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Some notes on the paper "The equivalence of cone metric spaces and metric spaces"

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Abstract
In this article, we shall show that the metrics defined by Feng and Mao, and Du are equivalent. We also provide some examples for one of the metrics.

1 Introduction and preliminary

Let E be a topological vector space (t.v.s.) with zero vector θ. A nonempty subset K of E is called a convex cone if $K + K \subseteq K$ and $\lambda K \subseteq K$ for each $\lambda \geq 0$. A convex cone K is said to be pointed if $K \cap - K = \{\theta\}$. For a given cone $K \subseteq E$, we can define a partial ordering $\preccurlyeq$ with respect to K by

$$x \preccurlyeq y \Leftrightarrow y - x \in K.$$

$x < y$ will stand for $x \preccurlyeq y$ and $x \neq y$ while $x \ll y$ stands for $y - x \in K^\circ$, where K° denotes the interior of K. In the following, we shall always assume that Y is a locally convex Hausdorff t.v.s. with zero vector θ, K is a proper, closed, and convex pointed cone in Y with $K^\circ \neq \emptyset$, $e \in K^\circ$ and $\preccurlyeq$ a partial ordering with respect to K. The nonlinear scalarization function $\xi_e : Y \to \mathbb{R}$ is defined by

$$\xi_e(y) = \inf\{r \in \mathbb{R} : y \in re - K\}$$

for all $y \in Y$.

We will use P instead of K when E is a real Banach spaces.

Lemma 1.1 [1]*For each $r \in R$ and $y \in Y$, the following statements are satisfied:*

(i) $\xi_e(y) \leq r \Leftrightarrow y \in re - K$.

(ii) $\xi_e(y) > r \Leftrightarrow y \notin re - K$.

(iii) $\xi_e(y) \geq r \Leftrightarrow y \notin re - K^\circ$.

(iv) $\xi_e(y) < r \Leftrightarrow y \in re - K^\circ$.

(v) $\xi_e(.)$ is positively homogeneous and continuous on Y .

(vi) $y_1 \in y_2 + K \Rightarrow \xi_e(y_2) \leq \xi_e(y_1)$

(vii) $\xi_e(y_1 + y_2) \leq \xi_e(y_1) + \xi_e(y_2)$ for all $y_1, y_2 \in Y$.

Definition 1.2 [1]*Let X be a nonempty set. A vector-valued function $d : X \times X \to Y$ is said to be a TVS-cone metric, if the following conditions hold:*

(C1) $\theta \preccurlyeq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ iff $x = y$

(C2) $d(x, y) = d(y, x)$ for all $x, y \in X$

(C3)$d(x, y) \preccurlyeq (x, z) + d(z, y)$ for all $x, y, z \in X$.

The pair (X, d) *is then called a TVS-cone metric space.*

Huang and Zhang [2] discuss the case in which Y is a real Banach space and call a vector-valued function $d : X \times X \to Y$ a cone metric if d satisfies (C1)-(C3). Clearly, a cone metric space, in the sense of Huang and Zhang, is a special case of a TVS-cone metric space.

In the following, some conclusions are listed.

Lemma 1.3 [3]*Let* (X, D) *be a cone metric space. Then*

$$d(x,y) = \inf_{\{u \in P | D(x,y) \preccurlyeq u\}} ||u||, \quad x, y \in X$$

is a metric on X.

Theorem 1.4 [3]*The metric space* (X, d) *is complete if and only if the cone metric space* (X, D) *is complete* .

Theorem 1.5 [1]*Let* (X, D) *be a TVS-cone metric space. Then* $d_2 : X \times X \to [0, \infty)$ *defined by* $d_2(x, y) = \xi_e(D(x, y))$ *is a metric.*

2 Main results

We first show that the metrics introduced the Lemma 1.3 and the Theorem 1.5 are equivalent. Then, we provide some examples involving the metric defined in Lemma 1.3.

Theorem 2.1 *For every cone metric* $D : X \times X \to E$ *there exists a metric* $d : X \times X \to \mathbb{R}^+$ *which is equivalent to* D *on* X.

Proof. Define $d(x, y) = \inf \{||u||: D(x, y) \preccurlyeq u\}$. By the Lemma 1.3 d is a metric. We shall now show that each sequence $\{x_n\} \subseteq X$ which converges to a point $x \in X$ in the (X, d) metric also converges to x in the (X, D) metric, and conversely. We have

$$\forall n,\ m \in \mathbb{N} \quad \exists u_{nm} \text{ such that } \quad ||u_{nm}|| < d(x_n,\ x) + \frac{1}{m}, \quad D(x_n,\ x) \preccurlyeq u_{nm}.$$

Put $v_n := u_{nn}$ then $||v_n|| < d(x_n,\ x) + \frac{1}{n}$ and $D(x_n, x) \preccurlyeq v_n$. Now if $x_n \to x$ in (X, d) then $d(x_n, x) \to 0$ and so $v_n \to 0$ too, therefore for all $c \gg 0$ there exists $N \in \mathbb{N}$ such that $v_n \ll c$ for all $n \geq N$. This implies that $D(x_n, x) \ll c$ for all $n \geq N$. Namely $x_n \to x$ in (X, D).

Conversely, for every real $\varepsilon > 0$, choose $c \in E$ with $c \gg 0$ and $||c|| < \varepsilon$. Then there exists $N \in \mathbb{N}$ such that $D(x_n, x) \ll c$ for all $n \geq N$. This means that for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) \leq ||c|| < \varepsilon$ for all $n \geq N$. Therefore $d(x_n, x) \to 0$ as $n \to \infty$ so $x_n \to x$ in (X, d).

□

Theorem 2.2 *If* $d_1(x, y) = \inf \{||u||: D(x, y) \preccurlyeq u\}$ *and* $d_2(x, y) = \xi_e(D(x, y))$ *where* D *is a cone metric on* X. *Then* d_1 *is equivalent with* d_2.

Proof. Let $x_n \xrightarrow{d_1} x$ then $d_1(x_n, x) \xrightarrow{\mathbb{R}} 0$ so by Theorem 2.1 in $x_n \xrightarrow{D} x$ so

$$\forall \varepsilon > 0, \quad \forall e \succ\succ 0 \quad \exists N \quad \forall n (n \geq N \Rightarrow D(x_n,\ x) \prec\prec \varepsilon e),$$

and or $\varepsilon e - D(x_n, x) \in K^\circ$ for all $n \geq N$. So $D(x_n, x) \in e - K^\circ$ for $n \geq N$. Now by [[1], Lemma 1.1 (iv)] $\xi_e(D(x_n, x)) < \varepsilon$ for all $n \geq N$. Namely $d_2(x_n, x) < \varepsilon$ for all $n \geq N$ therefore $d_2(x_n, x) \xrightarrow{\mathbb{R}} 0$ or $x_n \xrightarrow{d_2} x$.

Conversely, $x_n \underset{\rightarrow}{d_2} x$ hence $d_2(x_n, x) \underset{\rightarrow}{\mathbb{R}} 0$ so $\xi_e(D(x_n, x)) \underset{\rightarrow}{\mathbb{R}} 0$, therefore

$$\forall \varepsilon > 0 \quad \exists N \quad \forall n(n \geq N \Rightarrow \xi_e(D(x_n,\ x)) < \varepsilon).$$

So $D(x_n, x) \in \varepsilon e - K^\circ$ for $n \geq N$ by [[1], Lemma 1.1 (iv)]. Hence, $D(x_n, x) = \varepsilon e - k$ for some $k \in K^\circ$, so $D(x_n, x) \prec\prec \varepsilon e$ for $n \geq N$ this implies that $x_n \xrightarrow{D} x$ and again by Theorem 2.1 $x_n \xrightarrow{d_1} x$. □

In the following examples, we use the metric of Lemma 1.3.

Example 2.3 *Let* $0 \neq a \in P \subseteq \mathbb{R}^n$ *with* $||a|| = 1$ *and for every* $x, y \in \mathbb{R}^n$ *define*

$$D(x,\ y) = \begin{cases} a, & x \neq y; \\ 0, & x = y. \end{cases}$$

Then D is a cone metric on $\mathbb{R}^n$ *and its equivalent metric d is*

$$d(x,\ y) = \begin{cases} 1, & x \neq y; \\ 0, & x = y, \end{cases}$$

which is a discrete metric.

Example 2.4 *Let a, b* ≥ 0 *and consider the cone metric* $D : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ *with*

$$D(x,\ y) = (ad_1(x,\ y),\ bd_2(x,\ y))$$

where d_1, d_2 *are metrics on* $\mathbb{R}$. *Then its equivalent metric is*

$$d(x,\ y) = \sqrt{a^2 + b^2}||(d_1(x,\ y),\ d_2(x,\ y))||.$$

In particular if $d_1(x, y) := |x - y|$ *and* $d_2(x, y) := \alpha|x - y|$, *where* $\alpha \geq 0$ *then D is the same famous cone metric which has been introduced in [*[2]*, Example 1] and its equivalent metric is*

$$d(x,\ y) = \sqrt{1 + \alpha^2}|x - y|.$$

Example 2.5 *For* $q > 0$, $b > 1$, $E = l^q$, $P = \{\{x_n\}_{n\geq 1} : x_n \geq 0,$ *for all n*$\}$ *and* (X, ρ) *a metric space, define* $D : X \times X \to E$ *which is the same cone metric as [*[4]*, Example 1.3] by*

$$D(x,\ y) = \left\{ \left(\frac{\rho(x, y)}{b^n} \right)^{\frac{1}{q}} \right\}_{n \geq 1}.$$

Then its equivalent metric on × is

$$d(x,\ y) = \left\| \left\{ \left(\frac{\rho(x, y)}{b^n} \right)^{\frac{1}{q}} \right\}_{n \geq 1} \right\|_{l^q} = \left(\sum_{n=1}^{\infty} \frac{\rho(x, y)}{b^n} \right)^{\frac{1}{q}} = \left(\frac{\rho(x, y)}{b - 1} \right)^{\frac{1}{q}}.$$

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Authors' contributions
All authors have read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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