Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Coupled coincidence and common fixed point theorems for hybrid pair of mappings

Mujahid Abbas[1*], Ljubomir Ćirić[2], Bosko Damjanović[3] and Muhammad Ali Khan[1]

* Correspondence: mujahid@lums.edu.pk
[1]Department of Mathematics, Lahore University of Management Sciences, Lahore - 54792, Pakistan
Full list of author information is available at the end of the article

Abstract

Bhaskar and Lakshimkantham proved the existence of coupled fixed point for a single valued mapping under weak contractive conditions and as an application they proved the existence of a unique solution of a boundary value problem associated with a first order ordinary differential equation. Recently, Lakshmikantham and Ćirić obtained a coupled coincidence and coupled common fixed point of two single valued maps. In this article, we extend these concepts to multi-valued mappings and obtain coupled coincidence points and common coupled fixed point theorems involving hybrid pair of single valued and multi-valued maps satisfying generalized contractive conditions in the frame work of a complete metric space. Two examples are presented to support our results.

2000 Mathematics Subject Classification: 47H10; 47H04; 47H07.

Keywords: coupled common fixed point, coupled coincidence point, coupled point of coincidence, *w*-compatible mappings, *F*-weakly commuting mappings

1 Introduction and preliminaries

Let (X, d) be a metric space. For $x \in X$ and $A \subseteq X$, we denote $d(x, A) = \inf\{d(x, A): y \in A\}$. The class of all nonempty bounded and closed subsets of X is denoted by $CB(X)$. Let H be the Hausdorff metric induced by the metric d on X, that is,

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\},$$

for every $A, B \in CB(X)$.

Lemma 1 [1]*Let $A, B \in CB(X)$, and $\alpha > 1$. Then, for every $a \in A$, there exists $b \in B$ such that $d(a, b) \leq \alpha H(A, B)$.*

Lemma 2 [2]*Let $A, B \in CB(X)$, then for any $a \in A$, $d(a, B) \leq H(A, B)$.*

Definition 3 *Let X be a nonempty set, $F : X \times X \to 2^X$ (collection of all nonempty subsets of X) and $g : X \to X$. An element $(x, y) \in X \times X$ is called (i) coupled fixed point of F if $\times \in F(x, y)$ and $y \in F(y, x)$ (ii) coupled coincidence point of a hybrid pair $\{F, g\}$ if $g(x) \in F(x, y)$ and $g(y) \in F(y, x)$ (iii) coupled common fixed point of a hybrid pair $\{F, g\}$ if $\times = g(x) \in F(x, y)$ and $y = g(y) \in F(y, x)$.*

We denote the set of coupled coincidence point of mappings F and g by $C(F, g)$. Note that if $(x, y) \in C(F, g)$, then (y, x) is also in $C(F, g)$.

Definition 4 *Let $F : X \times X \to 2^X$ be a multi-valued mapping and g be a self map on X. The hybrid pair $\{F, g\}$ is called w- compatible if $g(F(x, y)) \subseteq F(gx, gy)$ whenever $(x, y) \in C(F, g)$.*

Definition 5 *Let $F : X \times X \to 2^X$ be a multi-valued mapping and g be a self-mapping on X. The mapping g is called F- weakly commuting at some point $(x, y) \in X \times X$ if $g^2(x) \in F(gx, gy)$ and $g^2(y) \in F(gy, gx)$.*

Bhaskar and Lakshmikantham [3] introduced the concept of coupled fixed point of a mapping F from $X \times X$ to X and established some coupled fixed point theorems in partially ordered sets. As an application, they studied the existence and uniqueness of solution for a periodic boundary value problem associated with a first order ordinary differential equation. Ćirić et al. [4] proved coupled common fixed point theorems for mappings satisfying nonlinear contractive conditions in partially ordered complete metric spaces and generalized the results given in [3]. Sabetghadam et al. [5] employed these concepts to obtain coupled fixed point in the frame work of cone metric spaces. Lakshmikantham and Ćirić [4] introduced the concepts of coupled coincidence and coupled common fixed point for mappings satisfying nonlinear contractive conditions in partially ordered complete metric spaces. The study of fixed points for multi-valued contractions mappings using the Hausdorff metric was initiated by Nadler [1] and Markin [6]. Later, an interesting and rich fixed point theory for such maps was developed which has found applications in control theory, convex optimization, differential inclusion and economics (see [7] and references therein). Klim and Wardowski [8] also obtained existence of fixed point for set-valued contractions in complete metric spaces. Dhage [9,10] established hybrid fixed point theorems and gave some applications (see also [11]). Hong in his recent study [12] proved hybrid fixed point theorems involving multi-valued operators which satisfy weakly generalized contractive conditions in ordered complete metric spaces. The study of coincidence point and common fixed points of hybrid pair of mappings in Banach spaces and metric spaces is interesting and well developed. For applications of hybrid fixed point theory we refer to [13-16]. For a survey of fixed point theory and coincidences of multimaps, their applications and related results, we refer to [16-22].

The aim of this article is to obtain coupled coincidence point and common fixed point theorems for a pair of multi-valued and single valued mappings which satisfy generalized contractive condition in complete metric spaces. It is to be noted that to find coupled coincidence points, we do not employ the condition of continuity of any mapping involved therein. Our results unify, extend, and generalize various known comparable results in the literature.

2 Main results

In the following theorem we obtain coupled coincidence and common fixed point for hybrid pair of mappings satisfying a generalized contractive condition.

Theorem 6 *Let (X, d) be a metric space, $F : X \times X \to CB(X)$ and $g : X \to X$ be mappings satisfying*

$$\begin{aligned} H(F(x,y),F(u,v)) \leq\ & a_1 d(gx,gu) + a_2 d(F(x,y),gx) + a_3 d(gy,gv) \\ & a_4 d(F(u,v),gu) + a_5 d(F(x,y),gu) + a_6 d(F(u,v),gx) \end{aligned} \tag{1}$$

for all $x, y, u, v \in X$, where $a_i = a_i(x, y, u, v)$, $i = 1, 2, \ldots, 6$, are nonnegative real numbers such that

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \leq h < 1, \tag{2}$$

where h is a fixed number. If $F(X \times X) \subseteq g(X)$ and $g(X)$ is complete subset of X, then F and g have coupled coincidence point. Moreover F and g have coupled common fixed point if one of the following conditions holds.

(a) F and g are w- compatible, $\lim_{n\to\infty} g^n x = u$ and $\lim_{n\to\infty} g^n y = v$ for some $(x, y) \in C(F, g)$, $u, v \in X$ and g is continuous at u and v.
(b) g is F- weakly commuting for some $(x, y) \in C(g, F)$, $g^2x = gx$ and $g^2y = gy$.
(c) g is continuous at x, y for some $(x, y) \in C(g, F)$ and for some $u, v \in X$, $\lim_{n\to\infty} g^n v = y$ and $\lim_{n\to\infty} g^n v = y$.
(d) $g(C(g, F))$ is singleton subset of $C(g, F)$.

Proof. *Let* $x_0, y_0 \in X$ be arbitrary. Then $F(x_0, y_0)$ and $F(y_0, x_0)$ are well defined. Choose $gx_1 \in F(x_0, y_0)$ and $gy_1 \in F(y_0, x_0)$. This can be done because $F(X \times X) \subseteq g(X)$. If $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 0$, then

$$d(gx_1, F(x_1, y_1)) \leq H(F(x_0, y_0), F(x_1, y_1)) = 0.$$

Hence $d(gx_1, F(x_1, y_1)) = 0$. Since $F(x_1, y_1)$ is closed, $gx_1 \in F(x_1, y_1)$. Similarly $gy_1 \in F(y_1, x_1)$. Thus (x_1, y_1) is a coupled coincidence point of $\{F, g\}$ and so we finish the proof. Now assume that $a_i > 0$, for some $i = 1, \ldots, 6$. Then $h > 0$ and so there exist $z_1 \in F(x_1, y_1)$ and $z_2 \in F(y_1, x_1)$ such that

$$d(gx_1, z_1) \leq H(F(x_0, y_0), F(x_1, y_1)) + \frac{h}{2},$$
$$d(gy_1, z_2) \leq H(F(y_0, x_0), F(y_1, x_1)) + \frac{h}{2}.$$

Since $F(X \times X) \subseteq g(X)$, there exist x_2 and y_2 in X such that $z_1 = gx_2$ and $z_2 = gy_2$. Thus

$$d(gx_1, gx_2) \leq H(F(x_0, y_0), F(x_1, y_1)) + \frac{h}{2},$$
$$d(gy_1, gy_2) \leq H(F(y_0, x_0), F(y_1, x_1)) + \frac{h}{2}.$$

Continuing this process, one obtains two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$gx_{n+1} \in F(x_n, y_n) \text{ and } gy_{n+1} \in F(y_n, x_n),$$

$$d(gx_n, gx_{n+1}) \leq H(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) + \frac{h^n}{2},$$
$$d(gy_n, gy_{n+1}) \leq H(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) + \frac{h^n}{2}.$$

From (1), we have

$$
\begin{aligned}
&d(gx_n, gx_{n+1}) \\
&\leq H(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) + \frac{h^n}{2} \\
&\leq a_1 d(gx_{n-1}, gx_n) + a_2 d(F(x_{n-1}, y_{n-1}), gx_{n-1}) + a_3 d(gy_{n-1}, gy_n) + a_4 d(F(x_n, y_n), gx_n) \\
&\quad + a_5 d(F(x_{n-1}, y_{n-1}), gx_n) + a_6 d(F(x_n, y_n), gx_{n-1}) + \frac{h^n}{2} \\
&\leq a_1 d(gx_{n-1}, gx_n) + a_2 d(gx_n, gx_{n-1}) + a_3 d(gy_{n-1}, gy_n) + a_4 d(gx_{n+1}, gx_n) \\
&\quad + a_6 d(gx_{n+1}, gx_{n-1}) + \frac{h^n}{2} \\
&\leq a_1 d(gx_{n-1}, gx_n) + a_2 d(gx_n, gx_{n-1}) + a_3 d(gy_{n-1}, gy_n) + a_4 d(gx_{n+1}, gx_n) \\
&\quad + a_6 d(gx_{n+1}, gx_n) + a_6 d(gx_n, gx_{n-1}) + \frac{h^n}{2} \\
&= (a_1 + a_2 + a_6) d(gx_{n-1}, gx_n) + a_3 d(gy_{n-1}, gy_n) + (a_4 + a_6) d(gx_n, gx_{n+1}) + \frac{h^n}{2},
\end{aligned}
$$

and it follows that

$$(1 - a_4 - a_6)\, d(gx_n, gx_{n+1}) \leq (a_1 + a_2 + a_6)\, d(gx_{n-1}, gx_n) + a_3 d(gy_{n-1}, gy_n) + \frac{h^n}{2}. \tag{3}$$

Similarly it can be shown that,

$$(1 - a_4 - a_6)\, d(gy_n, gy_{n+1}) \leq (a_1 + a_2 + a_6)\, d(gy_{n-1}, gy_n) + a_3 d(gx_{n-1}, gx_n) + \frac{h^n}{2}. \tag{4}$$

Again,

$$
\begin{aligned}
&d(gx_{n+1}, gx_n) \\
&= H(F(x_n, y_n), F(x_{n-1}, y_{n-1})) + \frac{h^n}{2} \\
&\quad \leq a_1 d(gx_n, gx_{n-1}) + a_2 d(F(x_n, y_n), gx_n) + a_3 d(gy_n, gy_{n-1}) + a_4 d(F(x_{n-1}, y_{n-1}), gx_{n-1}) \\
&\quad + a_5 d(F(x_n, y_n), gx_{n-1}) + a_6 d(F(x_{n-1}, y_{n-1}), gx_n) + \frac{h^n}{2} \\
&\leq a_1 d(gx_n, gx_{n-1}) + a_2 d(gx_{n+1}, gx_n) + a_3 d(gy_n, gy_{n-1}) + a_4 d(gx_n, gx_{n-1}) \\
&\quad + a_5 d(gx_{n+1}, gx_{n-1}) + \frac{h^n}{2} \\
&\leq a_1 d(gx_n, gx_{n-1}) + a_2 d(gx_{n+1}, gx_n) + a_3 d(gy_n, gy_{n-1}) + a_4 d(gx_n, gx_{n-1}) \\
&\quad + a_5 d(gx_{n+1}, gx_n) + a_5 d(gx_n, gx_{n-1}) + \frac{h^n}{2}.
\end{aligned}
$$

Hence,

$$(1 - a_2 - a_5)\, d(gx_{n+1}, gx_n) \leq (a_1 + a_4 + a_5)\, d(gx_{n-1}, gx_n) + a_3 d(gy_n, gy_{n-1}) + \frac{h^n}{2} \tag{5}$$

and

$$(1 - a_2 - a_5)\, d(gy_{n+1}, gy_n) \leq (a_1 + a_4 + a_5)\, d(gy_{n-1}, gy_n) + a_3 d(gx_n, gx_{n-1}) + \frac{h^n}{2}. \tag{6}$$

Let

$$\delta_n = d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}).$$

Now, from (3) and (4), and respectively (5) and (6), we obtain:

$$(1 - a_4 - a_6)\,\delta_n \le (a_1 + a_2 + a_3 + a_6)\,\delta_{n-1} + \frac{h^n}{2}, \tag{7}$$

$$(1 - a_2 - a_5)\,\delta_n \le (a_1 + a_3 + a_4 + a_5)\,\delta_{n-1} + \frac{h^n}{2}. \tag{8}$$

Adding (7) and (8) we get

$$(2 - a_2 - a_4 - a_5 - a_6)\,\delta_n \le (2a_1 + a_2 + 2a_3 + a_4 + a_5 + a_6)\,\delta_{n-1} + h^n. \tag{9}$$

Since by (2), $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \le h < 1$, so we have

$$\begin{aligned} 2a_1 + a_2 + 2a_3 + a_4 + a_5 + a_6 &= 2\left(a_1 + a_2 + a_3 + a_4 + a_5 + a_6\right) - a_2 - a_4 - a_5 - a_6 \\ &\le 2h - \left(a_2 + a_4 + a_5 + a_6\right) \\ &\le 2h - h\left(a_2 + a_4 + a_5 + a_6\right) \\ &= h\,(2 - a_2 - a_4 - a_5 - a_6)\,. \end{aligned}$$

Thus from (9) we get

$$(2 - a_2 - a_4 - a_5 - a_6)\,\delta_n \le h\,(2 - a_2 - a_4 - a_5 - a_6)\,\delta_{n-1} + h^n.$$

Hence, as $1/(2 - a_2 - a_4 - a_5 - a_6) < 1$,

$$\delta_n \le h\delta_{n-1} + h^n.$$

Thus we have

$$\delta_n \le h\left(h\delta_{n-2} + h^{n-1}\right) + h^n = h^2\delta_{n-2} + 2h^n.$$

Continuing this process we obtain

$$\delta_n \le h^n\delta_0 + nh^n. \tag{10}$$

By the triangle inequality and (10), for $m, n \in N$ with $m > n$, we have

$$\begin{aligned} &d(gx_n, gx_{m+n}) + d(gy_n, gy_{m+n}) \\ &\le d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + \cdots + d(gx_{n+m-1}, gx_{m+n}) \\ &\quad + d(gy_n, gy_{n+1}) + d(gy_{n+1}, gy_{n+2}) \ldots + d(gy_{n+m-1}, gy_{m+n}) \\ &\le (h^n\delta_0 + nh^n) + (h^{n+1}\delta_0 + (n+1)h^{n+1}) + \cdots + (h^{n+m-1}\delta_0 + (n+m-1)h^{n+m-1}) \\ &\quad + (h^n\delta_0 + nh^n) + (h^{n+1}\delta_0 + (n+1)h^{n+1}) + \cdots + (h^{n+m-1}\delta_0 + (n+m-1)h^{n+m-1}). \end{aligned}$$

Thus

$$d(gx_n, gx_{m+n}) + d(gy_n, gy_{m+n}) \le \sum_{i=n}^{n+m-1} \delta_0 h^i + \sum_{i=n}^{n+m-1} ih^i.$$

Since $h < 1$, we conclude that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$. Since $g(X)$ is complete, there exist $x, y \in X$ such that $gx_n \to gx$ and $gy_n \to gy$. Then from (1), we obtain

$$\begin{aligned}
&d(F(x,y),gx)\\
&\le d(F(x,y),gx_{n+1})+d(gx_{n+1},gx)\\
&\le H(F(x,y),F(x_n,y_n))+d(gx_{n+1},gx)\\
&\le a_1d(gx,gx_n)+a_2d(F(x,y),gx)+a_3d(gy,gy_n)+a_4d(F(x_n,y_n),gx_n)\\
&\quad +a_5d(F(x,y),gx_n)+a_6d(F(x_n,y_n),gx)+d(gx_{n+1},gx)\\
&\le a_1d(gx,gx_n)+a_2d(F(x,y),gx)+a_3d(gy,gy_n)+a_4d(gx_{n+1},gx_n)\\
&\quad +a_5d(F(x,y),gx_n)+a_6d(gx_{n+1},gx)+d(gx_{n+1},gx).
\end{aligned}$$

On taking limit as $n \to \infty$, we have

$$d(F(x,y),gx) \le (a_2+a_5)d(F(x,y),gx),$$

which implies that $d(F(x, y), gx) = 0$ and hence $F(x, y) = gx$. Similarly, $F(y, x) = gy$. Hence (x, y) is coupled coincidence point of the mappings F and g. Suppose now that (a) holds. Then for some $(x, y) \in C(F, g)$, $\lim_{n\to\infty} g^n x = u$ and $\lim_{n\to\infty} g^n y = v$, where $u, v \in X$. Since g is continuous at u and v, so we have that u and v are fixed points of g. As F and g are w- compatible, $g^n x \in C(F, g)$ for all $n \ge 1$ and $g^n x \in F(g^{n-1}x, g^{n-1}y)$.

Using (1), we obtain,

$$\begin{aligned}
d(gu,F(u,v)) &\le d(gu,g^nx)+d(g^nx,F(u,v))\\
&\le d(gu,g^nx)+H(F(g^{n-1}x,g^{n-1}y),F(u,v))\\
&\le d(gu,g^nx)+a_1d(g^nx,gu)+a_2d(F(g^{n-1}x,g^{n-1}y),g^nx)+a_3d(g^ny,gv)+a_4d(F(u,v),gu)\\
&\quad +a_5d(g^nx,gu)+a_6d(F(u,v),g^nx).
\end{aligned}$$

On taking limit as $n \to \infty$, we have

$$d(gu,F(u,v)) \le (a_4+a_6)d(gu,F(u,v)),$$

which implies $d(gu, F(u, v)) = 0$ and hence $gu \in F(u, v)$. Similarly, $gv \in F(v, u)$. Consequently $u = gu \in F(u, v)$ and $v = gv \in F(v, u)$. Hence (u, v) is a coupled fixed point of F and g. Suppose now that (b) holds.

If for some $(x, y) \in C(F, g)$, g is F- commuting, $g^2x = gx$ and $g^2y = gy$, then $gx = g^2x \in F(gx, gy)$ and $gy = g^2y \in F(gy, gx)$. Hence (gx, gy) is a coupled fixed point of F and g. Suppose now that (c) holds and assume that for some $(x, y) \in C(g, F)$ and for some $u, v \in X$, $\lim_{n\to\infty} g^n u = x$ and $\lim_{n\to\infty} g^n v = y$. By the continuity of g at x and y, we get $x = gx \in F(x, y)$ and $y = gy \in F(y, x)$. Hence (x, y) is coupled fixed point of F and g. Finally, suppose that (d) holds. Let $g(C(F, g)) = \{(x, x)$. Then $\{x\} = \{gx\} = F(x, x)$. Hence (x, x) is coupled fixed point of F and g. ■

Now we present following example to support our Theorem 8.

Example 9. Let $X = [1, 5]$ and $F : X \times X \to CB(X)$, $g : X \to X$ be defined as follows:

$$F(x,y) = [2,3] \text{ for all } x,y \in X,$$

$$g(x) = 5 - \frac{3}{5}x, \text{ for all } x \in X.$$

Then $H(F(x, y), F(u, v)) = 0$ for all $x, y, u, v \in X$. Therefore, F and g satisfy (1) for any $a_i \in [0, 1)$, $i = 1, 2, ..., 6$. Also $(4, 5) \in X \times X$ is a coupled coincidence point of hybrid pair $\{F, g\}$. Note that F and g do not satisfy anyone of the conditions from (a)-(d) of Theorem 8 and do not have a coupled common fixed point.

If in Theorem 8 $g = I$ (I = the identity mapping), then we have the following result.

Corollary 10. Let (X, d) be a complete metric space, $F : X \times X \to CB(X)$ be a mapping satisfying

$$\begin{aligned}&H(F(x,y),F(u,v)) \le a_1 d(x,u) + a_2 d(F(x,y),x) + a_3 d(y,v)\\&+a_4 d(F(u,v),u) + a_5 d(F(x,y),u) + a_6 d(F(u,v),x)\end{aligned}$$

for all $x, y, u, v \in X$, where $a_i = a_i(x, y, u, v)$, $i = 1, 2, ..., 6$, satisfy (2). Then F has a coupled fixed point.

Corollary 11. Let (X, d) be a metric space, $F : X \times X \to CB(X)$ and $g : X \to X$ be mappings satisfying

$$H((F(x,y),F(u,v)) \le \frac{k}{2}[d(gx,gu) + d(gy,gv)] \tag{11}$$

for all $x, y, u, v \in X$, where $k \in [0, 1)$. If $F(X \times X) \subseteq g(X)$ and $g(X)$ is complete subset of X, then F and g have a coupled coincidence point in X. Moreover, F and g have a coupled common fixed point if anyone of the conditions (a)-(d) of Theorem 8 holds.

Example 12. Let $X = [0, 1]$, $F : X \times X \to CB(X)$ and $g : X \to X$ be given as

$$F(x,y) = [0, \frac{\sin x + \sin y}{8}] \text{ for all } x, y \in X,$$

and

$$g(x) = \frac{x}{2} \text{ for all } x \in X.$$

Case (i) If $\sin x + \sin y = \sin u + \sin v$, then

$$H(F(x,y),F(u,v)) = 0.$$

Case (ii) If $\sin x + \sin y \neq \sin u + \sin v$, then

$$\begin{aligned}H(F(x,y),F(u,v)) &= \frac{1}{8}\left|(\sin x + \sin y) - (\sin u + \sin v)\right|\\&\le \frac{1}{8}\left(|\sin x - \sin u| + |\sin y - \sin v|\right)\\&\le \frac{1}{8}\left(|x-u| + |y-v|\right)\\&\le \frac{3}{16}\left(|x-u| + |y-v|\right)\\&= \frac{3}{8}\left(\left|\frac{x}{2} - \frac{u}{2}\right| + \left|\frac{y}{2} - \frac{v}{2}\right|\right)\\&= \frac{3}{8}(d(gx,gu), d(gy,gv)) = \frac{\frac{3}{4}}{2}[d(gx,gu) + d(gy,gv)].\end{aligned}$$

Therefore F and g satisfy all the conditions of Corollary 11 with $k = \frac{3}{4}$. Moreover, (0, 0) is a coupled common fixed point of F and g.

Corollary 13. Let (X, d) be a complete metric space, $F : X \times X \to CB(X)$ be a mapping satisfying

$$H((F(x,y),F(u,v)) \le \frac{k}{2}[d(x,u) + d(y,v)]$$

for all $x, y, u, v \in X$, where $k \in [0, 1)$, then F has a coupled fixed point in X.

Theorem 14. Let (X, d) be a metric space. Suppose that the mappings $F : X \times X \to CB(X)$ and $g : X \to X$ satisfy

$$H(F(x,y),F(u,v)) \le h\max\{d(gx,gu),d(gy,gv),d(F(x,y),gx),\frac{d(F(x,y),gu)+d(F(u,v),gx)}{2},d(F(u,v),gu)\} \tag{12}$$

for all $x, y, u, v \in X$, where $h \in [0, 1)$. If $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subset of X, then F and g have a coupled coincidence point in X. Moreover, F and g have a coupled common fixed point if one of the conditions (a)-(d) of Theorem 8 holds.

Proof. Let x_0 and y_0 be two arbitrary points in X. Choose $gx_1 \in F(x_0, y_0)$ and $gy_1 \in F(y_0, x_0)$. This can be done because $F(X \times X) \subseteq g(X)$. If $h = 0$, then

$$d(gx_1,F(x_1,y_1)) \le H(F(x_0,y_0),F(x_1,y_1)) = 0$$

gives that $d(gx_1, F(x_1, y_1)) = 0$, and $gx_1 \in F(x_1, y_1)$. Similarly $gy_1 \in F(y_1, x_1)$. Hence (x_1, y_1) is a coupled coincidence point of $\{F, g\}$. Now assume that $h > 0$. Set $k = \frac{1}{\sqrt{h}}$. Then $k > 1$ and so there exists $z_1 \in F(x_1, y_1)$ and $z_2 \in F(y_1, x_1)$ such that $gx_2 \in F(x_1, y_1)$, $gy_2 \in F(y_1, x_1)$ and such that

$$\begin{aligned}
&d(gx_1,z_1) \le kH(F(x_0,y_0),F(x_1,y_1)),\\
&d(gy_1,z_2) \le kH(F(y_0,x_0),F(y_1,x_1)).
\end{aligned}$$

Since $F(X \times X) \subseteq g(X)$, there exist x_2 and y_2 in X such that $z_1 = gx_2$ and $z_2 = gy_2$. Also,

$$\begin{aligned}
&d(gx_1,gx_2) \le kH(F(x_0,y_0),F(x_1,y_1)),\\
&d(gy_1,gy_2) \le kH(F(y_0,x_0),F(y_1,x_1)).
\end{aligned}$$

Continuing this process, one obtains two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $gx_{n+1} \in F(x_n, y_n)$, $gy_{n+1} \in F(y_n, x_n)$ and

$$\begin{aligned}
&d(gx_n,gx_{n+1}) \le kH(F(x_{n-1},y_{n-1}),F(x_n,y_n)),\\
&d(gy_n,gy_{n+1}) \le kH(F(y_{n-1},x_{n-1}),F(y_n,x_n)).
\end{aligned}$$

For each n, using (12), we have

$$\begin{aligned}
&d(gx_n,gx_{n+1})\\
&\le kH(F(x_{n-1},y_{n-1}),F(x_n,y_n))\\
&\le \sqrt{h}\max\Big\{d(gx_{n-1},gx_n),d(gy_{n-1},gy_n),d(F(x_{n-1},y_{n-1}),gx_{n-1}),\\
&\quad\frac{d(F(x_{n-1},y_{n-1}),gx_n)+d(F(x_n,y_n),gx_{n-1})}{2},d(F(x_n,y_n),gx_n)\Big\}\\
&\le \sqrt{h}\max\left\{d(gx_{n-1},gx_n),d(gy_{n-1},gy_n),\frac{d(gx_{n+1},gx_{n-1})}{2},d(gx_{n+1},gx_n)\right\}\\
&\le \sqrt{h}\max\left\{d(gx_{n-1},gx_n),d(gy_{n-1},gy_n),\frac{d(gx_{n+1},gx_n)+d(gx_n,gx_{n-1})}{2},d(gx_{n+1},gx_n)\right\}\\
&= \sqrt{h}\max\left\{d(gx_{n-1},gx_n),d(gy_{n-1},gy_n),d(gx_{n+1},gx_n)\right\}.
\end{aligned}$$

Hence, if we suppose that $d(gx_n,gx_{n+1}) \le \sqrt{h}d(gx_n,gx_{n+1})$, then $d(gx_n, gx_{n+1}) = 0$. Therefore,

$$d(gx_n,gx_{n+1})\sqrt{h}\max\{d(gx_{n-1},gx_n),d(gy_{n-1},gy_n)\}. \tag{13}$$

Similarly,

$$d(gy_n, gy_{n+1}) \leq \sqrt{h} \max\{d(gy_{n-1}, gy_n), d(gx_{n-1}, gx_n)\}. \tag{14}$$

Using (13) and (14), we obtain

$$d(gx_n, gx_{n+1}) \leq (\sqrt{h})^n \delta$$

and

$$d(gy_n, gy_{n+1}) \leq (\sqrt{h})^n \delta,$$

where $\delta = \max\{d(gx_0, gx_1), d(gy_0, gy_1)\}$. Thus for $m, n \in N$ with $m > n$,

$$\begin{aligned} d(gx_n, gx_{m+n}) &\leq d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + \cdots + d(gx_{n+m-1}, gx_{m+n}) \\ &+ (\sqrt{h})^n \delta + (\sqrt{h})^{n+1} \delta + \cdots + (\sqrt{h})^{n+m-1} \delta. \end{aligned}$$

Therefore

$$d(gx_n, gx_{m+n}) \leq \sum_{i=n}^{n+m-1} (\sqrt{h})^i \delta.$$

Hence we conclude that $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Similarly we obtain that $\{gy_n\}$ is a Cauchy sequence in $g(X)$. Since $g(X)$ is complete, so there exists $x, y \in X$ such that $gx_n \to gx$ and $gy_n \to gy$.

Thus from (12),

$$\begin{aligned} d(F(x,y), gx_{n+1}) &\leq H(F(x,y), F(x_n, y_n)) \\ &\leq h \max\left\{d(gx, gx_n), d(gy, gy_n), d(F(x,y), gx), \right. \\ &\quad \left. \frac{d(F(x,y), gx_n) + d(F(x_n, y_n), gx)}{2}, d(F(x_n, y_n), gx_n)\right\} \\ &\leq h \max\{d(gx, gx_n), d(gy, gy_n), d(F(x,y), gx), \\ &\quad \left. \frac{d(F(x,y), gx_n) + d(gx_{n+1}, gx)}{2}, d(gx_{n+1}, gx_n)\right\} \end{aligned}$$

On taking limit as $n \to \infty$, we obtain

$$d(F(x,y), gx) \leq hd(F(x,y), gx),$$

which implies $d(F(x, y), gx) = 0$. As $F(x, y)$ is closed, so $gx \in F(x, y)$. Similarly, $gy \in F(y, x)$. Therefore (x, y) is a coupled coincidence point of F and g.

(a) Suppose that $\lim_{n\to\infty} gx = u$ and $\lim_{n\to\infty} gy = v$, for some $(x, y) \in C(F, g)$; $u, v \in X$. Since g is continuous at u and v, so u and v are fixed points of g. Also since F and g are w- compatible, $g^n(x) \in C(F, g)$ for all $n \geq 1$ and $g^n(x) \in F(g^{n-1}(x), g^{n-1}(y))$. Using (12), we obtain

$$\begin{aligned} d(gu, F(u,v)) &\le d(gu, g^n(x)) + d(g^n(x), F(u,v)) \\ &\le d(gu, g^n(x)) + H(F(g^{n-1}x, g^{n-1}y), F(u,v)) \\ &\le d(gu, g^n(x)) + h \max \Big\{ d(g(g^{n-1}x), gu), d(g(g^{n-1}y), gv), \\ &\quad d(F(g^{n-1}x, g^{n-1}y), g(g^{n-1}x)), \\ &\quad \frac{d(F(g^{n-1}x, g^{n-1}y), gu) + d(F(u,v), g(g^{n-1}x))}{2}, d(F(u,v), gu) \Big\} \\ &\le d(gu, g^n(x)) + h \max \Big\{ d(g^n x, gu) + d(g^n y, gv), \\ &\quad \frac{d(g^n x, gu) + d(F(u,v), g^n x)}{2}, d(F(u,v), gu) \Big\}. \end{aligned}$$

Hence, taking limit as $n \to \infty$, we get

$$d(gu, F(u,v)) \le hd(gu, F(u,v)).$$

Hence $d(gu, F(u, v)) = 0$ and therefore $gu \in F(u, v)$. Similarly, $gv \in F(v, u)$. Consequently $u = gu \in F(u, v)$ and $v = gv \in F(v, u)$. Hence (u, v) is a coupled fixed point of F and g.

If the pair $\{F, g\}$ satisfies condition (b)-(d) of Theorem 8, then result follow using arguments similar to those given in the proof of Theorem 8. ■

Author details
[1]Department of Mathematics, Lahore University of Management Sciences, Lahore - 54792, Pakistan [2]Faculty of Mechanical Engineering, Kraljice Marije 16, 11 000 Belgrade, Serbia [3]Department of Mathematics, Faculty of Agriculture, Nemanjina 6, 11 000 Belgrade, Serbia

Authors' contributions
All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 3 October 2011 Accepted: 9 January 2012 Published: 9 January 2012

References

1. Nadler, S: Multi-valued contraction mappings. Pacific J Math. **20**(2), 475–488 (1969)
2. Dube, LS: A theorem on common fixed points of multivalued mappings. Ann Soc Sci Bruxelles. **84**(4), 463–468 (1975)
3. Bhashkar, TG, Lakshmikantham, V: Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Anal TMA. **65**(7), 1379–1393 (2006). doi:10.1016/j.na.2005.10.017
4. Lakshmikantham, V, Ćirić, L: Coupled fixed point theorems for nonlinear contractions in partially ordered metric space. Nonlinear Anal TMA. **70**, 4341–4349 (2009). doi:10.1016/j.na.2008.09.020
5. Sabetghadam, F, Masiha, HP, Sanatpour, AH: Some coupled fixed point theorems in cone metric space. Fixed Point Theory Appl **2009** (2009). Article ID 125426, 8
6. Markin, JT: Continuous dependence of fixed point sets. Proc Am Math Soc. **38**, 545–547 (1973). doi:10.1090/S0002-9939-1973-0313897-4
7. Gorniewicz, L: Topological Fixed Point Theory of Multivalued Mappings. Kluwer Academic Pubisher, Dordrecht, The Netherlands (1999)
8. Klim, D, Wardowski, D: Fixed Point Theorems for Set-Valued Contractions in Complete Metric Spaces. J Math Anal Appl. **334**, 132–139 (2007). doi:10.1016/j.jmaa.2006.12.012
9. Dhage, BC: Hybrid fixed point theory for strictly monotone increasing multivalued mappings with applications. Comput Math Appl. **53**, 803–824 (2007). doi:10.1016/j.camwa.2006.10.020
10. Dhage, BC: A fixed point theorem for multivalued mappings on ordered banach spaces with applications. Nonlinear Anal Forum. **10**, 105–126 (2005)

11. Dhage, BC: A general multivalued hybrid fixed point theorem and perturbed differential inclusions. Nonlinear Anal TMA. **64**, 2747–2772 (2006). doi:10.1016/j.na.2005.09.013
12. Hong, SH: Fixed points of multivalued operators in ordered metric spaces with applications. Nonlinear Anal TMA. **72**, 3929–3942 (2010). doi:10.1016/j.na.2010.01.013
13. Hong, SH: Fixed points for mixed monotone multivalued operators in banach spaces with applications. J Math Anal Appl. **337**, 333–342 (2008). doi:10.1016/j.jmaa.2007.03.091
14. Hong, SH, Guan, D, Wang, L: Hybrid Fixed Points of Multivalued Operators in Metric Spaces with Applications. Nonlinear Anal TMA. **70**, 4106–4117 (2009). doi:10.1016/j.na.2008.08.020
15. Hong, SH: Fixed points of discontinuous multivalued increasing operators in Banach spaces with applications. J Math Anal Appl. **282**, 151–162 (2003). doi:10.1016/S0022-247X(03)00111-2
16. Al-Thagafi, MA, Shahzad, N: Coincidence points, generalized *I*- nonexpansive multimaps and applications. Nonlinear Anal TMA. **67**(7), 2180–2188 (2007). doi:10.1016/j.na.2006.08.042
17. Ćirić, Lj, Cakić, N, Rajović, M, Ume, JS: Monotone generalized nonlinear contractions in partially ordered metric spaces. Fixed Point Theory Appl **2008** (2008). Article ID 131294, 11
18. Samet, B: Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces. Nonlinear Anal. **72**, 4508–4517 (2010). doi:10.1016/j.na.2010.02.026
19. Samet, B, Vetro, C: Coupled fixed point theorems for multi-valued nonlinear contraction mappings in partially ordered metric spaces. Nonlinear Anal. **74**, 4260–4268 (2011). doi:10.1016/j.na.2011.04.007
20. Altun, I, Damjanović, B, Djorić, D: Fixed point and common fixed point theorems on ordered cone metric spaces. Appl Math Lett. **23**, 310–316 (2010). doi:10.1016/j.aml.2009.09.016
21. Khan, AR, Domlo, AA, Hussain, N: Coincidences of Lipschitz type hybrid maps and invariant approximation. Numer Funct Anal Optim. **28**(9-10), 2180–2188 (2007)
22. Khan, AR: Properties of fixed point set of a multivaled map. J Appl Math Stoch Anal. **3**, 323–332 (2005)

doi:10.1186/1687-1812-2012-4
Cite this article as: Abbas *et al.*: Coupled coincidence and common fixed point theorems for hybrid pair of mappings. *Fixed Point Theory and Applications* 2012 **2012**:4.