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Common fixed point theorems for weakly increasing mappings on ordered orbitally complete metric spaces

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Abstract

In this article, we prove existence results for common fixed points of two or three relatively asymptotically regular mappings satisfying the orbital continuity of one of the involved maps on ordered orbitally complete metric spaces. We furnish suitable examples to demonstrate the validity of the hypotheses of our results.

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1 Introduction and preliminaries

Browder and Petryshyn introduced the concept of asymptotic regularity of a self-map at a point in a metric space.

Definition 1 [1] A self-map $\mathcal{T}$ on a metric space $(\mathcal{X}, d)$ is said to be asymptotically regular at a point $x \in \mathcal{X}$ if $\lim_{n\to\infty} d(\mathcal{T}^n x, \mathcal{T}^{n+1}x) = 0$.

Recall that the set $\mathcal{O}(x_0; \mathcal{T}) = \{\mathcal{T}^n x_0 : n = 0, 1, 2, \ldots\}$ is called the orbit of the self-map $\mathcal{T}$ at the point $x_0 \in \mathcal{X}$.

Definition 2 [2] A metric space $(\mathcal{X}, d)$ is said to be $\mathcal{T}$-orbitally complete if every Cauchy sequence contained in $\mathcal{O}(x; \mathcal{T})$ (for some x in $\mathcal{X}$) converges in $\mathcal{X}$.

Here, it can be pointed out that every complete metric space is $\mathcal{T}$-orbitally complete for any $\mathcal{T}$, but a $\mathcal{T}$-orbitally complete metric space need not be complete.

Definition 3 [1] A self-map $\mathcal{T}$ defined on a metric space $(\mathcal{X}, d)$ is said to be orbitally continuous at a point z in $\mathcal{X}$ if for any sequence $\{x_n\} \subset \mathcal{O}(x; \mathcal{T})$ (for some $x \in \mathcal{X}$), $x_n \to z$ as $n \to \infty$ implies $\mathcal{T}x_n \to \mathcal{T}_z$ as $n \to \infty$.

Clearly, every continuous self-mapping of a metric space is orbitally continuous, but not conversely.

Sastry et al. [3] extended the above concepts to two and three mappings and employed them to prove common fixed point results for commuting mappings. In what follows, we collect such definitions for three maps.

Definition 4 Let $\mathcal{S}, \mathcal{T}, \mathcal{R}$ be three self-mappings defined on a metric space $(\mathcal{X}, d)$.

1. If for a point $x_0 \in \mathcal{X}$, there exits a sequence $\{x_n\}$ in $\mathcal{X}$ such that $\mathcal{R}x_{2n+1} = \mathcal{S}x_{2n}, \mathcal{R}x_{2n+2} = \mathcal{T}x_{2n+1}, n = 0, 1, 2, \ldots,$ then the set $\mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R}) = \{\mathcal{R}x_n : n = 1, 2, \ldots\}$ is called the orbit of $(\mathcal{S}, \mathcal{T}, \mathcal{R})$ at x_0.

2. The space $(\mathcal{X}, d)$ is said to be $(\mathcal{S}, \mathcal{T}, \mathcal{R})$-orbitally complete at x_0 if every Cauchy sequence in $\mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R})$ converges in $\mathcal{X}$.

3. The map $\mathcal{R}$ is said to be orbitally continuous at x_0 if it is continuous on $\mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R})$.

4. The pair $(\mathcal{S}, \mathcal{T})$ is said to be asymptotically regular (in short a.r.) with respect to $\mathcal{R}$ at x_0 if there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that $\mathcal{R}x_{2n+1} = \mathcal{S}x_{2n}, \mathcal{R}x_{2n+2} = \mathcal{T}x_{2n+1}, n = 0, 1, 2, \ldots$, and $d(\mathcal{R}x_n, \mathcal{R}x_{n+1}) \to 0$ as $n \to \infty$.

5. If $\mathcal{R}$ is the identity mapping on $\mathcal{X}$, we omit "$\mathcal{R}$" in respective definitions.

On the other hand, fixed point theory has developed rapidly in metric spaces endowed with a partial ordering. The first result in this direction was given by Ran and Reurings [4] who presented its applications to matrix equations. Subsequently, Nieto and López [5] extended this result for nondecreasing mappings and applied it to obtain a unique solution for a first-order ordinary differential equation with periodic boundary conditions. Thereafter, several authors obtained many fixed point theorems in ordered metric spaces. For more details, see [6-15] and the references cited therein.

Recently, Nashine and Altun (HK Nashine and I Altun, unpublished work) proved the following ordered version of a result of Zhang [16]:

Theorem 1 *Let $(\mathcal{X}, d, \preccurlyeq)$ be a complete partially ordered metric space and let $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ be two weakly increasing mappings such that*

$$F(d(\mathcal{T}x, \mathcal{S}y)) \leq \psi(F(\Theta[\mathcal{T}, \mathcal{S}](x, y)))$$

holds for each comparable $x, y \in \mathcal{X}$, where $F, \psi : [0, +\infty) \to [0, +\infty)$ are functions such that

(i) F is nondecreasing, continuous, and $F(0) = 0 < F(t)$ for every $t > 0$;

(ii) ψ is nondecr easing, right continuous, and $\psi(t) < t$ for every $t > 0$, and

$$\Theta[\mathcal{T}, \mathcal{S}](x, y) = \max\{d(x, y), d(x, \mathcal{T}x), d(y, \mathcal{S}y), \frac{1}{2}[d(x, \mathcal{S}y) + d(y, \mathcal{T}x)]\}.$$

If $\mathcal{S}$ or $\mathcal{T}$ is continuous, then $\mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point.

In this article, we generalize this theorem of Nashine and Altun (HK Nashine and I Altun, unpublished work) (and, hence, some other related common fixed point results) in two directions. The first is treated in Section 3, where a pair of asymptotically regular mappings in an orbitally complete ordered metric space is considered. The existence and (under additional assumptions) uniqueness of their common fixed point is obtained under assumptions that these mappings are strictly weakly isotone increasing, one is orbitally continuous and they satisfy a generalized weakly contractive condition.

In Section 4, we consider the case of three self-mappings $\mathcal{S}, \mathcal{T}, \mathcal{R}$ where the pair $\mathcal{S}, \mathcal{T}$ is $\mathcal{R}$-relatively asymptotically regular and relatively weakly increasing, while the contractive condition is given with the help of two control functions.

We furnish suitable examples to demonstrate the validity of the hypotheses of our results.

2 Notation and definitions

First, we introduce some further notation and definitions that will be used later.

If $(\mathcal{X}, \preccurlyeq)$ is a partially ordered set then $x, y \in \mathcal{X}$ are called comparable if $x \preccurlyeq y$ or $y \preccurlyeq x$ holds. A subset $\mathcal{K}$ of $\mathcal{X}$ is said to be well ordered if every two elements of $\mathcal{K}$ are comparable. If $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is such that, for $x, y \in \mathcal{X}$, $x \preccurlyeq y$ implies $\mathcal{T}x \preccurlyeq \mathcal{T}y$, then the mapping $\mathcal{T}$ is said to be nondecreasing.

Definition 5 Let $(\mathcal{X}, \preccurlyeq)$ be a partially ordered set and $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$.

1. The mapping $\mathcal{T}$ is called dominating if $x \preccurlyeq \mathcal{T}x$ for each $x \in \mathcal{X}$[17].

2. The pair $(\mathcal{S}, \mathcal{T})$ is called weakly increasing if $\mathcal{S}x \preccurlyeq \mathcal{T}\mathcal{S}x$ and $\mathcal{T}x \preccurlyeq \mathcal{S}\mathcal{T}x$ for all $x \in \mathcal{X}$[18,19].

3. The mapping $\mathcal{S}$ is said to be $\mathcal{T}$-weakly isotone increasing if for all $x \in \mathcal{X}$ we have $\mathcal{S}x \preccurlyeq \mathcal{T}\mathcal{S}x \preccurlyeq \mathcal{S}\mathcal{T}\mathcal{S}x$[18-20].

4. The mapping $\mathcal{S}$ is said to be $\mathcal{T}$-strictly weakly isotone increasing if, for all $x \in \mathcal{X}$ such that $x \prec \mathcal{S}x$, we have $\mathcal{S}x \prec \mathcal{T}\mathcal{S}x \prec \mathcal{S}\mathcal{T}\mathcal{S}x$ (HK Nashine, B Samet, and C Vetro, unpublished work).

5. Let $\mathcal{R} : \mathcal{X} \to \mathcal{X}$ be such that $\mathcal{T}\mathcal{X} \subseteq \mathcal{R}\mathcal{X}$ and $\mathcal{S}\mathcal{X} \subseteq \mathcal{R}\mathcal{X}$, and denote $\mathcal{R}^{-1}(x) := \{u \in \mathcal{X} : \mathcal{R}u = x\}$, for $x \in \mathcal{X}$. We say that $\mathcal{T}$ and $\mathcal{S}$ are weakly increasing with respect to $\mathcal{R}$ if and only if for all $x \in \mathcal{X}$, we have [10]:

$$\mathcal{T}x \preccurlyeq \mathcal{S}y, \quad \forall y \in \mathcal{R}^{-1}(\mathcal{T}x)$$

and

$$\mathcal{S}x \preccurlyeq \mathcal{T}y, \quad \forall y \in \mathcal{R}^{-1}(\mathcal{S}x).$$

Example 1 [17] Let $\mathcal{X} = [0, 1]$ be endowed with the usual ordering. Let $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be defined by $\mathcal{T}x = \sqrt[n]{x}$. Since $x \leq \sqrt[n]{x} = \mathcal{T}x$ for all $x \in \mathcal{X}, \mathcal{T}$ is a dominating map.

Remark 1(1) None of two weakly increasing mappings need be nondecreasing. There exist some examples to illustrate this fact in [21].

(2) If $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ are weakly increasing, then $\mathcal{S}$ is $\mathcal{T}$-weakly isotone increasing.
(3) $\mathcal{S}$ can be $\mathcal{T}$-strictly weakly isotone increasing, while some of these two mappings can be not strictly increasing (see the following example).
(4) If $\mathcal{R}$ is the identity mapping ($\mathcal{R}x = x$ for all $x \in \mathcal{X}$), then $\mathcal{T}$ and $\mathcal{S}$ are weakly increasing with respect to $\mathcal{R}$ if and only if they are weakly increasing mappings.

Example 2 Let $\mathcal{X} = [0, +\infty)$ be endowed with the usual ordering and define $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ as

$$\mathcal{S}x = \begin{cases} 2x, \text{ if } x \in [0, 1], \\ 3x, \text{ if } x > 1; \end{cases} \qquad \mathcal{T}x = \begin{cases} 2, \quad \text{if } x \in [0, 1], \\ 2x, \text{ if } x > 1. \end{cases}$$

Clearly, we have $x \prec \mathcal{S}x \prec \mathcal{T}\mathcal{S}x \prec \mathcal{S}\mathcal{T}\mathcal{S}x$ for all $x \in \mathcal{X}$, and so, $\mathcal{S}$ is $\mathcal{T}$-strictly weakly isotone increasing; $\mathcal{T}$ is not strictly increasing.

Definition 6 [22,23]. Let $(\mathcal{X}, d)$ be a metric space and $f, g : \mathcal{X} \to \mathcal{X}$.

1. If $w = fx = gx$, for some $x \in \mathcal{X}$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.If $w = x$, then x is a common fixed point of f and g.

2. The mappings f and g are said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in \mathcal{X}$.

Definition 7 Let $\mathcal{X}$ be a nonempty set. Then $(\mathcal{X}, d, \preccurlyeq)$ is called an ordered metric space if

(i) $(\mathcal{X}, d)$ is a metric space,

(ii) $(\mathcal{X}, \preccurlyeq)$ is a partially ordered set.

The space $(\mathcal{X}, d, \preccurlyeq)$ is called regular if the following hypothesis holds: if $\{z_n\}$ is a nondecreasing sequence in $\mathcal{X}$ with respect to $\preccurlyeq$ such that $z_n \to z \in \mathcal{X}$ as $n \to \infty$, then $z_n \preccurlyeq z$.

3 Common fixed points for $\mathcal{T}$-strictly weakly isotone increasing mappings

In this section, we improve the results of Nashine and Altun (HK Nashine and I Altun, unpublished work) by considering the following:

1. a pair of asymptotically regular mappings;
2. orbital continuity of one of the involved maps;
3. strictly weakly isotone increasing condition;
4. generalized weakly contractive condition, and
5. an ordered orbitally complete metric space.

We will denote by $\mathcal{F}$ and Ψ the set of functions F, $\psi : [0, +\infty) \to [0, +\infty)$, respectively, such that:

(i) F is nondecreasing, continuous, and $F(0) = 0 < F(t)$ for every $t > 0$;

(ii) ψ is nondecreasing, right continuous, and $\psi(0) = 0$.

The first main result of this section is as follows:

Theorem 2 *Let $(\mathcal{X}, d, \preccurlyeq)$ be an ordered metric space. Let $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ be two mappings satisfying*

$$F(d(\mathcal{T}x, \mathcal{S}y)) \leq F(\Theta[\mathcal{T}, \mathcal{S}](x, y))) - \psi(F(\Theta[\mathcal{T}, \mathcal{S}](x, y))) \tag{3.1}$$

for all $x, y \in \overline{\mathcal{O}(x_0; \mathcal{S}, \mathcal{T})}$ (for some x_0) such that x and y are comparable, where $F \in \mathcal{F}$, $\psi \in \Psi$ and

$$\Theta[\mathcal{T}, \mathcal{S}](x, y) = \max\{d(x, y), d(x, \mathcal{T}x), d(y, \mathcal{S}y), \frac{1}{2}(d(x, \mathcal{S}y) + d(y, \mathcal{T}x))\}. \tag{3.2}$$

We assume the following hypotheses:

(i) $(\mathcal{T}, \mathcal{S})$ is a.r. at x_0;

(ii) $\mathcal{X}$ is $(\mathcal{S}, \mathcal{T})$ -orbitally complete at x_0;

(iii) $\mathcal{S}$ or $\mathcal{T}$ is $(\mathcal{S}, \mathcal{T})$-orbitally continuous at x_0;

(iv) $\mathcal{S}$ is $\mathcal{T}$-strictly weakly isotone increasing;

(v) there exists an $x_0 \in \mathcal{X}$ such that $x_0 \prec \mathcal{S}x_0$.

Then $\mathcal{S}$ and $\mathcal{T}$ have a common fixed point. Moreover, the set of common fixed points of $\mathcal{S}, \mathcal{T}$ in $\overline{\mathcal{O}(x_0; \mathcal{S}, \mathcal{T})}$ is well ordered if and only if it is a singleton.

Proof First of all we show that, if $\mathcal{S}$ or $\mathcal{T}$ has a fixed point, then it is a common fixed point of $\mathcal{S}$ and $\mathcal{T}$. Indeed, let z be a fixed point of $\mathcal{S}$. Now assume $d(z, \mathcal{T}z) > 0$. If we use the inequality (3.1), for $x = y = z$, we have

$$\begin{aligned} F(d(\mathcal{T}z, z)) = F(d(\mathcal{T}z, \mathcal{S}z)) &\leq F(\Theta[\mathcal{T}, \mathcal{S}](z, z)) - \psi(F(\Theta[\mathcal{T}, \mathcal{S}](z, z))) \\ &= F(d(\mathcal{T}z, z)) - \psi(F(d(\mathcal{T}z, z))), \end{aligned}$$

wherefrom $\psi(F(d(\mathcal{T}z, z))) = 0$, which is a contradiction. Thus $d(z, \mathcal{T}z) = 0$ and so z is a common fixed point of $\mathcal{S}$ and $\mathcal{T}$. Analogously, one can observe that if z is a fixed point of $\mathcal{T}$, then it is a common fixed point of $\mathcal{S}$ and $\mathcal{T}$.

Since $(\mathcal{T}, \mathcal{S})$ is a.r. at x_0 in $\mathcal{X}$, there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that

$$x_{2n+1} = \mathcal{S}x_{2n} \text{ and } x_{2n+2} = \mathcal{T}x_{2n+1} \text{ for } n \in \{0, 1, \ldots\} \tag{3.3}$$

and

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.4}$$

If $x_{n_0} = \mathcal{S}x_{n_0}$ or $x_{n_0} = \mathcal{T}x_{n_0}$ for some n_0, then the proof is finished. So assume $x_n \neq x_{n+1}$ for all n.

Since $\mathcal{S}$ is $\mathcal{T}$-strictly weakly isotone increasing, we have

$$\begin{aligned} x_1 &= \mathcal{S}x_0 \prec \mathcal{T}\mathcal{S}x_0 = \mathcal{T}x_1 = x_2 \prec \mathcal{S}\mathcal{T}\mathcal{S}x_0 = \mathcal{S}\mathcal{T}x_1 = \mathcal{S}x_2 = x_3, \\ x_3 &= \mathcal{S}x_2 \prec \mathcal{T}\mathcal{S}x_2 = \mathcal{T}x_3 = x_4 \prec \mathcal{S}\mathcal{T}\mathcal{S}x_2 = \mathcal{S}\mathcal{T}x_3 = \mathcal{S}x_4 = x_5, \end{aligned}$$

and continuing this process we get

$$x_1 \prec x_2 \prec \cdots \prec x_n \prec x_{n+1} \prec \cdots. \tag{3.5}$$

Next, we claim that $\{x_n\}$ is a Cauchy sequence in the metric space $\mathcal{O}(x_0; \mathcal{S}, \mathcal{T})$. We proceed by negation and suppose that $\{x_n\}$ is not a Cauchy sequence. That is, there exists $\varepsilon > 0$ such that $d(x_n, x_m) \geq \varepsilon$ for infinitely many values of m and n with $m < n$. This assures that there exist two sequences $\{m(k)\}$, $\{n(k)\}$ of natural numbers, with $m(k) < n(k)$, such that for each $k \in \mathbb{N}$

$$d(x_{2m(k)}, x_{2n(k)+1}) > \varepsilon. \tag{3.6}$$

It is not restrictive to suppose that $n(k)$ is the least positive integer exceeding $m(k)$ and satisfying (3.6). We have

$$\begin{aligned} \varepsilon &< d(x_{2m(k)}, x_{2n(k)+1}) \\ &\leq d(x_{2m(k)}, x_{2n(k)-1}) + d(x_{2n(k)-1}, x_{2n(k)}) + d(x_{2n(k)}, x_{2n(k)+1}) \\ &\leq \varepsilon + d(x_{2n(k)-1}, x_{2n(k)}) + d(x_{2n(k)}, x_{2n(k)+1}) \end{aligned}$$

and letting $k \to \infty$, we have $d(x_{2m(k)}, x_{2n(k)+1}) \to \varepsilon$. We note that

$$\begin{aligned} d\left(x_{2m(k)}, x_{2n(k)+1}\right) &- d\left(x_{2m(k)}, x_{2m(k)+1}\right) - d\left(x_{2n(k)+2}, x_{2n(k)+1}\right) \\ &\leq d\left(x_{2m(k)+1}, x_{2n(k)+2}\right) \\ &\leq d\left(x_{2m(k)}, x_{2n(k)+1}\right) + d\left(x_{2m(k)}, x_{2m(k)+1}\right) + d\left(x_{2n(k)+2}, x_{2n(k)+1}\right), \end{aligned}$$

and thus $d\left(x_{2m(k)+1}, x_{2n(k)+2}\right) \to \varepsilon$ as $k \to \infty$. We have

$$\begin{aligned} \Theta\,[\mathcal{T}, \mathcal{S}]\left(x_{2n(k)+1}, x_{2m(k)}\right) &= \max\Big\{d\left(x_{2n(k)+1}, x_{2m(k)}\right), d\left(x_{2n(k)+1}, x_{2n(k)+2}\right), d\left(x_{2m(k)}, x_{2m(k)+1}\right), \\ &\qquad \tfrac{1}{2}\left[d\left(x_{2n(k)+1}, x_{2m(k)+1}\right) + d\left(x_{2m(k)}, x_{2n(k)+2}\right)\right]\Big\} \\ &\leq \max\Big\{d\left(x_{2n(k)+1}, x_{2m(k)}\right), d\left(x_{2n(k)+1}, x_{2n(k)+2}\right), d\left(x_{2m(k)}, x_{2m(k)+1}\right), \\ &\qquad d\left(x_{2n(k)+1}, x_{2m(k)}\right) + \tfrac{1}{2}\left[d\left(x_{2m(k)}, x_{2m(k)+1}\right) + d\left(x_{2n(k)+1}, x_{2n(k)+2}\right)\right]\Big\} \end{aligned}$$

and so letting $k \to \infty$, we have $\lim_{k\to\infty} \Theta\,[\mathcal{T}, \mathcal{S}]\left(x_{2n(k)+1}, x_{2m(k)}\right) \leq \varepsilon$. Therefore, we have

$$\begin{aligned} F\left(d\left(x_{2m(k)+1}, x_{2n(k)+2}\right)\right) &= F\left(d\left(\mathcal{S}x_{2m(k)}, \mathcal{T}x_{2n(k)+1}\right)\right) \\ &\leq F\left(\Theta[\mathcal{T}, \mathcal{S}]\left(x_{2n(k)+1}, x_{2m(k)}\right)\right) - \psi\left(F\left(\Theta\,[\mathcal{T}, \mathcal{S}]\left(x_{2n(k)+1}, x_{2m(k)}\right)\right)\right) \end{aligned}$$

and letting $k \to \infty$ in the above equation, F being continuous and ψ right continuous, we get

$$F(\varepsilon) \le F(\varepsilon) - \psi(F(\varepsilon)) < F(\varepsilon),$$

a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence in $\mathcal{O}(x_0; \mathcal{S}, \mathcal{T})$. Since $\mathcal{X}$ is $(\mathcal{T}, \mathcal{S})$-orbitally complete at x_0, there exists $z \in \mathcal{X}$ with $\lim_{n\to\infty} x_n = z$.

If $\mathcal{S}$ or $\mathcal{T}$ is orbitally continuous, then clearly $z = \mathcal{S}z = \mathcal{T}z$

Theorem 3 *Let* $(\mathcal{X}, d, \preccurlyeq)$ *and* $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ *satisfy all the conditions of Theorem 2, except that condition* (iii) *is substituted by*

(iii') $\mathcal{X}$ *is regular.*

Then the same conclusions as in Theorem 2 hold.

Proof Following the proof of Theorem 2, we have that $\{x_n\}$ is a Cauchy sequence in $(\mathcal{X}, d)$ which is $(\mathcal{S}, \mathcal{T})$-orbitally complete at x_0. Then, there exists $z \in \mathcal{X}$ such that

$$\lim_{n\to\infty} x_n = z.$$

Now suppose that $d(z, \mathcal{S}z) > 0$. From regularity of $\mathcal{X}$, we have $x_{2n} \preccurlyeq z$ for all $n \in \mathbb{N}$. Hence, we can apply the considered contractive condition. Then, setting $x = x_{2n}$ and $y = z$ in (3.1), we obtain:

$$\begin{aligned} F(d(x_{2n+2}, \mathcal{S}z)) &= F(d(\mathcal{T}x_{2n+1}, \mathcal{S}z)) \\ &\le F\left(\Theta[\mathcal{T}, \mathcal{S}]\left(x_{2n+1,z}\right) - \psi\left(F\Theta[\mathcal{T}, \mathcal{S}]\left(x_{2n+1,z}\right)\right)\right), \end{aligned}$$

where

$$\begin{aligned} &\Theta[\mathcal{T}, \mathcal{S}]\left(x_{2n+1,z}\right) \\ &\quad = \max\left\{d\left(x_{2n+1,z}\right), d(x_{2n+1}, \mathcal{T}x_{2n+1}), d(z, \mathcal{S}z), \right. \\ &\qquad \left. \tfrac{1}{2}\left[d(x_{2n+1}, \mathcal{S}z) + d(z, \mathcal{T}x_{2n+1})\right]\right\} \\ &\quad = \max\left\{d(x_{2n+1}, z), d(x_{2n+1}, x_{2n+2}), d(z, \mathcal{S}z), \right. \\ &\qquad \left. \tfrac{1}{2}\left[d(x_{2n+1}, \mathcal{S}z) + d(z, x_{2n+2})\right]\right\}. \end{aligned}$$

Letting $n \to \infty$ in the above inequality and using the continuity of F and right continuity of ψ, we have

$$F(d(z, \mathcal{S}z)) \le F(d(z, \mathcal{S}z)) - \psi(F(d(z, \mathcal{S}z))) < F(d(z, \mathcal{S}z))$$

a contradiction. Therefore, $d(z, \mathcal{S}z) = 0$ and thus $z = \mathcal{S}z$. Hence, z is a common fixed point of $\mathcal{T}$ and $\mathcal{S}$.

Corollary 1 *Let* $(\mathcal{X}, d, \preccurlyeq)$ *be an ordered metric space. Let* $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ *be a mapping satisfying*

$$F(d(\mathcal{T}x, \mathcal{T}y)) \le F(\Theta_1[\mathcal{T}](x, y)) - \psi(F(\Theta_1[\mathcal{T}](x, y))) \tag{3.7}$$

for all $x, y \in \overline{\mathcal{O}(x_0; \mathcal{T})}$ *(for some* x_0*) such that* x *and* y *are comparable, where* $F \in \mathcal{F}$, $\psi \in \Psi$ *and*

$$\Theta_1[\mathcal{T}](x, y) = \max\left\{d(x, y), d(\mathcal{T}x, x), d(\mathcal{T}y, y), \frac{1}{2}\left(d(x, \mathcal{T}y) + d(\mathcal{T}x, y)\right)\right\}.$$

We assume the following hypotheses:

(i) $\mathcal{T}$ *is a.r. at some point* x_0;

(ii) $\mathcal{X}$ *is* $\mathcal{T}$ *-orbitally complete at* x_0;

(iii) $\mathcal{T}$ *is orbitally continuous at* x_0 *or* $\mathcal{X}$ *is regular.*

Also suppose that $\mathcal{T}x \prec \mathcal{T}(\mathcal{T}x)$ *for all* $x \in \mathcal{X}$ *such that* $x \prec \mathcal{T}x$ *and there exists an* $x_0 \in \mathcal{X}$ *such that* $x_0 \prec \mathcal{T}x_0$. *Then* $\mathcal{T}$ *has a fixed point. Moreover, the set of fixed points of* $\mathcal{T}$ *in* $\overline{\mathcal{O}(x_0;\mathcal{T})}$ *is well ordered if and only if it is a singleton.*

We also state a corollary of Theorem 2 involving a contraction of integral type.

Corollary 2 *Let* $\mathcal{S}$ *and* $\mathcal{T}$ *satisfy the conditions of Theorem 2, except that condition* (3.1) *is replaced by the following: there exists a positive Lebesgue integrable function u on* $\mathbb{R}_+$ *such that* $\int_0^\varepsilon u(t)dt > 0$ *for each* $\varepsilon > 0$ *and that*

$$\int_0^{F(d(\mathcal{S}x,\mathcal{T}y))} u(t)dt \le \int_0^{F(\Theta[\mathcal{T},\mathcal{S}](x,y))} u(t)dt - \int_0^{\psi(F(\Theta[\mathcal{T},\mathcal{S}](x,y)))} u(t)dt.$$

Then, $\mathcal{S}$ *and* $\mathcal{T}$ *have a common fixed point. Moreover, the set of common fixed points of* $\mathcal{S}$ *and* $\mathcal{T}$ *in* $\overline{\mathcal{O}(x_0;\mathcal{S},\mathcal{T})}$ *is well ordered if and only if it is a singleton.*

We present an example showing how our results can be used.

Example 3 Let $\mathcal{X} = \{0\} \cup A \cup B$, where $A = \{\frac{1}{n} | n \in \mathbb{N}\}$ and $B = (1, +\infty)$, be equipped with Euclidean metric d and the order $\preccurlyeq$ given by

$$x \preccurlyeq y \Leftrightarrow x = y \text{ or } \left(x, y \in A \text{ and } x \ge y\right).$$

Consider the mappings $\mathcal{S}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$ given by

$$\mathcal{S}x = \begin{cases} 0, & x = 0, \\ \frac{1}{n+1}, & x = \frac{1}{n},\ n \in \mathbb{N}, \\ 3x, & x \in B, \end{cases} \qquad \mathcal{T}x = \begin{cases} 0, & x = 0, \\ \frac{1}{n+1}, & x = \frac{1}{n},\ n \in \mathbb{N}, \\ 2x, & x \in B. \end{cases}$$

It is easy to check that $\mathcal{S}$ and $\mathcal{T}$ satisfy conditions (i)-(v) of Theorem 2 with $x_0 = \frac{1}{2}$. Take $F \in \mathcal{F}$ defined by

$$F(t) = \begin{cases} 0, & t = 0, \\ t^{1/t}, & 0 < t < 1, \\ t, & t \ge 1, \end{cases}$$

and $\psi \in \Psi$, given as $\psi(t) = \frac{1}{2}t$. In order to check the contractive condition (3.1), take $x, y \in \overline{\mathcal{O}(x_0;\mathcal{S},\mathcal{T})}$ with, say $x \prec y$, i.e., $x > y$ (the case $x = y$ is trivial). Then $x = \frac{1}{n}$ and $y = \frac{1}{m}$ for some $m, n \in \mathbb{N}$, $m > n$. We get that $d\left(\mathcal{T}x, \mathcal{S}y\right) = \frac{1}{n+1} - \frac{1}{m+1} = \frac{m-n}{(n+1)(m+1)}$ and

$$\begin{aligned} F\left(d\left(\mathcal{T}x, \mathcal{S}y\right)\right) &= \left(\frac{m-n}{(n+1)(m+1)}\right)^{\frac{(n+1)(m+1)}{m-n}} \\ &= \left(\frac{m-n}{(n+1)(m+1)}\right)^{\frac{n+m+1}{m-n}} \left(\frac{nm}{(n+1)(m+1)}\right)^{\frac{nm}{m-n}} \left(\frac{m-n}{nm}\right)^{\frac{nm}{m-n}} \\ &< \frac{1}{2} \cdot 1 \cdot \left(\frac{m-n}{nm}\right)^{\frac{nm}{m-n}} = \frac{1}{2} F\left(d(x,y)\right) \le \frac{1}{2} F\left(\Theta[\mathcal{T},\mathcal{S}](x,y)\right) \\ &= F\left(\Theta[\mathcal{T},\mathcal{S}](x,y)\right) - \psi\left(F\left(\Theta[\mathcal{T},\mathcal{S}](x,y)\right)\right). \end{aligned}$$

Hence, (3.1) is fulfilled. Applying Theorem 2, we conclude that $\mathcal{S}$ and $\mathcal{T}$ have a (unique) common fixed point ($z = 0$).

Note that $\mathcal{S}$ and $\mathcal{T}$ do not satisfy the contractive condition for arbitrary $x, y \in \mathcal{X}$.

4 Common fixed points for relatively weakly increasing mappings

In this section, we improve and generalize the results of Nashine and Altun (HK Nashine and I Altun, unpublished work) by taking into account the following for three maps $\mathcal{R}, \mathcal{S}, \mathcal{T}$:

1. $(\mathcal{S}, \mathcal{T})$ is a pair of asymptotically regular mappings with respect to $\mathcal{R}$;
2. orbital continuity of one of the involved maps;
3. $(\mathcal{S}, \mathcal{T})$ is a pair of weakly increasing mappings with respect to $\mathcal{R}$;
4. $(\mathcal{S}, \mathcal{T})$ is a pair of dominating maps;
5. $(\mathcal{S}, \mathcal{T})$ is a pair of compatible maps, and
6. the basic space is an ordered orbitally complete metric space.

We will denote by Φ the set of functions $\phi : [0 + \infty) \to [0, +\infty)$, such that ϕ is right continuous, $\phi(0) = 0$ and $\phi(t) < t$ for every $t > 0$.

The first result of this section is the following.

Theorem 4 *Let $(\mathcal{X}, d, \preccurlyeq)$ be a regular ordered metric space and let $\mathcal{T}, \mathcal{S}$ and $\mathcal{R}$ be self-maps on $\mathcal{X}$ satisfying*

$$F(d(\mathcal{T}x, \mathcal{S}y)) \leq \varphi(F(M[\mathcal{T}, \mathcal{S}, \mathcal{R}](x, y))) \tag{4.1}$$

for all $x, y \in \overline{\mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R})}$ (for some x_0) such that $\mathcal{R}x$ and $\mathcal{R}y$ are comparable, where $F \in \mathcal{F}$, $\phi \in \Phi$ and

$$\begin{aligned} M[\mathcal{T}, \mathcal{S}, \mathcal{R}](x, y) = \max\Big\{ & d(\mathcal{R}x, \mathcal{R}y), d(\mathcal{T}x, \mathcal{R}x), d(\mathcal{S}y, \mathcal{R}y), \\ & \frac{1}{2}\left[d(\mathcal{R}x, \mathcal{S}y) + d(\mathcal{T}x, \mathcal{R}y)\right]. \Big\} \end{aligned} \tag{4.2}$$

We assume the following hypotheses:

(i) $(\mathcal{S}, \mathcal{T})$ is a.r. with respect to $\mathcal{R}$ at $x_0 \in \mathcal{X}$;

(ii) $\mathcal{X}$ is $(\mathcal{S}, \mathcal{T}, \mathcal{R})$-orbitally complete at x_0;

(iii) $\mathcal{T}$ and $\mathcal{S}$ are weakly increasing with respect to $\mathcal{R}$;

(iv) $\mathcal{T}$ and $\mathcal{S}$ are dominating maps;

(v) $\mathcal{R}$ is monotone and orbitally continuous at x_0.

Assume either

(a) $\mathcal{S}$ and $\mathcal{R}$ are compatible; or

(b) $\mathcal{T}$ and $\mathcal{R}$ are compatible.

Then $\mathcal{S}, \mathcal{T}$ and $\mathcal{R}$ have a common fixed point. Moreover, the set of common fixed points of $\mathcal{S}, \mathcal{T}$ and $\mathcal{R}$ in $\overline{\mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R})}$ is well ordered if and only if it is a singleton.

Proof Since $(\mathcal{S}, \mathcal{T})$ is a.r. with respect to $\mathcal{R}$ at x_0 in $\mathcal{X}$, there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that

$$\mathcal{R}x_{2n+1} = \mathcal{S}x_{2n}, \quad \mathcal{R}x_{2n+2} = \mathcal{T}x_{2n+1}, \quad \forall n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}, \tag{4.3}$$

and

$$\lim_{n\to\infty} d(\mathcal{R}x_n, \mathcal{R}x_{n+1}) = 0 \tag{4.4}$$

holds. We claim that

$$\mathcal{R}x_n \preccurlyeq \mathcal{R}x_{n+1}, \quad \forall n \in \mathbb{N}_0. \tag{4.5}$$

To this aim, we will use the increasing property with respect to $\mathcal{R}$ satisfied by the mappings $\mathcal{T}$ and $\mathcal{S}$. From (4.3), we have

$$\mathcal{R}x_1 = \mathcal{S}x_0 \preccurlyeq \mathcal{T}y, \quad \forall y \in \mathcal{R}^{-1}(\mathcal{S}x_0).$$

Since $\mathcal{R}x_1 = \mathcal{S}x_0$, then $x_1 \in \mathcal{R}^{-1}(\mathcal{S}x_0)$, and we get

$$\mathcal{R}x_1 = \mathcal{S}x_0 \preccurlyeq \mathcal{T}x_1 = \mathcal{R}x_2.$$

Again,

$$\mathcal{R}x_2 = \mathcal{T}x_1 \preccurlyeq \mathcal{S}y, \ \forall y \in \mathcal{R}^{-1}(\mathcal{T}x_1).$$

Since $x_2 \in \mathcal{R}^{-1}(\mathcal{T}x_1)$, we get

$$\mathcal{R}x_2 = \mathcal{T}x_1 \preccurlyeq \mathcal{S}x_2 = \mathcal{R}x_3.$$

Hence, by induction, (4.5) holds. Therefore, we can apply (4.1) for $x = x_p$ and $y = x_q$ for all p and q.

Now, we assert that $\{\mathcal{R}x_n\}$ is a Cauchy sequence in the metric space $\mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R})$. We proceed by negation and suppose that $\{\mathcal{R}x_{2n}\}$ is not Cauchy. Then, there exists $\varepsilon > 0$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k,

$$n(k) > m(k) > k, \ \ d\left(\mathcal{R}x_{2m(k),}\mathcal{R}x_{2n(k)}\right) \geq \varepsilon, \ \ d\left(\mathcal{R}x_{2m(k)}, \mathcal{R}x_{2n(k)-2}\right) < \varepsilon. \tag{4.6}$$

From (4.6) and using the triangular inequality, we get

$$\begin{aligned}\varepsilon &\leq d\left(\mathcal{R}x_{2m(k)}, \mathcal{R}x_{2n(k)}\right)\\ &\leq d\left(\mathcal{R}_{2m(k)}, \mathcal{R}x_{2n(k)-2}\right) + d\left(\mathcal{R}x_{2n(k)-2}, \mathcal{R}x_{2n(k)-1}\right) + d\left(\mathcal{R}x_{2n(k)-1}, \mathcal{R}x_{2n(k)}\right)\\ &< \varepsilon + d\left(\mathcal{R}x_{2n(k)-2}, \mathcal{R}x_{2n(k)-1}\right) + d\left(\mathcal{R}x_{2n(k)-1}, \mathcal{R}x_{2n(k)}\right).\end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (4.4), we obtain

$$\lim_{k\to\infty} d\left(\mathcal{R}x_{2m(k)}, \mathcal{R}x_{2n(k)}\right) = \varepsilon. \tag{4.7}$$

Again, the triangular inequality gives us

$$\left|d\left(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2m(k)-1}\right) - d\left(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2m(k)}\right)\right| \leq d\left(\mathcal{R}x_{2m(k)-1}, \mathcal{R}x_{2m(k)}\right).$$

Letting $k \to \infty$ in the above inequality and using (4.4) and (4.7), we get:

$$\lim_{k\to+\infty} d\left(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2m(k)-1}\right) = \varepsilon. \tag{4.8}$$

On the other hand, we have

$$\begin{aligned}d\left(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2m(k)}\right) &\leq d\left(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2n(k)+1}\right) + d\left(\mathcal{R}x_{2n(k)+1}, \mathcal{R}x_{2m(k)}\right)\\ &= d\left(\mathcal{R}x_{2n(k)}, \mathcal{R}x_{2n(k)+1}\right) + d\left(\mathcal{S}x_{2n(k)}, \mathcal{T}x_{2m(k)-1}\right).\end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (4.4), (4.7) and properties of $F \in \mathcal{F}$, we have

$$F(\varepsilon) \leq \lim_{k\to\infty} F\left(d\left(\mathcal{S}x_{2n(k)}, \mathcal{T}x_{2m(k)-1}\right)\right). \tag{4.9}$$

Applying (4.1), we get:

$$F\left(d\left(\mathcal{S}x_{2n(k)}, \mathcal{T}x_{2m(k)-1}\right)\right) \leq \varphi\left(F\left(M\left[\mathcal{T}, \mathcal{S}, \mathcal{R}\right]\left(x_{2m(k)-1}, x_{2n(k)}\right)\right)\right). \tag{4.10}$$

One can check easily that for k large enough, we have:

$$M\left[\mathcal{T},\mathcal{S},\mathcal{R}\right]\left(x_{2m(k)-1},x_{2n(k)}\right)=d\left(\mathcal{R}x_{2n(k)},\mathcal{R}x_{2m(k)-1}\right)+d_k,$$

where $d_k \geq 0$ and $d_k \to 0$ as $k \to \infty$. From (4.10), for k large enough, we have

$$F\left(d\left(\mathcal{S}x_{2n(k)},\mathcal{T}x_{2m(k)-1}\right)\right)\leq\varphi\left(F\left(d\left(\mathcal{R}x_{2n(k)},\mathcal{R}x_{2m(k)-1}\right)+d_k\right)\right). \tag{4.11}$$

Letting $k \to \infty$ in (4.11) and using properties of F and ϕ, we have

$$\lim_{k\to+\infty}F\left(d\left(\mathcal{S}x_{2n(k)},\mathcal{T}x_{2m(k)-1}\right)\right)\leq\varphi\left(F\left(\varepsilon\right)\right)<F(\varepsilon). \tag{4.12}$$

Combining (4.9) and (4.12), we get $F(\varepsilon) < F(\varepsilon)$, a contradiction.

Hence, we deduce that $\{\mathcal{R}x_n\}$ is a Cauchy sequence in $\mathcal{O}(x_0;\mathcal{S},\mathcal{T},\mathcal{R})$. Since $\mathcal{X}$ is $(\mathcal{S},\mathcal{T},\mathcal{R})$-orbitally complete at x_0, there exists some $z \in \mathcal{X}$ such that

$$\mathcal{R}x_n\to z\text{ as }n\to\infty. \tag{4.13}$$

We will prove that z is a common fixed point of the three mappings $\mathcal{S},\mathcal{T}$ and $\mathcal{R}$.

We have

$$\mathcal{S}x_{2n}=\mathcal{R}x_{2n+1}\to z\text{ as }n\to\infty \tag{4.14}$$

and

$$\mathcal{T}x_{2n+1}=\mathcal{R}x_{2n+2}\to z\text{ as }n\to\infty. \tag{4.15}$$

Suppose that (a) holds, i.e., $\mathcal{S}$ and $\mathcal{R}$ are compatible. Then, using condition (v),

$$\lim_{n\to\infty}\mathcal{S}\mathcal{R}x_{2n+2}=\lim_{n\to\infty}\mathcal{R}\mathcal{S}x_{2n+2}=\mathcal{R}z. \tag{4.16}$$

From (4.13) and the orbitally continuity of $\mathcal{R}$, we have also

$$\mathcal{R}\left(\mathcal{R}x_n\right)\to\mathcal{R}z\text{ as }n\to\infty. \tag{4.17}$$

Now, using (iv), $x_{2n+1} \preccurlyeq \mathcal{T}x_{2n+1} = \mathcal{R}x_{2n+2}$ and since $\mathcal{R}$ is monotone, $\mathcal{R}x_{2n+1}$ and $\mathcal{R}\mathcal{R}x_{2n+2}$ are comparable. Thus, we can apply (4.1) to obtain

$$F\left(d\left(\mathcal{S}\mathcal{R}x_{2n+2},\mathcal{T}x_{2n+1}\right)\right)\leq\varphi\left(F\left(M[\mathcal{T},\mathcal{S},\mathcal{R}]\left(\mathcal{R}x_{2n+2},x_{2n+1}\right)\right)\right), \tag{4.18}$$

where

$$\begin{aligned}M[\mathcal{T},&\mathcal{S},\mathcal{R}](\mathcal{R}x_{2n+2},x_{2n+1})\\&=\max\{d(\mathcal{R}\mathcal{R}x_{2n+2},\mathcal{R}x_{2n+1}),d(\mathcal{R}\mathcal{R}x_{2n+2},\mathcal{S}\mathcal{R}x_{2n+2}),d(\mathcal{R}x_{2n+1},\mathcal{T}x_{2n+1}),\\&\quad\tfrac{1}{2}[d(\mathcal{R}\mathcal{R}x_{2n+2},\mathcal{T}x_{2n+1})+d(\mathcal{S}\mathcal{R}x_{2n+2},\mathcal{R}x_{2n+1})]\}.\end{aligned}$$

Letting $n \to \infty$ in (4.18), using (4.13)-(4.17), we obtain

$$F\left(d\left(\mathcal{R}z,z\right)\right)\leq\varphi\left(F\left(d\left(\mathcal{R}z,z\right)\right)\right)<F\left(d\left(\mathcal{R}z,z\right)\right),$$

unless

$$\mathcal{R}z=z. \tag{4.19}$$

Now, $x_{2n+1} \preccurlyeq \mathcal{T}x_{2n+1}$ and $\mathcal{T}x_{2n+1} \to z$ as $n \to \infty$, so by the assumption we have $x_{2n+1} \preccurlyeq z$ and $\mathcal{R}x_{2n+1}$ and $\mathcal{R}z$ are comparable. Hence (4.1) gives

$$F(d(\mathcal{S}z, \mathcal{T}x_{2n+1})) \leq \varphi(F(\max\{d(\mathcal{R}z, \mathcal{R}x_{2n+1}), d(\mathcal{S}z, \mathcal{R}z), d(\mathcal{T}x_{2n+1}, \mathcal{R}x_{2n+1}),$$
$$\tfrac{1}{2}[d(\mathcal{R}z, \mathcal{T}x_{2n+1}) + d(\mathcal{S}z, \mathcal{R}x_{2n+1})]\})).$$

Passing to the limit as $n \to \infty$ in the above inequality and using (4.19), it follows that

$$\begin{aligned} F(d(\mathcal{S}z, z)) &\leq \varphi(F(\max\{0, d(\mathcal{S}z, z), 0, \tfrac{1}{2}d(\mathcal{S}z, z)\})) \\ &\leq \varphi(F(d(\mathcal{S}z, z))) < F(d(\mathcal{S}z, z)), \end{aligned}$$

which holds unless

$$\mathcal{S}z = z. \tag{4.20}$$

Similarly, $x_{2n} \preccurlyeq \mathcal{S}x_{2n}$ and $\mathcal{S}x_{2n} \to z$ as $n \to \infty$, implies that $x_{2n} \preccurlyeq z$, hence $\mathcal{R}x_{2n}$ and $\mathcal{R}z$ are comparable. From (4.1) we get

$$F(d(\mathcal{S}x_{2n}, \mathcal{T}z)) \leq \varphi(F(\max\{d(\mathcal{R}x_{2n}, \mathcal{R}z), d(\mathcal{R}x_{2n}, \mathcal{S}x_{2n}), d(\mathcal{R}z, \mathcal{T}z),$$
$$\tfrac{1}{2}(d(\mathcal{R}x_{2n}, \mathcal{T}z) + d(\mathcal{S}x_{2n}, \mathcal{R}z))\})).$$

Passing to the limit as $n \to \infty$, we have

$$\begin{aligned} F(d(z, \mathcal{T}z)) &\leq \varphi(F(\max\{0, 0, d(z, \mathcal{T}z), d(z, \mathcal{T}z)\})) \\ &\leq \varphi(F(d(z, \mathcal{T}z))) < F(d(z, \mathcal{T}z)), \end{aligned}$$

which gives that

$$z = \mathcal{T}z. \tag{4.21}$$

Therefore, $\mathcal{S}z = \mathcal{T}z = \mathcal{R}z = z$, hence z is a common fixed point of $\mathcal{R}, \mathcal{S}$ and $\mathcal{T}$.

Similarly, the result follows when condition (b) holds.

Now, suppose that the set of common fixed points of $\mathcal{S}, \mathcal{T}$ and $\mathcal{R}$ in $\overline{\mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R})}$ is well ordered. We claim that there is a unique common fixed point of $\mathcal{S}, \mathcal{T}$ and $\mathcal{R}$ in $\overline{\mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R})}$. Assume to the contrary that $\mathcal{S}u = \mathcal{T}u = \mathcal{R}u = u$ and $\mathcal{S}v = \mathcal{T}v = \mathcal{R}v = v$ but $u \neq v$. By supposition, we can replace x by u and y by v in (4.1) to obtain

$$\begin{aligned} F(d(u, v)) &= F(d(\mathcal{S}u, \mathcal{T}v)) \\ &\leq \varphi(F(\max\{d(\mathcal{R}u, \mathcal{R}v), d(\mathcal{R}u, \mathcal{S}u), d(\mathcal{R}v, \mathcal{T}v) \\ &\qquad \tfrac{1}{2}[d(\mathcal{R}u, \mathcal{T}v) + d(\mathcal{S}u, \mathcal{R}v)]\})) \\ &= \varphi(F(\max\{d(u, v), 0, 0, d(u, v)\})) < F(d(u, v)), \end{aligned}$$

a contradiction. Hence, $u = v$. The converse is trivial.

We obtain the following corollaries from Theorem 4.

Corollary 3 *Let $(\mathcal{X}, d, \preccurlyeq)$ be a regular ordered metric space and let $\mathcal{T}$ and $\mathcal{S}$ be self-maps on $\mathcal{X}$ satisfying*

$$F\left(d\left(\mathcal{T}x, \mathcal{S}y\right)\right) \leq \varphi\left(F\left(M_1[\mathcal{T}, \mathcal{S}](x, y)\right)\right),$$

for all $x, y \in \overline{\mathcal{O}(x_0; \mathcal{S}, \mathcal{T})}$ (for some x_0) such that x and y are comparable, where $F \in \mathcal{F}$, $\phi \in \Phi$ and

$$M_1[\mathcal{T}, \mathcal{S}](x, y) = \max\{d(x, y), d(\mathcal{T}x, x), d(\mathcal{S}y, y), \frac{1}{2}[d(x, \mathcal{S}y) + d\left(\mathcal{T}x, y\right)]\}.$$

We assume the following hypotheses:

(i) $(\mathcal{S}, \mathcal{T})$ is a.r. at some point $x_0 \in \mathcal{X}$;

(ii) $\mathcal{X}$ *is* $(\mathcal{S}, \mathcal{T})$ *-orbitally complete at* x_0;

(iii) $\mathcal{T}$ *and* $\mathcal{S}$ *are weakly increasing;*

(iv) $\mathcal{T}$ *and* $\mathcal{S}$ *are dominating maps.*

Then $\mathcal{T}$ *and* $\mathcal{S}$ *have a common fixed point. Moreover, the set of common fixed points of* $\mathcal{T}$ *and* $\mathcal{S}$ *in* $\overline{\mathcal{O}(x_0; \mathcal{S}, \mathcal{T})}$ *is well ordered if and only if it is a singleton.*

Corollary 4 *Let* $(\mathcal{X}, d, \preccurlyeq)$ *be a regular ordered metric space and let* $\mathcal{T}$ *and* $\mathcal{R}$ *be self-maps on* $\mathcal{X}$ *satisfying*

$$F(d(\mathcal{T}x, \mathcal{T}y)) \leq \varphi(F(M_2[\mathcal{T}, \mathcal{R}](x, y))),$$

for all $x, y \in \overline{\mathcal{O}\,(x_0; \mathcal{T}, \mathcal{T}, \mathcal{R})}$ *(for some* x_0*) such that* $\mathcal{R}x$ *and* $\mathcal{R}y$ *are comparable, where* $F \in \mathcal{F}$, $\phi \in \Phi$ *and*

$$M_2[\mathcal{T}, \mathcal{R}](x, y) = \max\{d(\mathcal{R}x, \mathcal{R}y), d(\mathcal{T}x, \mathcal{R}x), d(\mathcal{T}y, \mathcal{R}y), \frac{1}{2}[d(\mathcal{R}x, \mathcal{T}y) + d(\mathcal{T}x, \mathcal{R}y)]\}.$$

We assume the following hypotheses:

(i) $\mathcal{T}$ *is a.r. with respect to* $\mathcal{R}$ *at* $x_0 \in \mathcal{X}$;

(ii) $\mathcal{X}$ *is* $(\mathcal{T}, \mathcal{R})$ *-orbitally complete at* x_0;

(iii) $\mathcal{T}$ *is weakly increasing with respect to* $\mathcal{R}$;

(iv) $\mathcal{T}$ *is a dominating map;*

(v) $\mathcal{R}$ *is monotone and orbitally continuous at* x_0.

Then $\mathcal{T}$ *and* $\mathcal{R}$ *have a common fixed point. Moreover, the set of common fixed points of* $\mathcal{T}$ *and* $\mathcal{R}$ *in* $\overline{\mathcal{O}\,(x_0; \mathcal{T}, \mathcal{T}, \mathcal{R})}$ *is well ordered if and only if it is a singleton.*

Corollary 5 *Let* $(\mathcal{X}, d, \preccurlyeq)$ *be a regular ordered metric space and let* $\mathcal{T}$ *be a self-map on* $\mathcal{X}$ *satisfying for all* $x, y \in \overline{\mathcal{O}(x_0; \mathcal{T})}$ *such that x and y are comparable,*

$$F(d(\mathcal{T}x, \mathcal{T}y)) \leq \varphi(F(M_3[\mathcal{T}](x, y))),$$

where $F \in \mathcal{F}$, $\phi \in \Phi$ *and*

$$M_3[\mathcal{T}](x, y) = \max\{d(x, y), d(\mathcal{T}x, x), d(\mathcal{T}y, y), \frac{1}{2}[d(x, \mathcal{T}y) + d(\mathcal{T}x, y)]\}.$$

We assume the following hypotheses:

(i) $\mathcal{T}$ *is a.r. at some point* x_0 *of* $\mathcal{X}$;

(ii) $\mathcal{X}$ *is* $\mathcal{T}$ *-orbitally complete at* x_0;

(iii) $\mathcal{T}x \preccurlyeq \mathcal{T}(\mathcal{T}x)$ *for all* $x \in \mathcal{X}$;

(iv) $\mathcal{T}$ *is a dominating map.*

Then $\mathcal{T}$ *has a fixed point. Moreover, the set of fixed points of* $\mathcal{T}$ *in* $\overline{\mathcal{O}(x_0; \mathcal{T})}$ *is well ordered if and only if it is a singleton.*

We also state a corollary of Theorem 4 involving a contraction of integral type.

Corollary 6 *Let* $\mathcal{S}, \mathcal{T}$ *and* $\mathcal{R}$ *satisfy the conditions of Theorem 4, except that condition* (4.1) *is replaced by the following: there exists a positive Lebesgue integrable function u on* $\mathbb{R}_+$ *such that* $\int_0^{\varepsilon} u(t)dt > 0$ *for each* $\varepsilon > 0$ *and that*

$$\int_0^{F(d(\mathcal{S}x, \mathcal{T}y))} u(t)dt \leq \int_0^{\varphi(F(M[\mathcal{T}, \mathcal{S}, \mathcal{R}](x, y)))} u(t)dt.$$

Then, $\mathcal{S}, \mathcal{T}$ and $\mathcal{R}$ have a common fixed point. Moreover, the set of common fixed points of $\mathcal{S}, \mathcal{T}$ and $\mathcal{R}$ in $\overline{\mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R})}$ is well ordered if and only if it is a singleton.

Example 4 Let the set $\mathcal{X} = [0, +\infty)$ be equipped with the usual metric d and the order defined by

$$x \preccurlyeq y \Leftrightarrow x \geq y.$$

Consider the following self-mappings on $\mathcal{X}$:

$$\mathcal{R}x = 6x, \quad \mathcal{S}x = \begin{cases} \frac{1}{2}x, & 0 \leq x \leq \frac{1}{2} \\ x, & x > \frac{1}{2}, \end{cases}, \qquad \mathcal{T}x = \begin{cases} \frac{1}{3}x, & 0 \leq x \leq \frac{1}{3} \\ x, & x > \frac{1}{3}. \end{cases},$$

Take $x_0 = \frac{1}{2}$. Then it is easy to show that

$$\mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R}) \subset \left\{ \frac{1}{2^k \cdot 3^l} : k, l \in \mathbb{N} \right\}$$

and $\overline{\mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R})} = \mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R}) \cup \{0\}$, and all the conditions (i)-(v) and (a)-(b) of Theorem 4 are fulfilled (condition (iii) on $\overline{\mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R})}$. Take $\psi(t) = \frac{1}{6}t$ and $F \in \mathcal{F}$ of the form $F(t) = kt$, $k > 0$. Then contractive condition (4.1) takes the form

$$\left| \frac{1}{2}x - \frac{1}{3}y \right| \leq \frac{1}{6} \max \left\{ |6x - 6y|, \frac{11}{2}x, \frac{17}{3}y, \frac{1}{2}\left[\left| 6x - \frac{1}{3}y \right| + \left| 6y - \frac{1}{2}x \right| \right] \right\},$$

for $x, y \in \overline{\mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R})}$. Using substitution $y = tx$, $t \geq 0$, the last inequality reduces to

$$|3 - 2t| \leq \max \left\{ 6|1 - t|, \tfrac{11}{2}, \tfrac{17}{3}t, \tfrac{1}{2}\left[\left| 6 - \tfrac{1}{3}t \right| + \left| 6t - \tfrac{1}{2} \right| \right] \right\},$$

and can be checked by discussion on possible values for $t \geq 0$. Hence, all the conditions of Theorem 4 are satisfied and $\mathcal{S}, \mathcal{T}, \mathcal{R}$ have a unique common fixed point in $\overline{\mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R})}$ (which is 0).

Remark 2 It was shown by examples in [24] that (in similar situations):

(1) if the contractive condition is satisfied just on $\mathcal{O}(x_0; \mathcal{S}, \mathcal{T}, \mathcal{R})$, there might not exist a (common) fixed point;

(2) under the given hypotheses (common) fixed point might not be unique in the whole space $\mathcal{X}$.

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Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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