Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

A new iterative method for a common solution of fixed points for pseudo-contractive mappings and variational inequalities

Tanom Chamnarnpan and Poom Kumam*

* Correspondence: poom.kum@kmutt.ac.th
Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Bangmod, Thrungkru, Bangkok 10140, Thailand

Abstract

In this article, we introduce a new iterative scheme for finding a common element of the set of fixed points for a continuous pseudo-contractive mapping and the solution set of a variational inequality problem governed by continuous monotone mappings. Strong convergence for the proposed iterative scheme is proved. Our results improve and extend some recent results in the literature.

2000 Mathematics Subject Classification: 46C05; 47H09; 47H10.

Keywords: monotone mapping, nonexpansive mapping, pseudo-contractive mappings, variational inequality

1. Introduction

The theory of variational inequalities represents, in fact, a very natural generalization of the theory of boundary value problems and allows us to consider new problems arising from many fields of applied mathematics, such as mechanics, physics, engineering, the theory of convex programming, and the theory of control. While the variational theory of boundary value problems has its starting point in the method of orthogonal projection, the theory of variational inequalities has its starting point in the projection on a convex set.

Let C be a nonempty closed and convex subset of a real Hilbert space H. The classical variational inequality problem is to find a $u \in C$ such that $\langle v-u, Au\rangle \geq 0$ for all $v \in C$, where A is a nonlinear mapping. The set of solutions of the variational inequality is denoted by $VI(C, A)$. The variational inequality problem has been extensively studied in the literature, see [1-5] and the reference therein. In the context of the variational inequality problem, this implies that $u \in VI(C, A) \Leftrightarrow u = P_C(u - \lambda Au)$, $\forall\lambda > 0$, where P_C is a metric projection of H into C.

Let A be a mapping from C to H, then A is called *monotone* if and only if for each x, $y \in C$,

$$\langle x - y, Ax - Ay\rangle \geq 0. \tag{1.1}$$

An operator A is said to be strongly positive on H if there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H.$$

A mapping A of C into itself is called L-Lipschitz continuous if there exits a positive and number L such that

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

A mapping A of C into H is called *α-inverse-strongly monotone* if there exists a positive real number α such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2,$$

for all x, $y \in C$; see [2,6-10]. If A is an α-inverse strongly monotone mapping of C into H, then it is obvious that A is $\frac{1}{\alpha}$-Lipschitz continuous, that is, $\|Ax - Ay\| \leq \frac{1}{\alpha}\|x - y\|$ for all x, $y \in C$. Clearly, the class of monotone mappings include the class of α-inverse strongly monotone mappings.

Recall that a mapping T of C into H is called *pseudo-contractive* if for each x, $y \in C$, we have

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2. \tag{1.2}$$

T is said to be a k-strict pseudo-contractive mapping if there exists a constant $0 \leq k \leq 1$ such that

$$\langle x - y, Tx - Ty\rangle \leq \|x - y\|^2 - k\|(I - T)x - (I - T)y\|^2, \text{ for all } x,\ y \in D(T).$$

A mapping T of C into itself is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$, for all x, $y \in C$. We denote by $F(T)$ the set of fixed points of T. Clearly, the class of pseudo-contractive mappings include the class of nonexpansive and strict pseudo-contractive mappings.

For finding an element of $F(T)$, where T is a nonexpansive mapping of C into itself, Halpern [11] was the first to study the convergence of the following scheme:

$$x_{n+1} = \alpha_{n+1}u + (1 - \alpha_{n+1})T(x_n), \quad n \geq 0, \tag{1.3}$$

where u, $x_0 \in C$ and a sequence $\{\alpha_n\}$ of real numbers in (0,1) in the framework of Hilbert spaces. Lions [12] improved the result of Halpern by proving strong convergence of $\{x_n\}$ to a fixed point of T provided that the real sequence $\{\alpha_n\}$ satisfies certain mild conditions. In 2000, Moudafi [13] introduced viscosity approximation method and proved that if H is a real Hilbert space, for given $x_0 \in C$, the sequence $\{x_n\}$ generated by the algorithm

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)T(x_n), \quad n \geq 0, \tag{1.4}$$

where $f : C \to C$ is a contraction mapping with a constant $\beta \in (0,1)$ and $\{\alpha_n\} \subset (0,1)$ satisfies certain conditions, converges strongly to fixed point of Moudafi [13] generalizes Halpern's theorems in the direction of viscosity approximations. In [14,15], Zegeye and Shahzad extended Moudafi's result to Banach spaces which more general than Hilbert spaces. For other related results, see [16-18]. Viscosity approximations are very important because they are applied to convex optimization, linear programming, monotone inclusion and elliptic differential equations. Marino and Xu [19], studied the viscosity approximation method for nonexpansive mappings and considered the following general iterative method:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \quad n \geq 0. \tag{1.5}$$

They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.5) converges strongly to the unique solution of the variational inequality

$$\left\langle \left(A - \gamma f\right) x^*, x - x^* \right\rangle \geq 0, x \in C,$$

which is the optimality condition for the minimization problem

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

For finding an element of $F(T) \cap VI(C, A)$, where T is nonexpansive and A is α-inverse strongly monotone, Takahashi and Toyoda [20] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)TP_C(x_n - \lambda_n Ax_n), \quad n \geq 0. \tag{1.6}$$

where $x_0 \in C$, $\{\alpha_n\}$ is a sequence in (0,1), and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$, and obtained weak convergence theorem in a Hilbert space H. Iiduka and Takahashi [7] proposed a new iterative scheme $x_1 = x \in C$ and

$$x_{x+1} = \alpha_n x + (1 - \alpha_n)TP_C(x_n - \lambda_n Ax_n), \quad n \geq 0, \tag{1.7}$$

and obtained strong convergence theorem in a Hilbert space.

Motivated and inspired by the work mentioned above which combined from Equations (1.5) and (1.6), in this article, we introduced a new iterative scheme (3.1) below which converges strongly to common element of the set of fixed points of continuous pseudo-contractive mappings which more general than nonexpansive mappings and the solution set of the variational inequality problem of continuous monotone mappings which more general than α-inverse strongly monotone mappings. As a consequence, we provide an iterative scheme which converges strongly to a common element of set of fixed points of finite family continuous pseudo-contractive mappings and the solutions set of finite family of variational inequality problems for continuous monotone mappings. Our theorems extend and unify most the results that have been proved for these important class of nonlinear operators.

2. Preliminaries

Let H be a nonempty closed and convex subset of a real Hilbert space H. Let A be a mapping from C into H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \left\|x - y\right\|, \forall y \in C.$$

PC is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping of H onto C.

Lemma 2.1. *Let H be a real Hilbert space. The following identity holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in H.$$

Lemma 2.2. *Let C be a closed convex subset of a Hilbert space H. Let $x \in H$ and $x_0 \in C$. Then $x_0 = P_C x$ if and only if*

$$\langle z - x_0, x_0 - x\rangle, \ \forall z \in C.$$

Lemma 2.3. [21] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation*

$$a_{n+1} \leq (1-\gamma_n)a_n + \sigma_n, \quad n \geq 0,$$

where,

(i) $\{\gamma_n\} \subset (0,1), \sum_{n=1}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \frac{\sigma_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\sigma_n| < \infty$.

Then, the sequence $\{a_n\} \to 0$ as $n \to \infty$.

Lemma 2.4. [22] *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $A : C \to H$ be a continuous monotone mapping. Then, for $r > 0$ and $x \in H$, there exist $z \in C$ such that*

$$\langle y - z, Az\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C. \tag{2.1}$$

Moreover, by a similar argument of the proof of Lemmas 2.8 and 2.9 in [23], *Zegeye* [22] *obtained the following lemmas:*

Lemma 2.5. [22] *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $A : C \to H$ be a continuous monotone mapping. For $r > 0$ and $x \in H$, define a mapping $F_r : H \to C$ as follows:*

$$F_r x := \left\{ z \in C : \langle y - z, Az\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C \right\}$$

for all $x \in H$. Then the following hold:

(1) *F_r is single-valued;*
(2) *F_r is a firmly nonexpansive type mapping, i.e., for all $x, y \in H$,*

$$\|F_r x - F_r y\|^2 \leq \langle F_r x - F_r y, x - y\rangle;$$

(3) *$F(F_r) = VI(C,A)$;*
(4) *$VI(C, A)$ is closed and convex.*

In the sequel, we shall make use of the following lemmas:

Lemma 2.6. [22] *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $T : C \to H$ be a continuous pseudo-contractive mapping. Then, for $r > 0$ and $x \in H$, there exist $z \in C$ such that*

$$\langle y - z, Tz \rangle - \frac{1}{r} \langle y - z, (1 + r)z - x \rangle \leq 0, \quad \forall y \in C. \tag{2.2}$$

Lemma 2.7.[22]*Let* C *be a nonempty closed and convex subset of a real Hilbert space* H. *Let* $T : C \to C$ *be a continuous pseudo-contractive mapping. For* $r > 0$ *and* $x \in H$, *define a mapping* $T_r : H \to C$ *as follows:*

$$T_r x := \left\{ z \in C : \langle y - z, Tz \rangle + \frac{1}{r} \langle y - z, (1 + r)z - x \rangle \leq 0, \quad \forall y \in C \right\}$$

for all $x \in H$. *Then the following hold:*

(1) T_r *is single - valued;*
(2) T_r *is a firmly nonexpansive type mapping, i.e., for all* $x, y \in H$,

$$\| T_r x - T_r y \|^2 \leq \langle T_r x - T_r y, x - y \rangle ;$$

(3) $F(T_r) = F(T)$;
(4) *F(T) is closed and convex.*

Lemma 2.8.[19]*Assume A is a strongly positive linear bounded operator on a Hilbert space H with coefficient* $\bar{\gamma} > 0$*and* $0 < \rho \leq \|A\|^{-1}$. *Then* $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.

Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a continuous pseudo-contractive mapping and $A : C \to H$ be a continuous monotone mapping. Then in what follows, T_{r_n} and F_{r_n} will be defined as follows: For $x \in H$ and $\{r_n\} \subset (0, \infty)$, defined

$$T_{r_n} x := \left\{ z \in C : \langle y - z, Tz \rangle - \frac{1}{r_n} \langle y - z, (1 - r_n)z - x \rangle \leq 0, \quad \forall y \in C \right\}$$

and

$$F_{r_n} x := \left\{ z \in C : \langle y - z, Az \rangle + \frac{1}{r_n} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C \right\}.$$

3. Strong convergence theorems

In this section, we will prove a strong convergence theorem for finding a common element of the set of fixed points for a continuous pseudo-contractive mapping and the solution set of a variational inequality problem governed by continuous monotone mappings.

Theorem 3.1.*Let C be a nonempty closed and convex subset of a real Hilbert space H. Let* $T : C \to C$ *be a continuous pseudo-contractive mapping and* $A : C \to H$ *be a continuous monotone mapping such that* $\mathfrak{F} := F(T) \cap VI(C, A) \neq \emptyset$. *Let f be a contraction of H into itself with a contraction constant* β *and let* $B : H \to H$ *be a strongly positive linear bounded self-adjoint operator with coefficients* $\bar{\beta} > 0$*and let* $\{x_n\}$ *be a sequence generated by* $x_1 \in C$ *and*

$$\begin{cases} y_n = F_{r_n} x_n \\ x_{n+1} = \alpha_n \gamma f(x_n) + \delta_n x_n + [(1 - \delta_n)I - \alpha_n B] T_{r_n} y_n, \end{cases} \tag{3.1}$$

where $\{\alpha_n\} \subset [0,1]$ *and* $\{r_n\} \subset (0, \infty)$ *such that*

(C1) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\lim_{n\to\infty} \delta_n = 0, \sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$;

(C3) $\liminf_{n\to\infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then, the sequence $\{x_n\}$ *converges strongly to* $z \in \mathfrak{F}$, *which is the unique solution of the variational inequality:*

$$\langle (B - \gamma f)z, x - z\rangle \geq 0, \quad \forall x \in \mathfrak{F}. \tag{3.2}$$

Equivalently, $z = P_{\mathfrak{F}}(I - B + \gamma f)z$, *which is the optimality condition for the minimization problem*

$$\min_{x\in C} \frac{1}{2} \langle Az, z\rangle - h(z),$$

where h *is a potential function for* γf *(i.e.,*$h'(z) = \gamma f(z)$ *for* $z \in H$*).*

Remark: (1) The variational inequality (3.2) has the unique solution; (see [19]). (2) It follows from condition (C1) that $(1 - \delta_n)I - \alpha_n B$ is positive and $\|(1 - \delta_n)I - \alpha_n B\| \leq I - \delta_n - \alpha_n \bar{\beta}$ for all $n \geq 1$; (see [24]).

Proof. We processed the proof with following four steps:

Step 1. First, we will prove that the sequence $\{x_n\}$ is bounded.

Let $v \in \mathfrak{F}$ and let $u_n = T_{r_n} y_n$ and $y_n = F_{r_n} x_n$. Then, from Lemmas 2.5 and 2.7 that

$$\|u_n - v\| = \|T_{r_n} y_n - T_{r_n} v\| \leq \|y_n - v\| = \|F_{r_n} x_n - F_{r_n} v\| \leq \|x_n - v\|. \tag{3.3}$$

Moreover, from (3.1) and (3.2), we compute

$$\begin{aligned}
\|x_{n+1} - v\| &= \|\alpha_n(\gamma(f(x_n) - Bv)) + \delta_n(x_n - v) + [(1 - \delta_n)I - \alpha_n B]T_{r_n} - v\|. \\
&\leq \alpha_n \|\gamma f(x_n) - Bv)\| + \delta_n \|(x_n - v)\| + \|(1 - \delta_n)I - \alpha_n B\| \|T_{r_n} - v\|. \\
&\leq \alpha_n \beta\gamma \|x_n - v\| + \alpha_n \|\gamma f(v) - Bv\| + \delta_n \|x_n - v\| + (1 + \delta_n - \alpha_n \bar{\beta}) \|T_{r_n} y_n - v\|. \\
&\leq \alpha_n \beta\gamma \|x_n - v\| + \alpha_n \|\gamma f(v) - Bv\| + \delta_n \|x_n - v\| + (1 + \delta_n - \alpha_n \bar{\beta}) \|u_n - v\|. \\
&\leq \alpha_n \beta\gamma \|x_n - v\| + \alpha_n \|\gamma f(v) - Bv\| + \delta_n \|x_n - v\| + (1 + \delta_n - \alpha_n \bar{\beta}) \|x_n - v\|. \\
&= \alpha_n \beta\gamma \|x_n - v\| + \alpha_n \|\gamma f(v) - Bv\| + \delta_n \|x_n - v\| + \|x_n - v\| \\
&- \delta_n \|x_n - v\| - \alpha_n \bar{\beta} \|x_n - v\|. \\
&= \alpha_n \beta\gamma \|x_n - v\| + \alpha_n \|\gamma f(v) - Bv\| + \|x_n - v\| - \alpha_n \bar{\beta} \|x_n - v\|. \\
&\leq (\alpha_n \beta\gamma + 1 - \alpha_n \bar{\beta}) \|x_n - v\| + \alpha_n \|\gamma f(v) - Bv\|. \\
&= (1 - \alpha_n(\bar{\beta} - \beta\gamma)) \|x_n - v\| + \alpha_n \|\gamma f(v) - Bv\|. \\
&\leq \max\left\{\|x_n - v\|, \frac{\|\gamma f(v) - Bv\|}{\bar{\beta} - \beta\gamma},\right\} \quad \forall n \geq 1.
\end{aligned}$$

Therefore, by the simple introduction, we have

$$\|x_n - v\| = \max\left\{\|x_1 - v\|, \frac{\|\gamma f(v) - Bv\|}{\bar{\beta} - \beta\gamma}\right\}, \forall n \geq 1$$

which show that $\{x_n\}$ is bounded, so $\{y_n\}$, $\{u_n\}$, and $\{f(x_n)\}$ are bounded.

Step 2. We will show that $\|x_{n+1} - x_n\| \to 0$ and $\|u_n - y_n\| \to 0$ as $n \to \infty$.

Notice that each T_{r_n} and F_{r_n} are firmly nonexpansive. Hence, we have

$$\|u_{n+1} - u_n\| = \left\|T_{r_n}y_{n+1} - T_{r_n}y_n\right\| \leq \left\|y_{n+1} - y_n\right\| = \left\|F_{r_n}x_{n+1} - F_{r_n}x_n\right\| \leq \|x_{n+1} - x_n\|.$$

From (3.1), we note that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \big\|\alpha_n\gamma f(x_n) + \delta_n x_n + [(1-\delta_n)I - \alpha_n B]T_{r_n}y_n \\
&\quad -\alpha_{n-1}\gamma f(x_{n-1}) - \delta_{n-1}x_{n-1} - [(1-\delta_{n-1})I - \alpha_{n-1}B]T_{r_n}y_{n-1}\big\|. \\
&= \big\|\alpha_n\gamma f(x_n) + \delta_n x_n + (I - \delta_n - \alpha_n B)u_n \\
&\quad -\alpha_{n-1}\gamma f(x_{n-1}) - \delta_{n-1}x_{n-1} - (I - \delta_{n-1} - \alpha_{n-1}B)u_{n-1}\big\|. \\
&\leq \big\|\alpha_n\gamma f(x_n) - \alpha_n\gamma f(x_{n-1}) + \alpha_n\gamma f(x_{n-1}) + \delta_n x_n - \delta_{n-1}x_{n-1} - \alpha_{n-1}\gamma f(x_{n-1}) \\
&\quad + (I - \delta_n - \alpha_n B)u_n - (I - \delta_n - \alpha_n B)u_{n-1} + (I - \delta_n - \alpha_n B)u_{n-1} \\
&\quad -(I - \delta_{n-1} - \alpha_{n-1}B)u_{n-1}\big\|. \\
&\leq \big\|\alpha_n\gamma f(x_n) - \alpha_n\gamma f(x_{n-1})\big\| + \big\|\alpha_n\gamma f(x_{n-1}) - \alpha_{n-1}\gamma f(x_{n-1})\big\| + \|\delta_n x_n - \delta_{n-1}x_{n-1}\| \\
&\quad + \big\|(I - \delta_n - \alpha_n B)u_n - (I - \delta_n - \alpha_n B)u_{n-1}\big\| + \big\|(I - \delta_n - \alpha_n B)u_{n-1} \\
&\quad -(I - \delta_{n-1} - \alpha_{n-1}B)u_{n-1}\big\|. \\
&= \alpha_n\gamma\left\|f(x_n) - f(x_{n-1})\right\| + |\alpha_n - \alpha_{n-1}|\left\|\gamma f(x_{n-1})\right\| + \delta_n x_n - \delta_{n-1}x_{n-1} + \delta_n x_{n-1} - \delta_n x_{n-1} \\
&\quad + (I - \delta_n - \alpha_n B)\|u_n - u_{n-1}\| + \left\|(I - \delta_n - \alpha_n B - I + \delta_{n-1} + \alpha_{n-1}B)u_{n-1}\right\|. \\
&= \alpha_n\gamma\beta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\gamma\left\|f(x_{n-1})\right\| + \delta_n\|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}|\|x_{n-1}\| \\
&\quad + |I - \delta_n - \alpha_n B|\|u_n - u_{n-1}\| + |\delta_{n-1} - \delta_n + \alpha_{n-1}B + \alpha_n B|\|u_{n-1}\|. \\
&\leq \alpha_n\gamma\beta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\gamma\left\|f(x_{n-1})\right\| + \delta_n\|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}|\|x_{n-1}\| \\
&\quad + |I - \delta_n - \alpha_n B|\|x_n - x_{n-1}\| + |\delta_{n-1} - \delta_n|\|u_{n-1}\| + |\alpha_{n-1}B + \alpha_n B|\|u_{n-1}\|. \\
&\leq \alpha_n\gamma\beta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\gamma\left\|f(x_{n-1})\right\| + \delta_n\|x_n - x_{n-1}\| + |\delta_n - \delta_{n-1}|\|x_{n-1}\| \\
&\quad + |I - \delta_n - \alpha_n B|\|x_n - x_{n-1}\| + |\delta_{n-1} - \delta_n|\|x_{n-1}\| - |\alpha_{n-1} - \alpha_n|B\|x_{n-1}\|. \\
&\leq \alpha_n\gamma\beta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\gamma\left\|f(x_{n-1})\right\| + \delta_n\|x_n - x_{n-1}\| \\
&\quad + |I - \delta_n - \alpha_n B|\|x_n - x_{n-1}\| - |\alpha_{n-1} - \alpha_n|B\|x_{n-1}\|. \\
&\leq \alpha_n\gamma\beta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\gamma\left\|f(x_{n-1})\right\| + |\delta_n + I - \delta_n - \alpha_n B|\|x_n - x_{n-1}\| \\
&\quad - |\alpha_{n-1} - \alpha_n|B\|x_{n-1}\|. \\
&\leq \alpha_n\gamma\beta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\gamma\left\|f(x_{n-1})\right\| + |I - \alpha_n B|\|x_n - x_{n-1}\| \\
&\quad - |\alpha_n - \alpha_{n-1}|B\|x_{n-1}\|. \\
&\leq \alpha_n\gamma\beta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\left\|\gamma f(x_{n-1}) - Bx_{n-1}\right\| + |I - \alpha_n B|\|x_n - x_{n-1}\| \\
&\leq \alpha_n\gamma\beta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\left\|\gamma f(x_{n-1}) - Bx_{n-1}\right\| + |I - \alpha_n B|\left\|y_n - y_{n-1}\right\| \\
&\leq \alpha_n\gamma\beta\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|K + |I - \alpha_n B|\left\|y_n - y_{n-1}\right\|,
\end{aligned} \tag{3.4}$$

where $K = \|\gamma f(x_{n-1}) - Bx_{n-1}\| = 2\sup\{\|f(x_n)\| + \|u_n\|: n \in N\}$. Moreover, since $y_n = F_{r_n}x_n$ and $y_{n+1} = F_{r_{n+1}}x_{n+1}$, we get

$$\langle y - y_n, Ay_n\rangle + \frac{1}{r_n}\langle y - y_n, y_n - x_n\rangle \geq 0, \quad \forall y \in C \tag{3.5}$$

and

$$\langle y - y_{n+1}, Ay_{n+1}\rangle + \frac{1}{r_{n+1}}\langle y - y_{n+1}, y_{n+1} - x_{n+1}\rangle \geq 0, \quad \forall y \in C. \tag{3.6}$$

Putting $y = y_{n+1}$ in (3.5) and $y = y_n$ in (3.6), we obtain

$$\langle y_{n+1} - y_n, Ay_n\rangle + \frac{1}{r_n}\langle y_{n+1} - y_n, y_n - x_n\rangle \geq 0 \tag{3.7}$$

and

$$\langle y_n - y_{n+1}, Ay_{n+1}\rangle + \frac{1}{r_{n+1}}\langle y_n - y_{n+1}, y_{n+1} - x_{n+1}\rangle \geq 0. \tag{3.8}$$

Adding (3.7) and (3.8), we have

$$\langle y_{n+1} - y_n, Ay_n - Ay_{n+1}\rangle + \left\langle y_{n+1} - y_n, \frac{y_n - x_n}{r_n} - \frac{y_{n+1} - x_{n+1}}{r_{n+1}}\right\rangle \geq 0$$

which implies that

$$-\langle y_{n+1} - y_n, Ay_{n+1} - Ay_n\rangle + \left\langle y_{n+1} - y_n, \frac{y_n - x_n}{r_n} - \frac{y_{n+1} - x_{n+1}}{r_{n+1}}\right\rangle \geq 0.$$

Using the fact that A is monotone, we get

$$\left\langle y_{n+1} - y_n, \frac{y_n - x_n}{r_n} - \frac{y_{n+1} - x_{n+1}}{r_{n+1}}\right\rangle \geq 0.$$

and hence

$$\left\langle y_{n+1} - y_n, y_n - y_{n+1} + y_{n+1} - y_n - \frac{r_n}{r_{n+1}}(y_{n+1} - x_{n+1})\right\rangle \geq 0.$$

We observe that

$$\begin{aligned} \|y_{n+1} - y_n\|^2 &\leq \left\langle y_{n+1} - y_n, x_{n+1} - x_n(1 - \frac{r_n}{r_{n+1}})(y_{n+1} - x_{n+1})\right\rangle \\ &\leq \|y_{n+1} - y_n\| \left\{\|x_{n+1} - x_n\| + \left|(1 - \frac{r_n}{r_{n+1}})\right| \|(y_{n+1} - x_{n+1})\|\right\}. \end{aligned} \tag{3.9}$$

Without loss of generality, let k be a real number such that $r_n > k > 0$ for all $n \in N$. Then, we have

$$\begin{aligned} \|y_{n+1} - y_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n| \|y_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{1}{k}|r_{n+1} - r_n| M, \end{aligned} \tag{3.10}$$

where $M = \sup\{\|y_n - x_n\|: n \in N\}$. Furthermore, from (3.4) and (3.10), we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \alpha_n\gamma\beta \|x_n - x_{n-1}\| + \|\alpha_n - \alpha_{n-1}\| K + (1 - \alpha_n)\left(\|x_n - x_{n-1}\| + \frac{1}{k}|r_n - r_{i-1}| M\right). \\ &= (1 - \alpha_n + \alpha_n\gamma\beta)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| K + \frac{1}{k}|r_n - r_{n-1}| M. \\ &= (1 - \alpha_n(1 - \gamma\beta))\|x_n - x_{n-1}\| + K|\alpha_n - \alpha_{n-1}| + \frac{M}{k}|r_n - r_{n-1}|. \end{aligned}$$

Using Lemma 2.3, and by the conditions (C1) and (C3), we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Consequently, from (3.10), we obtain

$$\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0. \tag{3.11}$$

Since $u_n = T_{r_n}y_n$ and $u_{n+1} = T_{r_{n+1}}y_{n+1}$, we have

$$\langle y - u_n, Tu_n\rangle - \frac{1}{r_n}\langle y - u_n, (1 - r_n)u_n - y_n\rangle \leq 0, \quad \forall y \in C \tag{3.12}$$

and

$$\langle y-u_{n+1},Tu_{n+1}\rangle-\frac{1}{r_{n+1}}\langle y-u_{n+1},(1-r_{n+1})u_{n+1}-y_{n+1}\rangle\leq 0,\quad \forall y\in C. \tag{3.13}$$

Putting $y := u_{n+1}$ in (3.12) and $y := u_n$ in (3.13), we get

$$\langle u_{n+1}-u_n,Tu_n\rangle-\frac{1}{r_n}\langle u_{n+1}-u_n,(1-r_n)u_n-y_n\rangle\leq 0. \tag{3.14}$$

and

$$\langle u_n-u_{n+1},Tu_{n+1}\rangle-\frac{1}{r_{n+1}}\langle u_n-u_{n+1},(1-r_{n+1})u_{n+1}-y_{n+1}\rangle\leq 0. \tag{3.15}$$

Adding (3.14) and (3.15), we have

$$\langle u_{n+1}-u_n,Tu_n-Tu_{n+1}\rangle-\left\langle u_{n+1}-u_n,\frac{(1-r_n)u_n-y_n}{r_n}-\frac{(1-r_{n+1})u_{n+1}-y_{n+1}}{r_{n+1}}\right\rangle\leq 0.$$

Using the fact that T is pseudo-contractive, we get

$$\left\langle u_{n+1}-u_n,\frac{u_n-y_n}{r_n}-\frac{u_{n+1}-y_{n+1}}{r_{n+1}}\right\rangle\geq 0$$

and hence

$$\left\langle u_{n+1}-u_n,u_n-u_{n+1}+u_{n+1}-y_n-\frac{r_n}{r_{n+1}}(u_{n+1}+y_{n+1})\right\rangle\geq 0.$$

Thus, using the methods in (3.9) and (3.10), we can obtain

$$\|u_{n+1}-u_n\|\leq\|y_{n+1}-y_n\|+\frac{1}{r_{n+1}}|r_{n+1}+r_n|M_1, \tag{3.16}$$

where $M_1 = \sup\{\|u_n - y_n\|: n \in N\}$. Therefore, from (3.11) and property of $\{r_n\}$, we get

$$\lim_{n\to\infty}\|u_{n+1}-u_n\|=0.$$

Furthermore, since $x_n=\alpha_{n-1}\gamma f(x_{n+1})+\delta_{n-1}x_{n-1}+[(1-\delta_{n-1})I-\alpha_{n-1}B]T_{r_n}y_{n-1}$, we have

$$\begin{aligned}\|x_n-u_n\| &\leq \|x_n-u_{n-1}\|+\|u_{n-1}-u_n\|\\ &= \|\alpha_{n-1}\gamma f(x_{n-1})+\delta_{n-1}x_{n-1}+[(1-\delta_{n-1})I-\alpha_{n-1}B]T_{r_n}y_{n-1}-u_{n-1}\|+\|u_{n-1}-u_n\|\\ &= \alpha_{n-1}\gamma f(x_{n-1})+\delta_{n-1}x_{n-1}+(I-\delta_{n-1}-\alpha_{n-1}B)u_{n-1}-u_{n-1}\|+\|u_{n-1}-u_n\|\\ &= \alpha_{n-1}\gamma f(x_{n-1})+\delta_{n-1}x_{n-1}+u_{n-1}-\delta_{n-1}u_{n-1}-\alpha_{n-1}Bu_{n-1}-u_{n-1}\|+\|u_{n-1}-u_n\|\\ &\leq \alpha_{n-1}\gamma f(x_{n-1})-\alpha_{n-1}Bu_{n-1}+\delta_{n-1}x_{n-1}-\delta_{n-1}u_{n-1}\|+\|u_{n-1}-u_n\|\\ &\leq \alpha_{n-1}\|\gamma f(x_{n-1})-Bu_{n-1}\|+\delta_{n-1}\|x_{n-1}-u_{n-1}\|+\|u_{n-1}-u_n\|.\end{aligned}$$

Thus, by (C1) and (C2), we obtain

$$\|x_n-u_n\|\to 0, n\to\infty. \tag{3.17}$$

For $v \in \mathfrak{F}$, using Lemma 2.5, we obtain

$$\begin{aligned}\left\|y_n - v\right\|^2 &= \left\|F_{r_n}y_n - F_{r_n}v\right\|^2\\ &\le \left\langle F_{r_n}x_n - F_{r_n}v, x_n - v\right\rangle\\ &\le \left\langle y_n - v, x_n - v\right\rangle\\ &= \frac{1}{2}\left(\left\|y_n - v\right\|^2 + \|x_n - v\|^2 - \left\|x_n - y_n\right\|^2\right)\end{aligned}$$

and

$$\left\|y_n - v\right\|^2 \le \|x_n - v\|^2 - \left\|x_n - y_n\right\|^2. \tag{3.18}$$

Therefore, from (3.1), the convexity of $\|\cdot\|^2$, (3.2) and (3.18), we get

$$\begin{aligned}\|x_{n+1} - v\|^2 &= \left\|\alpha_n\gamma f(x_n) + \delta_n x_n + [(1-\delta_n)I - \alpha_n B]T_{r_n}y_n - v\right\|^2\\ &= \left\|(1-\delta_n)(T_{r_n}y_n - v) + \delta_n(x_n - v) + \alpha_n(\gamma f(x_n) - BT_{r_n}y_n)\right\|^2\\ &\le \left\|(1-\delta_n)(T_{r_n}y_n - v) + \delta_n(x_n - v)\right\|^2 + 2\alpha_n\left\langle\gamma f(x_n) - BT_{r_n}y_n, x_{n+1} - v\right\rangle\\ &\le (1-\delta_n)\left\|(y_n - v)\right\|^2 + \delta_n\left\|(x_n - v)\right\|^2 + 2\alpha_n L^2\end{aligned}$$

and hence

$$(1-\delta_n)\left\|(y_n - v)\right\|^2 \le \delta_n\left\|(x_n - v)\right\|^2 - \left\|(x_{n+1} - v)\right\|^2 + 2\alpha_n L^2. \tag{3.19}$$

So, we have $\|y_n - v\| \to 0$ as $n \to \infty$. Consequently, from (3.16) and (3.18), we obtain

$$\left\|u_n - y_n\right\| \le \|u_n - x_n\| + \left\|x_n - y_n\right\| \to 0 \text{ as } n \to \infty.$$

Step 3. We will show that

$$\limsup_{n\to\infty}\left\langle(\gamma f - B)z, x_n - z\right\rangle \le 0. \tag{3.20}$$

Let $Q = P_{\mathfrak{F}}$, and since, $Q(I - B + \gamma f)$ is contraction on H into C (see also [[25], pp. 18]) and H is complete. Thus, by Banach Contraction Principle, then there exist a unique element z of H such that $z = Q(I - B + \gamma f)z$.

We choose subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\left\langle(\gamma f - B)z, x_n - z\right\rangle = \lim_{n\to\infty}\left\langle\gamma fz - Bz, x_{n_i} - z\right\rangle$$

Since $\{x_{n_i}\}$ is bounded, there exists a sequence $\{x_{n_{ij}}\}$ of $\{x_{n_i}\}$ and $y \in C$ such that $\{x_{n_{ij}}\} \rightharpoonup y$. Without loss of generality, we may assume that $x_{n_i} \rightharpoonup y$. Since C is closed and convex it is weakly closed and hence $y \in C$. Since $x_n - y_n \to 0$ as $n \to \infty$ we have that $y_{n_i} \rightharpoonup y$. Now, we show that $y \in \mathfrak{F}$. Since $y_n = F_{r_n}$, Lemma 2.5 and using (3.5), we get

$$\left\langle y - y_n, Ay_n\right\rangle + \left\langle y - y_n, \frac{y_n - x_n}{r_n}\right\rangle \ge 0, \quad \forall y \in C. \tag{3.21}$$

and

$$\left\langle y - y_{n_i}, Ay_{n_i}\right\rangle + \left\langle y - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \ge 0, \quad \forall y \in C. \tag{3.22}$$

Set $v_t = tv + (1- t)y$ for all $t \in (0,1]$ and $v \in C$. Consequently, we get $v_i \in C$. From (3.22), it follows that

$$\begin{aligned}\left\langle v_t - y_{n_i}\right\rangle &\geq \left\langle v_t - y_{n_i}, Av_t\right\rangle - \left\langle v_t - y_{n_i}, Av_t\right\rangle - \left\langle v_t - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{r_n}\right\rangle \\ &= \left\langle v_t - y_{n_i}, Av_t - Ay_{n_i}\right\rangle - \left\langle v_t - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{r_n}\right\rangle,\end{aligned}$$

from the fact that $y_{n_i} - x_{n_i} \to 0$ as $i \to \infty$, we obtain that $\frac{u_{n_i} - x_{n_i}}{r_n} \to 0$ as $i \to \infty$. Since A is monotone, we also have that $\left\langle v_t - y_{n_i}, Av_t - Ay_{n_i}\right\rangle \geq 0$. Thus, if follows that

$$0 \leq \lim_{i\to\infty} \left\langle v_t - y_{n_i}, Av_t\right\rangle = \langle v_t - w, Av_t\rangle ,$$

and hence $\left\langle v - y, Av_t\right\rangle \geq 0, \quad \forall v \in C.$

If $t \to 0$, the continuity of A gives that

$$\left\langle v - y, Ay\right\rangle \geq 0, \quad \forall v \in C.$$

This implies that $y \in VI(C, A)$.

Furthermore, since $u_n = T_{r_n} y_n$, Lemma 2.5 and using (3.12), we get

$$\left\langle y - u_{n_i}, Tu_{n_i}\right\rangle - \frac{1}{r_n}\left\langle y - u_{n_i}, (r_{n_i} + 1)u_{n_i} - y_{n+1}\right\rangle \leq 0, \quad \forall y \in C. \tag{3.23}$$

Put $z_t = t(v) + (1 - t)y$ for all $t \in (0,1]$ and $v \in C$. Then, $z_t \in C$ and from (3.23) and pseudo-contractivity of T, we get

$$\begin{aligned}\left\| u_{n_i} - z_t, Tz_t\right\| &= \left\langle u_{n_i} - z_t, Tz_t\right\rangle + \left\langle z_t - u_{n_i}, Tu_i\right\rangle - \frac{1}{r_n}\left\langle z_t - u_{n_i}, (1 + r_{n_i})u_{n_i} - y_{n_i}\right\rangle \\ &= -\left\langle z_t - u_{n_i}, Tz_t\right\rangle - \frac{1}{r_{n_i}}\left\langle z_t - u_{n_i}, u_{n_i} - y_{n_i}\right\rangle - \left\langle z_t - u_{n_i}, u_{n_i}\right\rangle \\ &\geq \left\| z_t - u_{n_i}\right\|^2 - \frac{1}{r_{n_i}}\left\langle z_t - u_{n_i}, u_{n_i} - y_{n_i}\right\rangle - \left\langle z_t - u_{n_i}, u_{n_i}\right\rangle \\ &= -\left\langle z_t - u_{n_i}, z_t\right\rangle - \left\langle z_t - u_{n_i}, \frac{u_{n_i} - y_{n_i}}{r_{n_i}}\right\rangle.\end{aligned}$$

Thus, since $u_n - y_n \to 0$, as $n \to \infty$ we obtain that $\frac{u_{n_i} - y_{n_i}}{r_{n_i}} \to 0$ as $i \to \infty$. Therefore, as $i \to \infty$, it follows that

$$\left\langle y - z_t, Tz_t\right\rangle \geq \left\langle y - z_t, z_t\right\rangle$$

and hence

$$-\left\langle v - y, Tz_t\right\rangle \geq -\left\langle v - y, z_t\right\rangle, \quad \forall v \in C.$$

Taking $t \to 0$ and since T is continuous we obtain

$$-\left\langle v - y, Ty\right\rangle \geq -\left\langle v - y, y\right\rangle, \quad \forall v \in C.$$

Now, we get $v = Ty$. Then we obtain that $y = Ty$ and hence $y \in F(T)$. Therefore, $y \in F(T) \cap VI(C, A)$ and since $z = P_{\mathfrak{F}}(I - B + \gamma f)z$, Lemma 2.2 implies that

$$\begin{aligned}\limsup_{n\to\infty}\left\langle(\gamma f-B)z,x_n-z\right\rangle &= \lim_{i\to\infty}\left\langle(I-B+\gamma f)z-z,x_{n_i}-z\right\rangle \\ &= \left\langle(\gamma f-B)z,y-z\right\rangle \le 0.\end{aligned} \tag{3.24}$$

Step 4. Finally, we will show that $x_n \to z$ as $n \to \infty$, where $z = P_{\mathfrak{F}}(I - B + \gamma f)z$. From (3.1) and (3.2) we observe that

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \left\langle\alpha_n\gamma f(x_n)+\delta_n x_n+[(1-\delta_n)I-\alpha_n B]T_{r_n}y_n-z,x_{n+1}-z\right\rangle \\ &= \alpha_n\left\langle\gamma f(x_n)-Bz,x_{n+1}-z\right\rangle+\delta_n\left\langle x_n-z,x_{n+1}-z\right\rangle \\ &\quad+\left\langle[(1-\delta_n)I-\alpha_n B](T_{r_n}-z),x_{n+1}-z\right\rangle \\ &\le \alpha_n\gamma\left\langle f(x_n)-f(z),x_{n+1}-z\right\rangle+\alpha_n\left\langle\gamma f(z)-Bz,x_{n+1}-z\right\rangle \\ &\quad+\delta_n\|x_n-z\|\,\|x_{n+1}-z\|+(1-\delta_n-\alpha_n\bar{\beta})\|z_n-z\|\,\|x_{n+1}-z\| \\ &\le \alpha_n\gamma K\|x_n-z\|\,\|x_{n+1}-z\|+\alpha_n\left\langle\gamma f(z)-Bz,x_{n+1}-z\right\rangle \\ &\quad+\delta_n\|x_n-z\|\,\|x_{n+1}-z\|+(1-\delta_n-\alpha_n\bar{\beta})\|z_n-z\|\,\|x_{n+1}-z\| \\ &= \alpha_n\gamma K\|x_n-z\|\,\|x_{n+1}-z\|+\alpha_n\left\langle\gamma f(z)-Bz,x_{n+1}-z\right\rangle \\ &\quad+(1-\alpha_n\bar{\beta})\|x_n-z\|\,\|x_{n+1}-z\| \\ &\le \frac{\gamma k}{2}\alpha_n(\|x_n-z\|^2+\|x_{n+1}-z\|^2)+\alpha_n\left\langle\gamma f(z)-Bz,x_{n+1}-z\right\rangle \\ &\quad+(1-\alpha_n\bar{\beta})(\|x_n-z\|\,\|x_{n+1}-z\|) \\ &\le \frac{\gamma k}{2}\alpha_n(\|x_n-z\|^2+\|x_{n+1}-z\|^2)+\alpha_n\left\langle\gamma f(z)-Bz,x_{n+1}-z\right\rangle \\ &\quad+\frac{(1-\alpha_n\bar{\beta})}{2}(\|x_n-z\|^2+\|x_{n+1}-z\|^2) \\ &\le \frac{1-\alpha_n(\bar{\beta}-k\gamma)}{2}\|x_n-z\|^2+\frac{1}{2}\|x_{n+1}-z\|^2+\alpha_n\left\langle\gamma f(x)-Bz,x_{n+1}-z\right\rangle,\end{aligned}$$

which implies that

$$\|x_{n+1}+z\|^2 \le [1-\alpha_n(\bar{\beta}-k\gamma)]\|x_n-z\|^2+2\alpha_n\left\langle\gamma f(z)-Bz,x_{n+1}-z\right\rangle.$$

By the condition (C1), (3.24) and using Lemma 2.3, we see that $\lim_{n\to\infty}\|x_n - z\| = 0$. This complete to proof. □

If we take $f(x) = u$, $\forall x \in H$ and $\gamma = 1$, then by Theorem 3.1, we have the following corollary:

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a continuous pseudo-contractive mapping and $A : C \to H$ be a continuous monotone mapping such that $\mathfrak{F} := F(T) \cap VI(C, A) \neq \emptyset$. let $B : H \to H$ be a strongly positive linear bounded self-adjoint operator with coefficients $\bar{\beta} > 0$and let $\{x_n\}$ be a sequence generated by $x_1 \in H$ and*

$$\begin{cases} y_n = F_{r_n}x_n \\ x_{n+1} = \alpha_n u+\delta_n x_n+[(1-\delta_n)I-\alpha_n B]T_{r_n}y_n, \end{cases} \tag{3.25}$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0, \infty)$ such that

(C1) $\lim_{n\to\infty}\alpha_n = 0, \sum_{n=1}^{\infty}\alpha_n = \infty$;

(C2) $\lim_{n\to\infty}\delta_n = 0, \sum_{n=1}^{\infty}|\delta_{n+1}-\delta_n| < \infty$;

(C3) $\liminf_{n\to\infty} r_n > 0, \sum_{n=1}^{\infty}|r_{n+1}-r_n| < \infty$.

Then, the sequence $\{x_n\}$ *converges strongly to* $z \in \mathfrak{F}$, *which is the unique solution of the variational inequality:*

$$\langle (B - f)z, x - z \rangle \geq 0, \forall x \in \mathfrak{F}. \tag{3.26}$$

Equivalently, $z = P_{\mathfrak{F}}(I - B + f)z$.

If we take $T \equiv 0$, then $T_{r_n} \equiv I$ (the identity map on C). So by Theorem 3.1, we obtain the following corollary.

Corollary 3.3. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $A : C \to H$ *be a continuous monotone mapping such that* $\mathfrak{F} := VI(C, A) \neq \emptyset$. *Let* f *be a contraction of* H *into itself and let* $B : H \to H$ *be a strongly positive linear bounded self-adjoint operator with coefficients* $\bar{\beta} > 0$ *and let* $\{x_n\}$ *be a sequence generated by* $x_1 \in H$ *and*

$$x_{n+1} = \alpha_n \gamma f(x_n) + \delta_n x_n + [(1 - \delta_n)I - \alpha_n B]F_{r_n} x_n, \tag{3.27}$$

where $\{\alpha_n\} \subset [0,1]$ *and* $\{r_n\} \subset (0, \infty)$ *such that*

(C1) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\lim_{n\to\infty} \delta_n = 0, \sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$;

(C3) $\liminf_{n\to\infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then, the sequence $\{x_n\}$ *converges strongly to* $z \in \mathfrak{F}$, *which is the unique solution of the variational inequality:*

$$\langle (B - \gamma f)z, x - z \rangle \geq 0, \forall x \in \mathfrak{F} \tag{3.28}$$

Equivalently, $z = P_{\mathfrak{F}}(I - B + \gamma f)z$.

If we take $A \equiv 0$, then $F_{r_n} \equiv I$ (the identity map on C). So by Theorem 3.1, we obtain the following corollary.

Corollary 3.4. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $T : C \to C$ *be a continuous pseudo-contractive mapping such that* $\mathfrak{F} := F(T) \neq \emptyset$. *Let* f *be a contraction of* H *into itself and let* $B : H \to H$ *be a strongly positive linear bounded self-adjoint operator with coefficients* $\bar{\beta} > 0$ *and let* $\{x_n\}$ *be a sequence generated by* $x_1 \in H$ *and*

$$x_{n+1} = \alpha_n \gamma f(x_n) + \delta_n x_n + [(1 - \delta_n)I - \alpha_n B]T_{r_n} x_n, \tag{3.29}$$

where $\{\alpha_n\} \subset [0,1]$ *and* $\{r_n\} \subset (0, \infty)$ *such that*

(C1) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\lim_{n\to\infty} \delta_n = 0, \sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$;

(C3) $\liminf_{n\to\infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then, the sequence $\{x_n\}_{n\geq 1}$ converges strongly to $z \in \mathfrak{F}$, which is the unique solution of the variational inequality:

$$\langle (B - \gamma f)z, x - z\rangle \geq 0, \forall x \in \mathfrak{F}. \tag{3.30}$$

Equivalently, $z = P_{\mathfrak{F}}(I - B + \gamma f)z$.

If we take $C \equiv H$ in Theorem 3.1, then we obtain the following corollary.

Corollary 3.5. *Let H be a real Hilbert space. Let $T_n : H \to H$ be a continuous pseudo-contractive mapping and $A : H \to H$ be a continuous monotone mapping such that $\mathfrak{F} := F(T) \cap A^{-1}(0) \neq \emptyset$. Let f be a contraction of C into itself and let $B : H \to H$ be a strongly positive linear bounded self-adjoint operator with coefficients $\bar{\beta} > 0$and let $\{x_n\}$ be a sequence generated by $x_1 \in H$ and*

$$\begin{cases} y_n = F_{r_n}x_n \\ x_{n+1} = \alpha_n\gamma f(x_n) + \delta_n x_n + [(1-\delta_n)I - \alpha_n B]T_{r_n}y_n \end{cases} \tag{3.31}$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0, \infty)$ such that

(C1) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\lim_{n\to\infty} \delta_n = 0, \sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$;

(C3) $\liminf_{n\to\infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to $z \in \mathfrak{F}$, which is the unique solution of the variational inequality:

$$\langle (B - \gamma f)z, x - z\rangle \geq 0, \forall x \in \mathfrak{F} \tag{3.32}$$

Equivalently, $z = P_{\mathfrak{F}}(I - B + \gamma f)z$.

Proof. Since $D(A) = H$, we note that $VI(H, A) = A^{-1}(0)$. So, by Theorem 3.1, we obtain the desired result. □

Remark 3.6. Our results extend and unify most of the results that have been proved for these important classes of nonlinear operators. In particular, Theorem 3.1 extends Theorem 3.1 of Iiduka and Takahashi [7] and Zegeye et al. [26], Corollary 3.2 of Su et al. [27] in the sense that our convergence is for the more general class of continuous pseudo-contractive and continuous monotone mappings. Corollary 3.4 also extends Theorem 4.2 of Iiduka and Takahashi [7] in the sense that our convergence is for the more general class of continuous pseudo-contractive and continuous monotone mappings.

Acknowledgements

This study was supported by the Higher Education Research Promotion and National Research University Project of Thailand, Office of the Higher Education Commission (NRU-CSEC No. 54000267).

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

Received: 26 October 2011 Accepted: 24 April 2012 Published: 24 April 2012

References

1. Borwein, FE: Nonlinear monotone operators and convex sets in Banach spaces. Bull Am Math Soc. **71**, 780–785 (1965). doi:10.1090/S0002-9904-1965-11391-X
2. Zegeye, H, Shahzad, N: Strong convergence for monotone mappings and relatively weak nonexpansive mappings. Nonlinear Anal. **70**, 2707–2716 (2009). doi:10.1016/j.na.2008.03.058
3. Bruck, RE: On the weak convergence of an ergodic iteration for the solution of variational inequalities for monotone operators in Hilbert space. J Math Anal Appl. **61**, 159–164 (1977). doi:10.1016/0022-247X(77)90152-4
4. Takahashi, W: Nonlinear complementarity problem and systems of convex inequalities. J Optim Theory Appl. **24**, 493–508 (1978)
5. Noor, MA, Noor, KI, Al-Said, E: Iterative medthods for solving general quasi-varational inequalities. Optim Lett. **4**, 513–530 (2010). doi:10.1007/s11590-010-0180-3
6. Borwein, JM: Fifty years of maximal monotonicity. Optim Lett. **4**, 473–490 (2010). doi:10.1007/s11590-010-0178-x
7. Iiduka, H, Takahashi, W, Toyoda, M: Approximation of solutions of variational inequalities for monotone mappings. Pan Am Math J. **14**, 49–61 (2004)
8. Browder, FE, Petryshyn, WV: Construction of fixed points of nonlinear mappings in Hilbert spaces. J Math Anal Appl. **20**, 197–228 (1967). doi:10.1016/0022-247X(67)90085-6
9. Liu, F, Nashed, MZ: Regularization of nonlinear ill-posed variational inequalities and convergence rates. Set-Valued Anal. **6**, 313–344 (1998). doi:10.1023/A:1008643727926
10. Nakajo, K, Takahashi, W: Strong and weak convergence theorems by an improved spliting method. Commun Appl Nonlinear Anal. **9**, 99–107 (2002)
11. Halpern, B: Fixed points of nonexpanding maps. Bull Am Math Soc. **73**, 957–961 (1967). doi:10.1090/S0002-9904-1967-11864-0
12. Lions, PL: Approximation de points fixes de contractions. C R Acad Sci Ser A - B Paris. **284**, 1357–1359 (1977)
13. Moudafi, MA: Viscosity approximation methods for fixed point problems. J Math Anal Appl. **241**, 46–55 (2000). doi:10.1006/jmaa.1999.6615
14. Zegeye, H, Shahzad, N: Approximation methods for common fixed point of finite family of nonexpansive mappings. Numer Funct Anal. **28**(11-12), 1405–1419 (2007). doi:10.1080/01630560701749730
15. Zegeye, H, Shahzad, N: Viscosity approximation methods for a common fixed point of finite family of nonexpansive mappings. Appl Math Comput. **191**, 155–163 (2007). doi:10.1016/j.amc.2007.02.072
16. Nilsrakoo, W, Saejung, S: Equilibruim problems and Moudafi's viscosity approximaion methods in Hilbert spaces. Dyn Contin Discrete Impuls Syst Ser A Math Anal. **17**, 195–213 (2010)
17. Petrusel, A, Yao, JC: Viscosity approximation to common fixed points of families of nonexpansive mappings with generalized contractions mappings. Nonlinear Anal. **69**, 1100–1111 (2008). doi:10.1016/j.na.2007.06.016
18. Yao, Y, Yao, JC: On modified iterative method for nonexpansive mappings and monotone mappings. Appl Math Comput. **186**, 1551–1558 (2007). doi:10.1016/j.amc.2006.08.062
19. Marino, G, Xu, HK: A general iterative method for nonexpansive mapping in Hilbert space. J Math Anal Appl. **318**, 43–52 (2006). doi:10.1016/j.jmaa.2005.05.028
20. Takahashi, W, Toyoda, M: Weak convergence theorems for nonexpansive mappings and monotone mappings. J Optim Theory Appl. **118**, 417–428 (2003). doi:10.1023/A:1025407607560
21. Xu, HK: Iterative algorithms for nonlinearoperator. J Lond Math Soc. **66**, 240–256 (2002). doi:10.1112/S0024610702003332
22. Zegeye, H: An iterative approximation methods for a common fixed point of two pseudo-contractive mappings. ISRN Math Anal **2011**, 14 (2011). Article ID 621901, doi:10.5402/2011/621901
23. Takahashi, W, Zembayashi, K: Strong and weak convergence theorems for equilibriums problems and relatively nonexpansive mappings in Banach spaces. Nonlinear Anal. **70**, 45–57 (2009). doi:10.1016/j.na.2007.11.031
24. Qin, XL, Cho, SY, Kang, SM: Iterative algorithm for variational inequality and equilibrium problems with applications. J Glob Optim. **48**, 423–445 (2010). doi:10.1007/s10898-009-9498-8
25. Katchang, P, Kumam, P: A General Iterative Method of Fixed Points for Mixed Equilibrium Problems and Variational Inclusion Problems. Journal of Inequalities and Applications **2010**, 25 (2010). Article ID 370197, doi:10.1155/2010/370197
26. Zegeye, H, Shahzad, N: Strong convergence of an iterative method for pseudo-contractive and monotone mappings. J Glob Optim (2011). doi:10.1007/s10898-011-9755-5
27. Su, Y, Shang, M, Qin, X: An iterative method of solution for equilibrium and optimization problems. Nonlinear Anal. **69**, 2709–2719 (2008). doi:10.1016/j.na.2007.08.045

doi:10.1186/1687-1812-2012-67
Cite this article as: Chamnarnpan and Kumam: **A new iterative method for a common solution of fixed points for pseudo-contractive mappings and variational inequalities.** *Fixed Point Theory and Applications* 2012 **2012**:67.