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Fixed points of a new type of contractive mappings in complete metric spaces

Dariusz Wardowski

Correspondence: wardd@math.uni.lodz.pl
Department of Nonlinear Analysis, Faculty of Mathematics and Computer Science, University of Łódź, Banacha 22, 90-238 Łódź, Poland

Abstract

In the article, we introduce a new concept of contraction and prove a fixed point theorem which generalizes Banach contraction principle in a different way than in the known results from the literature. The article includes an example which shows the validity of our results, additionally there is delivered numerical data which illustrates the provided example.

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1 Introduction

Throughout the article denoted by $\mathbb{R}$ is the set of all real numbers, by $\mathbb{R}_+$ is the set of all positive real numbers and by $\mathbb{N}$ is the set of all natural numbers. (X, d), (X for short), is a metric space with a metric d.

In the literature, there are plenty of extensions of the famous Banach contraction principle [1], which states that every self-mapping T defined on a complete metric space (X, d) satisfying

$$\forall_{x,y\in X}\ \mathrm{d}(Tx, Ty) \le \lambda \mathrm{d}(x, y), \quad \text{where } \lambda \in (0, 1), \tag{1}$$

has a unique fixed point and for every $x_0 \in X$ a sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to the fixed point. Some of the extensions weaken right side of inequality in the condition (1) by replacing λ with a mapping, see e.g. [2,3]. In other results, the underlying space is more general, see e.g [4-7]. The Nadler's paper [8] started the invatigations concerning fixed point theory for set-valued contractions, see e.g. [9-20]. There are many theorems regarding asymptotic contractions, see e.g. [21-23], contractions of Meir-Keeler type [24], see e.g [19,23,25] and weak contractions, see e.g. [26-28]. There are also lots of different types of fixed point theorems not mentioned above extending the Banach's result.

In the present article, using a mapping F: $\mathbb{R}^+ \to \mathbb{R}$ we introduce a new type of contraction called F-contraction and prove a new fixed point theorem concerning F-contraction. For the concrete mappings F, we obtain the contractions of the type known from the literature, Banach contraction as well. The article includes the examples of F-contractions and an example showing that the obtained extension is significant. Theoretical considerations that we support by computational data illustrate the nature of F-contractions.

2 The result

Definition 2.1 Let $F: \mathbb{R}_+ \to \mathbb{R}$ be a mapping satisfying:

(F1) F is strictly increasing, i.e. for all $\alpha, \beta \in \mathbb{R}_+$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$;

(F2) For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

(F3) There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

A mapping $T: X \to X$ is said to be an F-contraction if there exists $\tau > 0$ such that

$$\forall_{x,y\in X}\ \left(\mathrm{d}(Tx, Ty) > 0 \Rightarrow \tau + F(\mathrm{d}(Tx, Ty)) \le F(\mathrm{d}(x, y))\right). \tag{2}$$

When we consider in (2) the different types of the mapping F then we obtain the variety of contractions, some of them are of a type known in the literature. See the following examples:

Example **2.1** Let $F : \mathbb{R}_+ \to \mathbb{R}$ be given by the formula $F(\alpha) = \ln \alpha$. It is clear that F satisfies (F1)-(F3) ((F3) for any $k \in (0, 1)$). Each mapping $T : X \to X$ satisfying (2) is an F-contraction such that

$$\mathrm{d}(Tx, Ty) \le e^{-\tau}\mathrm{d}(x, y), \quad \text{for all } x, y \in X,\ Tx \ne Ty. \tag{3}$$

It is clear that for $x, y \in X$ such that $Tx = Ty$ the inequality $\mathrm{d}(Tx, Ty) \le e^{-\tau}\mathrm{d}(x, y)$ also holds, i.e. T is a Banach contraction [1].

Example **2.2** If $F(\alpha) = \ln \alpha + \alpha$, $\alpha > 0$ then F satisfies (F1)-(F3) and the condition (2) is of the form

$$\frac{\mathrm{d}(Tx, Ty)}{\mathrm{d}(x, y)} e^{\mathrm{d}(Tx,Ty)-\mathrm{d}(x,y)} \le e^{-\tau}, \quad \text{for all } x, y \in X,\ Tx \ne Ty. \tag{4}$$

Example **2.3** Consider $F(\alpha) = -1/\sqrt{\alpha}$, $\alpha > 0$. F satisfies (F1)-(F3) ((F3) for any $k \in (1/2, 1)$). In this case, each F-contraction T satisfies

$$\mathrm{d}(Tx, Ty) \le \frac{1}{\left(1 + \tau\sqrt{\mathrm{d}(x, y)}\right)^2}\mathrm{d}(x, y), \quad \text{for all } x, y \in X,\ Tx \ne Ty.$$

Here, we obtained a special case of nonlinear contraction of the type $\mathrm{d}(Tx, Ty) \le \alpha(\mathrm{d}(x, y))\mathrm{d}(x, y)$. For details see [2,3].

Example **2.4** Let $F(\alpha) = \ln(\alpha^2 + \alpha)$, $\alpha > 0$. Obviously F satisfies (F1)-(F3) and for F-contraction T, the following condition holds:

$$\frac{\mathrm{d}(Tx, Ty)(\mathrm{d}(Tx, Ty) + 1)}{\mathrm{d}(x, y)(\mathrm{d}(x, y) + 1)} \le e^{-\tau}, \quad \text{for all } x, y \in X,\ Tx \ne Ty.$$

Let us observe that in Examples 2.1-2.4 the contractive conditions are satisfied for $x, y \in X$, such that $Tx = Ty$.

Remark 2.1 From (F1) and (2) it is easy to conclude that every F-contraction T is a contractive mapping, i.e.

$$\mathrm{d}(Tx, Ty) < \mathrm{d}(x, y), \quad \text{for all } x, y \in X, Tx \ne Ty.$$

Thus every F-contraction is a continuous mapping.

Remark 2.2 Let F_1, F_2 be the mappings satisfying (F1)-(F3). If $F_1(\alpha) \le F_2(\alpha)$ for all $\alpha > 0$ and a mapping $G = F_2 - F_1$ is nondecreasing then every F_1-contraction T is F_2-contraction.

Indeed, from Remark 2.1 we have $G(\mathrm{d}(Tx, Ty)) \le G(\mathrm{d}(x, y))$ for all $x, y \in X$, $Tx \neq Ty$. Thus, for all $x, y \in X$, $Tx \neq Ty$ we obtain

$$\begin{aligned}\tau + F_2(\mathrm{d}(Tx, Ty)) &= \tau + F_1(\mathrm{d}(Tx, Ty)) + G(\mathrm{d}(Tx, Ty)) \\ &\le F_1(\mathrm{d}(x, y)) + G(\mathrm{d}(x, y)) = F_2(\mathrm{d}(x, y)).\end{aligned}$$

Now we state the main result of the article.

Theorem 2.1 *Let* (X, d) *be a complete metric space and let* $T : X \to X$ *be an F-contraction. Then* T *has a unique fixed point* $x^* \in X$ *and for every* $x_0 \in X$ *a sequence* $\{T^n x_0\}_{n\in\mathbb{N}}$ *is convergent to* x^*.

Proof. First, let us observe that T has at most one fixed point. Indeed, if $x_1^*, x_2^* \in X$, $Tx_1^* = x_1^* \neq x_2^* = Tx_2^*$, then we get

$$\tau \le F(\mathrm{d}(x_1^*, x_2^*)) - F(\mathrm{d}(Tx_1^*, Tx_2^*)) = 0,$$

which is a contradiction.

In order to show that T has a fixed point let $x_0 \in X$ be arbitrary and fixed. We define a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$, $x_{n+1} = Tx_n$, $n = 0, 1, \ldots$. Denote $\gamma_n = \mathrm{d}(x_{n+1}, x_n)$, $n = 0, 1, \ldots$.

If there exists $n_0 \in \mathbb{N}$ for which $x_{n_0+1} = x_{n_0}$, then $Tx_{n_0} = x_{n_0}$ and the proof is finished.

Suppose now that $x_{n+1} \neq x_n$, for every $n \in \mathbb{N}$. Then $\gamma_n > 0$ for all $n \in \mathbb{N}$ and, using (2), the following holds for every $n \in \mathbb{N}$:

$$F(\gamma_n) \le F(\gamma_{n-1}) - \tau \le F(\gamma_{n-2}) - 2\tau \le \ldots \le F(\gamma_0) - n\tau. \tag{5}$$

From (5), we obtain $\lim_{n\to\infty} F(\gamma_n) = -\infty$ that together with (F2) gives

$$\lim_{n\to\infty} \gamma_n = 0. \tag{6}$$

From (F3) there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} {\gamma_n}^k F(\gamma_n) = 0. \tag{7}$$

By (5), the following holds for all $n \in \mathbb{N}$:

$${\gamma_n}^k F(\gamma_n) - {\gamma_n}^k F(\gamma_0) \le {\gamma_n}^k (F(\gamma_0) - n\tau) - {\gamma_n}^k F(\gamma_0) = -{\gamma_n}^k n\tau \le 0. \tag{8}$$

Letting $n \to \infty$ in (8), and using (6) and (7), we obtain

$$\lim_{n\to\infty} n{\gamma_n}^k = 0. \tag{9}$$

Now, let us observe that from (9) there exists $n_1 \in \mathbb{N}$ such that $n{\gamma_n}^k \le 1$ for all $n \ge n_1$. Consequently we have

$$\gamma_n \le \frac{1}{n^{1/k}}, \quad \text{for all } n \ge n_1. \tag{10}$$

In order to show that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence consider $m, n \in \mathbb{N}$ such that $m > n \ge n_1$. From the definition of the metric and from (10) we get $\mathrm{d}(x_m, x_n) \le \gamma_{m-1} + \gamma_{m-2} + \cdots + \gamma_n < \sum_{i=n}^{\infty} \gamma_i \le \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}$.

From the above and from the convergence of the series $\sum_{i=1}^{\infty} 1/i^{\frac{1}{k}}$ we receive that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

From the completeness of X there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. Finally, the continuity of T yields

$$\mathrm{d}(Tx^*, x^*) = \lim_{n\to\infty} \mathrm{d}(Tx_n, x_n) = \lim_{n\to\infty} \mathrm{d}(x_{n+1}, x_n) = 0,$$

which completes the proof. □

Note that for the mappings $F_1(\alpha) = \ln(\alpha)$, $\alpha > 0$, $F_2(\alpha) = \ln(\alpha) + \alpha$, $\alpha > 0$, $F_1 < F_2$ and a mapping $F_2 - F_1$ is strictly increasing. Hence, by Remark 2.2, we obtain that every Banach contraction (3) satisfies the contraction condition (4). On the other side in Example 2.5, we present a metric space and a mapping T which is not F_1-contraction (Banach contraction), but still is an F_2-contraction. Consequently, Theorem 2.1 gives the family of contractions which in general are not equivalent.

Example **2.5** Consider the sequence $\{S_n\}_{n\in\mathbb{N}}$ as follows:

$$\begin{aligned} &S_1 = 1,\\ &S_2 = 1 + 2,\\ &\ldots\\ &S_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}, n \in \mathbb{N},\\ &\ldots \end{aligned}$$

Let $X = \{S_n : n \in \mathbb{N}\}$ and $\mathrm{d}(x, y) = |x - y|$, $x, y \in X$. Then (X, d) is a complete metric space. Define the mapping $T : X \to X$ by the formulae:

$$\begin{aligned} &T(S_n) = S_{n-1} \quad \text{for } n > 1,\\ &T(S_1) = S_1. \end{aligned}$$

First, let us consider the mapping F_1 defined in Example 2.1. The mapping T is not the F_1-contraction in this case (which actually means that T is not the Banach contraction). Indeed, we get

$$\lim_{n\to\infty} \frac{\mathrm{d}(T(S_n), T(S_1))}{\mathrm{d}(S_n, S_1)} = \lim_{n\to\infty} \frac{S_{n-1} - 1}{S_n - 1} = 1.$$

On the other side taking F_2 as in Example 2.2, we obtain that T is F_2-contraction with $\tau = 1$. To see this, let us consider the following calculations:

First, observe that

$$\forall_{m,n\in\mathbb{N}}\ [T(S_m) \neq T(S_n) \Leftrightarrow ((m > 2 \wedge n = 1) \vee (m > n > 1))].$$

For every $m \in \mathbb{N}$, $m > 2$ we have

$$\begin{aligned} \frac{\mathrm{d}(T(S_m), T(S_1))}{\mathrm{d}(S_m, S_1)} e^{\mathrm{d}(T(S_m),T(S_1)) - \mathrm{d}(S_m,S_1)} &= \frac{S_{m-1} - 1}{S_m - 1} e^{S_{m-1} - S_m}\\ &= \frac{m^2 - m - 2}{m^2 + m - 2} e^{-m} < e^{-m} < e^{-1}. \end{aligned}$$

For every $m, n \in \mathbb{N}$, $m > n > 1$ the following holds

$$\begin{aligned} \frac{\mathrm{d}(T(S_m), T(S_n))}{\mathrm{d}(S_m, S_n)} e^{\mathrm{d}(T(S_m),T(S_n)) - \mathrm{d}(S_m,S_n)} &= \frac{S_{m-1} - S_{n-1}}{S_m - S_n} e^{S_n - S_{n-1} + S_{m-1} - S_m}\\ &= \frac{m + n - 1}{m + n + 1} e^{n-m} < e^{n-m} \leq e^{-1}. \end{aligned}$$

Table 1 The comparison of Banach contraction condition with F-contraction condition

n	x_n	$C_{F_1}(S_1, S_n)$	$C_{F_2}(S_1, S_n)$
3	378	0.91629	3.91629
4	351	0.58779	4.58779
5	325	0.44183	5.44183
6	300	0.35667	6.35667
7	276	0.30010	7.30010
8	253	0.25951	8.25951
9	231	0.22884	9.22884
10	210	0.20479	10.20479
11	190	0.18540	11.18540
12	171	0.16942	12.16942
13	153	0.15600	13.15600
14	136	0.14458	14.14458
15	120	0.13473	15.13473
16	105	0.12615	16.12615
17	91	0.11861	17.11861
18	78	0.11192	18.11192
19	66	0.10595	19.10595
20	55	0.10059	20.10059
21	45	0.09575	21.09575
22	36	0.09135	22.09135
23	28	0.08734	23.08734
24	21	0.08367	24.08367
25	15	0.08030	25.08030
26	10	0.07719	26.07719
27	6	0.07431	27.07431
28	3	0.07164	28.07164
29	1	0.06916	29.06916
30	1	0.06684	30.06684
$\vdots$	$\vdots$	$\vdots$	$\vdots$
3×10^4	1	6.66667×10^{-5}	30000.00007
$n \to \infty$	$T1 = 1$	tends to 0	$\geq \tau = 1$

The generated iterations start from a point $x_0 = S_{29} = 435$. $C_F(S_1, S_n)$ denotes $F(d(S_1, S_n)) - F(d(T(S_1), T(S_n)))$

Clearly S_1 is a fixed point of T. To see the computational data confirming the above calculations the reader is referred to Table 1.

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Competing interests
The author declares that he has no competing interests.

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