0$, $\|y_n - u^*\| \to 0$ as $n \to \infty$. Therefore, the operator equation $A(u,u) = bu$ has a unique solution u^* in $[Ru_0, v_0]$. Moreover, constructing successively the sequences

$$x_n = b^{-1}A(x_{n-1}, y_{n-1}), \qquad y_n = b^{-1}A(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots,$$

for any initial values $x_0, y_0 \in [Ru_0, v_0]$, we have $\|x_n - u^*\| \to 0$, $\|y_n - u^*\| \to 0$ as $n \to \infty$. □

Corollary 2.5 *Let E be a real Banach space and let P be a normal cone in E. $A : P \to P$ is an increasing operator which satisfies* (H_3) *and*

(H_6) *there exist $u_0, v_0 \in P$, $R \in (1, +\infty)$ such that $Au_0 \geq u_0$, $Av_0 \leq v_0$, $v_0 \geq Ru_0$.*

Then the operator equation $Au = bu$ has a unique solution u^* in $[Ru_0, v_0]$, where $b = R^{\alpha-1}$. Moreover, constructing successively the sequence

$$x_n = b^{-1}Ax_{n-1}, \quad n = 1, 2, \ldots,$$

for any initial value $x_0 \in [Ru_0, v_0]$, we have $\|x_n - u^*\| \to 0$ as $n \to \infty$.

Competing interests
The author declares that he has no competing interests.

Acknowledgements
The author was supported financially by the Youth Science Foundations of China (11201272) and Shanxi Province (2010021002-1).

Received: 7 January 2013 Accepted: 17 April 2013 Published: 3 May 2013

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doi:10.1186/1687-1812-2013-119
Cite this article as: Zhai: Fixed point theorems for a class of mixed monotone operators with convexity. *Fixed Point Theory and Applications* 2013 **2013**:119.