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Remarks on some recent fixed point theorems

Hassen Aydi[1], Erdal Karapınar[2] and Bessem Samet[3*]

* Correspondence: bessem.samet@gmail.com
[3]Department of Mathematics, King Saud University, Riyadh, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

The purpose of this article is to show that some recent fixed point theorems are particular results of previous existing theorems in the literature.

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1 Introduction

In this section, we recall some known results on fixed point theory.

We start with the well-known Banach contraction principle [1].

Theorem 1. *Let* (X, d) *be a complete metric space and let* $F: X \to X$ *be a mapping such that for each pair of points* $x, y \in X$,

$$d(Fx, Fy) \le k\, d(x, y),$$

where k *is a constant in* $[0, 1)$. *Then* F *has a unique fixed point.*

In 2008, Dutta and Choudhury [2] obtained the following generalization of the Banach contraction principle.

Theorem 2. *Let* (X, d) *be a complete metric space and let* $F: X \to X$ *be a mapping such that for each pair of points* $x, y \in X$,

$$\psi(d(Fx, Fy)) \le \psi(d(x, y)) - \varphi(d(x, y)),$$

where $\psi, \phi: [0, \infty) \to [0, \infty)$ *are continuous, non-decreasing and* $\psi^{-1}(\{0\}) = \phi^{1}(\{0\}) = \{0\}$. *Then* F *has a unique fixed.*

Remark 3. *Note that Theorem* 2 *remains true if* ϕ *satisfies only the following assumptions:* ϕ *is lower semi-continuous and* $\phi^{-1}(\{0\}) = \{0\}$ *(see, for example, Abbas and Dorić* [3]*and Dorić* [4]).

Using the above remark, we have also

Theorem 4. *Let* (X, d) *be a complete metric space and let* $F: X \to X$ *be a mapping such that for each pair of points* $x, y \in X$,

$$\psi(d(Fx, Fy)) \le \psi(d(x, y)) - \varphi(d(x, y)), \tag{1}$$

where $\psi: [0, \infty) \to [0, \infty)$ *is continuous, non-decreasing,* $\psi^{-1}(\{0\}) = \{0\}$, *and* $\phi: [0, \infty) \to [0, \infty)$ *is lower semi-continuous and* $\phi^{-1}(\{0\}) = \{0\}$. *Then* F *has a unique fixed.*

We have also an ordered version of Theorem 4 (see [3-5]).

Theorem 5. *Let* $(X,\preccurlyeq)$ *be a partially ordered set and suppose that there is a metric* d *on* X *such that* (X, d) *is a complete metric space. Let* $F: X \to X$ *be a continuous non-*

decreasing mapping such that

$$\psi(d(Fx, Fy)) \le \psi(d(x, y)) - \varphi(d(x, y)), \tag{2}$$

for all $x, y \in X$ *with* $x \preccurlyeq y$, *where* ψ: $[0, \infty) \to [0, \infty)$ *is continuous, non-decreasing,* $\psi^{-1}(\{0\}) = \{0\}$, *and* ϕ: $[0, \infty) \to [0, \infty)$ *is lower semi-continuous and* $\phi^{-1}(\{0\}) = \{0\}$. *If there exists* $x_0 \in X$ *such that* $x_0 \preccurlyeq Fx_0$, *then F has a fixed point.*

The following result was obtained by Olaleru [6].

Theorem 6. *Let* (X, d) *be a cone metric space with a cone P having non-empty interior. Let f, g:* $X \to X$ *be mappings such that*

$$d(fx, fu) \le \alpha_1 d(fx, gx) + \alpha_2 d(fu, gu) + \alpha_3 d(fu, gx) + \alpha_4 d(fx, gu) + \alpha_5 d(gx, gu) \tag{3}$$

for all $x, u \in X$, *where* $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \in [0, 1)$ *and* $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$. *Suppose that* $f(X) \subseteq g(X)$ *and* $g(X)$ *is a complete subspace of X. Then f and g have a coincidence point. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.*

The purpose of this article is to show that some recent fixed point theorems are particular cases of the above mentioned results. This article can be considered as a continuation of the recent work of Haghi et al. [7].

2 Main results

Beiranvand et al. [8] introduced a new class of mappings T: $X \to X$ as follows.

Definition 7. *The mapping* T: $X \to X$ *is said to be sequentially convergent, if the sequence* $\{y_n\}$ *in X is convergent whenever* $\{T y_n\}$ *is convergent.*

In the same article, the authors established the following result.

Theorem 8. *Let* (X, d) *be a complete metric space and T, f:* $X \to X$ *be two mappings satisfying*

$$d(T fx, \; T fy) \le kd(T x, T y)), \tag{4}$$

for all $x, y \in X$, *where k is a constant in* $[0, 1)$ *and T is continuous, injective and sequentially convergent. Then f has a unique fixed point.*

Theorem 8 has attracted the attention of many authors, see, for example, [9-15], where extensions and generalizations of Theorem 8 were considered.

We shall prove the following result.

Theorem 9. *Theorem* 1 *and Theorem* 8 *are equivalent.*

Proof. Clearly, if T is the identity mapping, Theorem 8 reduces to Theorem 1. Now, we shall prove that Theorem 8 can be deduced for the Banach contraction principle. Define the mapping δ: $X \times X \to [0, \infty)$ by

$$\delta(x, y) = d(Tx, Ty), \quad x, y \in X.$$

For all $x, y, z \in X$, we have $\delta(x, y) = \delta(y, x)$, $\delta(x, y) \le \delta(x, y) + \delta(y, z)$ and

$$\delta(x, y) = 0 \Leftrightarrow d(Tx, \; Ty) = 0 \Leftrightarrow Tx = Ty \Leftrightarrow x = y \,(\text{since } T \text{ is injective}).$$

Then δ is a metric on X. Moreover, (X, δ) is a complete metric space. Indeed, let $\{x_n\}$ be a Cauchy sequence in (X, δ). From the definition of δ, this implies that $\{T x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, there exists $y \in X$ such that $d(T x_n, y) \to 0$ as $n \to \infty$. But T is sequentially convergent, then there exists $x \in X$, such that $d(x_n, x) \to 0$ as $n \to \infty$. Since T is continuous, this implies that $d(T x_n, Tx) \to 0$ as $n \to \infty$, that is, $\delta(x_n, x) \to 0$ as $n \to \infty$. This proves that (X, δ) is complete. Now, condition (4) reduces to

$$\delta(fx, fy) \leq k\delta(x, y)$$

for all $x, y \in X$. Thus Theorem 8 follows immediately from the Banach contraction principle (Theorem 1). □

Recently, Eslamian and Abkar [16] etablished the following result.

Theorem 10. *Let (X, d) be a complete metric space and $f: X \to X$ be such that*

$$\psi(d(fx, fy)) \leq \alpha(d(x, y)) - \beta(d(x,y)), \tag{5}$$

for all $x, y \in X$, where $\psi, \alpha, \beta: [0, \infty) \to [0, \infty)$ are such that ψ is continuous and non-decreasing, α is continuous, β is lower semi-continuous,

$$\psi(t) = 0 \textit{ if and only if } t = 0, \quad \alpha(0) = \beta(0) = 0, \tag{6}$$

$$\textit{and } \psi(t) - \alpha(t) + \beta(t) > 0 \textit{ for all } t > 0. \tag{7}$$

Then f has a unique fixed point.

We shall prove the following result.

Theorem 11. *Theorem* 10 *and Theorem* 4 *are equivalent.*

Proof. Taking $\alpha = \psi$ in Theorem 10, we obtain immediately Theorem 4. Now, we shall prove that Theorem 10 can be deduced from Theorem 4. Indeed, let $f: X \to X$ be a mapping satisfying (5) with $\psi, \alpha, \beta: [0, \infty) \to [0, \infty)$ satisfy conditions (6) and (7). From (5), for all $x, y \in X$, we have

$$\begin{aligned}\psi\left(d\left(fx, fy\right)\right) &\leq \alpha\left(d\left(x, y\right)\right) - \beta\left(d\left(x, y\right)\right)\\ &= \psi\left(d\left(x, y\right)\right) - \left[\beta\left(d\left(x, y\right)\right) - \alpha\left(d\left(x, y\right)\right) + \psi\left(d\left(x, y\right)\right)\right].\end{aligned}$$

Define $\theta: [0, \infty) \to [0, \infty)$ by

$$\theta(t) = \beta(t) - \alpha(t) + \psi(t), \quad t \geq 0.$$

Then, we have

$$\psi\left(d\left(fx, fy\right)\right) \leq \psi\left(d\left(x, y\right)\right) - \theta\left(d\left(x, y\right)\right),$$

for all $x, y \in X$. Clearly, from (6) and (7), θ is lower semi-continuous and $\theta^{1}(\{0\}) = \{0\}$. Now, Theorem 10 follows immediately from Theorem 4. □

Binayak et al. [17] extended Theorem 10 to the ordered case.

Theorem 12. *Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $f: X \to X$ be a continuous non-decreasing mapping such that*

$$\psi(d(fx, fy)) \leq \alpha(d(x, y)) - \beta(d(x,y)),$$

for all $x, y \in X$ with $x \preccurlyeq y$, where $\psi, \alpha, \beta: [0, \infty) \to [0, \infty)$ are such that ψ is continuous and non-decreasing, α is continuous, β is lower semi-continuous,

$$\psi(t) = 0 \textit{ if and only if } t = 0, \quad \alpha(0) = \beta(0) = 0,$$

$$\text{and}\,\psi(t) - \alpha(t) + \beta(t) > 0 \quad \text{for all}\, t > 0.$$

If there exists $x_0 \in X$ such that $x_0 \preccurlyeq fx_0$, then f has a fixed point.

Following similar arguments as in the proof of Theorem 11, we obtain

Theorem 13. *Theorem* 5 *and Theorem* 12 *are equivalent.*

Abbas et al. [18] introduced the concept of *w*-compatibility for a pair of mappings $F: X \times X \to X$ and $g: X \to X$.

Definition 14. *The mappings $F: X \times X \to X$ and $g: X \to X$ are called w-compatible if $g(F(x, y)) = F(gx, gy)$ whenever $gx = F(x, y)$ and $gy = F(y, x)$.*

In the same article, the authors established the following result.

Theorem 15. *Let (X, d) be a cone metric space with a cone P having non-empty interior, $F: X \times X \to X$ and $g: X \to X$ be mappings satisfying*

$$d(F(x, y), F(u, v)) \le a_1 d(gx, gu) + a_2 d(F(x, y), gx) + a_3 d(gy, gu) \tag{8}$$

$$+a_4 d\left(F(u, v), gu\right) + \alpha_5 d\left(F(x, y), gu\right) + a_6 d\left(F(u, v), gx\right), \tag{9}$$

for all $x, y, u, v \in X$, where a_i, $i = 1, 2,\ldots, 6$ are nonnegative real numbers such that $\sum_{i=1}^{6} a_i < 1$. If $F(X \times X) \subseteq g(X)$ and $g(X)$ is complete subset of X, then F and g have a coupled coincidence point in X, that is, there exists $(x, y) \in X \times X$ such that $gx = F(x, y)$ and $gy = F(y, x)$. Moreover, if F and g are w-compatible, then there exists a unique $u \in X$ such that $u = gu = F(u, u)$.

We shall prove the following result.

Theorem 16. *Theorem* 6 *and Theorem* 15 *are equivalent.*

Proof. (i) Theorem 15 ⇒ Theorem 6. Let $f, g: X \to X$ be mappings satisfying the hypotheses of Theorem 6. Define the mapping $F: X \times X \to X$ by

$$F(x, y) = fx, \quad x, y \in X.$$

From (3), we get

$$\begin{aligned} d\left(F(x, y), F(u, v)\right) < {} & \alpha_1 d\left(F(x, y), gx\right) + \alpha_2 d\left(F(u, v), gu\right) + \alpha_3 d\left(F(u, v), gx\right) + \alpha_4 d\left(F(x, y), gu\right) \\ & + \alpha_5 d(gx, gu), \end{aligned}$$

for all $x, y, u, v \in X$. Then condition (8) is satisfied with $(a_1, a_2, a_3, a_4, a_5, a_6) = (\alpha_5, \alpha_1, 0, \alpha_2, \alpha_4, \alpha_3)$. On the other hand, from the definition of F, we have $F(X \times X) = f(X) \subseteq g(X)$. Also, $g(X)$ is a complete subspace of (X, d). Now, applying Theorem 15, we obtain that F and g have a coupled coincidence point in X, that is, there exists (x, y) $X \times X$ such that $gx = F(x, y)$ and $gy = F(y, x)$. From the definition of F, this implies that $gx = fx$, that is, x is a coincidence point of f and g. Suppose now that f and g are weakly compatible. Let $x, y \in X$ such that $gx = F(x, y)$. This implies that $gx = fx$. Since f and g are weakly compatible, we get $fgx = gfx$, that is, $F(gx, gy) = g(F(x, y))$. This implies that F and g are w-compatible. From Theorem 15, there exists a unique $u \in X$ such that $u = gu = F(u, u)$, that is, there exists a unique $u \llcorner X$ such that $u = gu = fu$. Then f and g have a unique common fixed point. Thus we proved Theorem 6.

(ii) Theorem 6 ⇒ Theorem 15.

Let $F: X \times X \to X$ and $g: X \to X$ be mappings satisfying the hypotheses of Theorem 15. Define the mapping $f: X \to X$ by

$$fx = F(x, x), \quad x \in X.$$

From (8), we have

$$d(fx, fu) \le \alpha_1 d(fx, gx) + \alpha_2 d(fu, gu) + \alpha_3 d(fu, gx) + \alpha_4 d(fx, gu) + \alpha_5 d(gx, gu),$$

for all $x, u \in X$, where $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (a_2, a_4, a_6, a_5, a_1 + a_3)$. Then condition (3) of Theorem 6 is satisfied. On the other hand, we have $f(X) \subseteq F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of (X, d). Applying Theorem 6, we obtain that there exists $x \in X$ (a coincidence point) such that $fx = gx$, that is, $F(x, x) = gx$. Moreover, if F and g are w-compatible, then f and g are weakly compatible. Applying again Theorem 6, we obtain that f and g have a unique common fixed point, that is, there exists a unique $u \in X$ such that $u = gu = fu = F(u, u)$. Thus we proved Theorem 15. □

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Author details
[1]Institut Supérieur d'Informatique et des Technologies de Communication de Hammam Sousse, Université de Sousse, Route GP1-4011, Hammam Sousse, Tunisie [2]Department of Mathematics, Atılım University, Incek, 06836, Ankara, Turkey [3]Department of Mathematics, King Saud University, Riyadh, Saudi Arabia

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Competing interests
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