Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

A best proximity point theorem for Geraghty-contractions

J Caballero, J Harjani and K Sadarangani*

*Correspondence:
ksadaran@dma.ulpgc.es
Departamento de Matemáticas, Universidad de Las Palmas de Gran Canaria, Campus de Tafira Baja, Las Palmas de Gran Canaria, 35017, Spain

Abstract

The purpose of this paper is to provide sufficient conditions for the existence of a unique best proximity point for Geraghty-contractions.

Our paper provides an extension of a result due to Geraghty (Proc. Am. Math. Soc. 40:604-608, 1973).

Keywords: fixed point; Geraghty-contraction; *P*-property; best proximity point

1 Introduction

Let A and B be nonempty subsets of a metric space (X, d).

An operator $T: A \to B$ is said to be a k-contraction if there exists $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for any $x, y \in A$. Banach's contraction principle states that when A is a complete subset of X and T is a k-contraction which maps A into itself, then T has a unique fixed point in A.

A huge number of generalizations of this principle appear in the literature. Particularly, the following generalization of Banach's contraction principle is due to Geraghty [1].

First, we introduce the class $\mathcal{F}$ of those functions $\beta: [0, \infty) \to [0, 1)$ satisfying the following condition:

$$\beta(t_n) \to 1 \quad \text{implies} \quad t_n \to 0.$$

Theorem 1.1 ([1]) *Let (X, d) be a complete metric space and $T: X \to X$ be an operator. Suppose that there exists $\beta \in \mathcal{F}$ such that for any $x, y \in X$,*

$$d(Tx, Ty) \leq \beta\big(d(x, y)\big) \cdot d(x, y). \tag{1}$$

Then T has a unique fixed point.

Since the constant functions $f(t) = k$, where $k \in [0, 1)$, belong to $\mathcal{F}$, Theorem 1.1 extends Banach's contraction principle.

Remark 1.1 Since the functions belonging to $\mathcal{F}$ are strictly smaller than one, condition (1) implies that

$$d(Tx, Ty) < d(x, y) \quad \text{for any } x, y \in X \text{ with } x \neq y.$$

Therefore, any operator $T\colon X \to X$ satisfying (1) is a continuous operator.

The aim of this paper is to give a generalization of Theorem 1.1 by considering a non-self map T.

First, we present a brief discussion about a best proximity point.

Let A be a nonempty subset of a metric space (X,d) and $T\colon A \to X$ be a mapping. The solutions of the equation $Tx = x$ are fixed points of T. Consequently, $T(A) \cap A \neq \emptyset$ is a necessary condition for the existence of a fixed point for the operator T. If this necessary condition does not hold, then $d(x,Tx) > 0$ for any $x \in A$ and the mapping $T\colon A \to X$ does not have any fixed point. In this setting, our aim is to find an element $x \in A$ such that $d(x,Tx)$ is minimum in some sense. The best approximation theory and best proximity point analysis have been developed in this direction.

In our context, we consider two nonempty subsets A and B of a complete metric space and a mapping $T\colon A \to B$.

A natural question is whether one can find an element $x_0 \in A$ such that $d(x_0,Tx_0) = \min\{d(x,Tx)\colon x \in A\}$. Since $d(x,Tx) \geq d(A,B)$ for any $x \in A$, the optimal solution to this problem will be the one for which the value $d(A,B)$ is attained by the real valued function $\varphi\colon A \to \mathbb{R}$ given by $\varphi(x) = d(x,Tx)$.

Some results about best proximity points can be found in [2–9].

2 Notations and basic facts

Let A and B be two nonempty subsets of a metric space (X,d).

We denote by A_0 and B_0 the following sets:

$$A_0 = \{x \in A\colon d(x,y) = d(A,B) \text{ for some } y \in B\},$$
$$B_0 = \{y \in B\colon d(x,y) = d(A,B) \text{ for some } x \in A\},$$

where $d(A,B) = \inf\{d(x,y)\colon x \in A \text{ and } y \in B\}$.

In [8], the authors present sufficient conditions which determine when the sets A_0 and B_0 are nonempty.

Now, we present the following definition.

Definition 2.1 Let A,B be two nonempty subsets of a metric space (X,d). A mapping $T\colon A \to B$ is said to be a Geraghty-contraction if there exists $\beta \in \mathcal{F}$ such that

$$d(Tx,Ty) \leq \beta\bigl(d(x,y)\bigr) \cdot d(x,y) \quad \text{for any } x,y \in A.$$

Notice that since $\beta\colon [0,\infty) \to [0,1)$, we have

$$d(Tx,Ty) \leq \beta\bigl(d(x,y)\bigr) \cdot d(x,y) < d(x,y) \quad \text{for any } x,y \in A \text{ with } x \neq y.$$

Therefore, every Geraghty-contraction is a contractive mapping.

In [10], the author introduces the following definition.

Definition 2.2 ([10]) Let (A,B) be a pair of nonempty subsets of a metric space (X,d) with $A_0 \neq \emptyset$. Then the pair (A,B) is said to have the P-property if and only if for any $x_1,x_2 \in A_0$

and $y_1, y_2 \in B_0$,

$$\left.\begin{aligned} d(x_1, y_1) &= d(A, B) \\ d(x_2, y_2) &= d(A, B) \end{aligned}\right\} \quad \Rightarrow \quad d(x_1, x_2) = d(y_1, y_2).$$

It is easily seen that for any nonempty subset A of (X, d), the pair (A, A) has the P-property.

In [10], the author proves that any pair (A, B) of nonempty closed convex subsets of a real Hilbert space H satisfies the P-property.

3 Main results

We start this section presenting our main result.

Theorem 3.1 *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $T: A \to B$ be a Geraghty-contraction satisfying $T(A_0) \subseteq B_0$. Suppose that the pair (A, B) has the P-property. Then there exists a unique x^* in A such that $d(x^*, Tx^*) = d(A, B)$.*

Proof Since A_0 is nonempty, we take $x_0 \in A$.

As $Tx_0 \in T(A_0) \subseteq B_0$, we can find $x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$. Similarly, since $Tx_1 \in T(A_0) \subseteq B_0$, there exists $x_2 \in A_0$ such that $d(x_2, Tx_1) = d(A, B)$. Repeating this process, we can get a sequence (x_n) in A_0 satisfying

$$d(x_{n+1}, Tx_n) = d(A, B) \quad \text{for any } n \in \mathbb{N}.$$

Since (A, B) has the P-property, we have that

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \quad \text{for any } n \in \mathbb{N}.$$

Taking into account that T is a Geraghty-contraction, for any $n \in \mathbb{N}$, we have that

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \beta\bigl(d(x_{n-1}, x_n)\bigr) \cdot d(x_{n-1}, x_n) < d(x_{n-1}, x_n). \tag{2}$$

Suppose that there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0+1}) = 0$.

In this case,

$$0 = d(x_{n_0}, x_{n_0+1}) = d(Tx_{n_0-1}, Tx_{n_0}),$$

and consequently, $Tx_{n_0-1} = Tx_{n_0}$.

Therefore,

$$d(A, B) = d(x_{n_0}, Tx_{n_0-1}) = d(x_{n_0}, Tx_{n_0})$$

and this is the desired result.

In the contrary case, suppose that $d(x_n, x_{n+1}) > 0$ for any $n \in \mathbb{N}$.

By (2), $(d(x_n, x_{n+1}))$ is a decreasing sequence of nonnegative real numbers, and hence there exists $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r.$$

In the sequel, we prove that $r = 0$.

Assume $r > 0$, then from (2) we have

$$0 < \frac{d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} \leq \beta\big(d(x_{n-1}, x_n)\big) < 1 \quad \text{for any } n \in \mathbb{N}.$$

The last inequality implies that $\lim_{n\to\infty} \beta(d(x_{n-1}, x_n)) = 1$ and since $\beta \in \mathcal{F}$, we obtain $r = 0$ and this contradicts our assumption.

Therefore,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3}$$

Notice that since $d(x_{n+1}, Tx_n) = d(A, B)$ for any $n \in \mathbb{N}$, for $p, q \in \mathbb{N}$ fixed, we have $d(x_p, Tx_{p-1}) = d(x_q, Tx_{q-1}) = d(A, B)$, and since (A, B) satisfies the P-property, $d(x_p, x_q) = d(Tx_{p-1}, Tx_{q-1})$.

In what follows, we prove that (x_n) is a Cauchy sequence.

In the contrary case, we have that

$$\limsup_{m,n\to\infty} d(x_n, x_m) > 0.$$

By using the triangular inequality,

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{m+1}) + d(x_{m+1}, x_m).$$

By (2) and since $d(x_{n+1}, x_{m+1}) = d(Tx_n, Tx_m)$, by the above mentioned comment, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(Tx_n, Tx_m) + d(x_{m+1}, x_m) \\ &\leq d(x_n, x_{n+1}) + \beta\big(d(x_n, x_m)\big) \cdot d(x_n, x_m) + d(x_{m+1}, x_m), \end{aligned}$$

which gives us

$$d(x_n, x_m) \leq \big(1 - \beta\big(d(x_n, x_m)\big)\big)^{-1}\big[d(x_n, x_{n+1}) + d(x_{m+1}, x_m)\big].$$

Since $\limsup_{m,n\to\infty} d(x_n, x_m) > 0$ and by (3), $\limsup_{n\to\infty} d(x_n, x_{n+1}) = 0$, from the last inequality it follows that

$$\limsup_{m,n\to\infty} \big(1 - \beta\big(d(x_n, x_m)\big)\big)^{-1} = \infty.$$

Therefore, $\limsup_{m,n\to\infty} \beta(d(x_n, x_m)) = 1$.

Taking into account that $\beta \in \mathcal{F}$, we get $\limsup_{m,n\to\infty} d(x_n, x_m) = 0$ and this contradicts our assumption.

Therefore, (x_n) is a Cauchy sequence.

Since $(x_n) \subset A$ and A is a closed subset of the complete metric space (X, d), we can find $x^* \in A$ such that $x_n \to x^*$.

Since any Geraghty-contraction is a contractive mapping and hence continuous, we have $Tx_n \to Tx^*$.

This implies that $d(x_{n+1}, Tx_n) \to d(x^*, Tx^*)$.

Taking into account that the sequence $(d(x_{n+1}, Tx_n))$ is a constant sequence with value $d(A, B)$, we deduce

$$d\bigl(x^*, Tx^*\bigr) = d(A, B).$$

This means that x^* is a best proximity point of T.

This proves the part of existence of our theorem.

For the uniqueness, suppose that x_1 and x_2 are two best proximity points of T with $x_1 \neq x_2$.

This means that

$$d(x_i, Tx_i) = d(A, B) \quad \text{for } i = 1, 2.$$

Using the P-property, we have

$$d(x_1, x_2) = d(Tx_1, Tx_2).$$

Using the fact that T is a Geraghty-contraction, we have

$$d(x_1, x_2) = d(Tx_1, Tx_2) \leq \beta\bigl(d(x_1, x_2)\bigr) \cdot d(x_1, x_2) < d(x_1, x_2),$$

which is a contradiction.

Therefore, $x_1 = x_2$.

This finishes the proof. □

4 Examples

In order to illustrate our results, we present some examples.

Example 4.1 Consider $X = \mathbb{R}^2$ with the usual metric.

Let A and B be the subsets of X defined by

$$A = \{0\} \times [0, \infty) \quad \text{and} \quad B = \{1\} \times [0, \infty).$$

Obviously, $d(A, B) = 1$ and A, B are nonempty closed subsets of X.

Moreover, it is easily seen that $A_0 = A$ and $B_0 = B$.

Let $T: A \to B$ be the mapping defined as

$$T(0, x) = \bigl(1, \ln(1 + x)\bigr) \quad \text{for any } (0, x) \in A.$$

In the sequel, we check that T is a Geraghty-contraction.

In fact, for $(0,x),(0,y)\in A$ with $x\neq y$, we have

$$\begin{aligned} d\big(T(0,x),T(0,y)\big) &= d\big(\big(1,\ln(1+x)\big),\big(1,\ln(1+y)\big)\big) \\ &= \big|\ln(1+x)-\ln(1+y)\big| \\ &= \left|\ln\left(\frac{1+x}{1+y}\right)\right|. \end{aligned} \tag{4}$$

Now, we prove that

$$\left|\ln\left(\frac{1+x}{1+y}\right)\right| \le \ln\big(1+|x-y|\big). \tag{5}$$

Suppose that $x>y$ (the same reasoning works for $y>x$).

Then, since $\phi(t)=\ln(1+t)$ is strictly increasing in $[0,\infty)$, we have

$$\ln\left(\frac{1+x}{1+y}\right)=\ln\left(\frac{1+y+x-y}{1+y}\right)=\ln\left(1+\frac{x-y}{1+y}\right)\le \ln(1+x-y)=\ln\big(1+|x-y|\big).$$

This proves (5).

Taking into account (4) and (5), we have

$$\begin{aligned} d\big(T(0,x),T(0,y)\big) &= \left|\ln\left(\frac{1+x}{1+y}\right)\right| \\ &\le \ln\big(1+|x-y|\big) \\ &= \frac{\ln(1+|x-y|)}{|x-y|}\cdot|x-y| \\ &= \frac{\phi(d((0,x),(0,y)))}{d((0,x),(0,y))}\cdot d\big((0,x),(0,y)\big) \\ &= \beta\big(d\big((0,x),(0,y)\big)\big)\cdot d\big((0,x),(0,y)\big), \end{aligned} \tag{6}$$

where $\phi(t)=\ln(1+t)$ for $t\ge 0$, and $\beta(t)=\frac{\phi(t)}{t}$ for $t>0$ and $\beta(0)=0$.

Obviously, when $x=y$, the inequality (6) is satisfied.

It is easily seen that $\beta(t)=\frac{\ln(1+t)}{t}\in\mathcal{F}$ by using elemental calculus.

Therefore, T is a Geraghty-contraction.

Notice that the pair (A,B) satisfies the P-property.

Indeed, if

$$\begin{aligned} d\big((0,x_1),(1,y_1)\big) &= \sqrt{1+(x_1-y_1)^2}=d(A,B)=1, \\ d\big((0,x_2),(1,y_2)\big) &= \sqrt{1+(x_2-y_2)^2}=d(A,B)=1, \end{aligned}$$

then $x_1=y_1$ and $x_2=y_2$ and consequently,

$$d\big((0,x_1),(0,x_2)\big)=|x_1-x_2|=|y_1-y_2|=d\big((1,y_1),(1,y_2)\big).$$

By Theorem 3.1, T has a unique best proximity point.

Obviously, this point is $(0,0)\in A$.

The condition A and B are nonempty closed subsets of the metric space (X,d) is not a necessary condition for the existence of a unique best proximity point for a Geraghty-contraction $T: A \to B$ as it is proved with the following example.

Example 4.2 Consider $X = \mathbb{R}^2$ with the usual metric and the subsets of X given by

$$A = \{0\} \times [0,\infty) \quad \text{and} \quad B = \{1\} \times \left[0, \frac{\pi}{2}\right).$$

Obviously, $d(A,B) = 1$ and B is not a closed subset of X.

Note that $A_0 = 0 \times [0, \frac{\pi}{2})$ and $B_0 = B$.

We consider the mapping $T: A \to B$ defined as

$$T(0,x) = (1, \arctan x) \quad \text{for any } (0,x) \in A.$$

Now, we check that T is a Geraghty-contraction.

In fact, for $(0,x), (0,y) \in A$ with $x \neq y$, we have

$$d\big(T(0,x), T(0,y)\big) = d\big((1,\arctan x),(1,\arctan y)\big) = |\arctan x - \arctan y|. \tag{7}$$

In what follows, we need to prove that

$$|\arctan x - \arctan y| \leq \arctan|x - y|. \tag{8}$$

In fact, suppose that $x > y$ (the same argument works for $y > x$).

Put $\arctan x = \alpha$ and $\arctan y = \beta$ (notice that $\alpha > \beta$ since the function $\phi(t) = \arctan t$ for $t \geq 0$ is strictly increasing).

Taking into account that

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha \cdot \tan\beta}$$

and since $\alpha, \beta \in [0, \frac{\pi}{2})$, we have that $\tan\alpha, \tan\beta \in [0,\infty)$, and consequently, from the last inequality it follows that

$$\tan(\alpha - \beta) \leq \tan\alpha - \tan\beta.$$

Applying ϕ (notice that $\phi(t) = \arctan t$) to the last inequality and taking into account the increasing character of ϕ, we have

$$\alpha - \beta \leq \arctan(\tan\alpha - \tan\beta),$$

or equivalently,

$$\arctan x - \arctan y = \alpha - \beta \leq \arctan(x - y),$$

and this proves (8).

By (7) and (8), we get

$$\begin{aligned} d\bigl(T(0,x),T(0,y)\bigr) &= |\arctan x - \arctan y| \\ &\le \arctan|x-y| \\ &= \frac{\arctan|x-y|}{|x-y|}\cdot|x-y| \\ &= \beta\bigl(d(0,x),d(0,y)\bigr)\cdot d\bigl((0,x),(0,y)\bigr), \end{aligned} \tag{9}$$

where $\beta(t)=\frac{\arctan t}{t}$ for $t>0$ and $\beta(0)=0$. Obviously, the inequality (9) is satisfied for $(0,x),(0,y)\in A$ with $x=y$.

Now, we prove that $\beta\in\mathcal{F}$.

In fact, if $\beta(t_n)=\frac{\arctan t_n}{t_n}\to 1$, then the sequence (t_n) is a bounded sequence since in the contrary case, $t_n\to\infty$ and thus $\beta(t_n)\to 0$. Suppose that $t_n\nrightarrow 0$. This means that there exists $\epsilon>0$ such that, for each $n\in\mathbb{N}$, there exists $p_n\ge n$ with $t_{p_n}\ge\epsilon$. The bounded character of (t_n) gives us the existence of a subsequence (t_{k_n}) of (t_{p_n}) with (t_{k_n}) convergent. Suppose that $t_{k_n}\to a$. From $\beta(t_n)\to 1$, we obtain $\frac{\arctan t_{k_n}}{t_{k_n}}\to\frac{\arctan a}{a}=1$ and, as the unique solution of $\arctan x = x$ is $x_0=0$, we obtain $a=0$.

Thus, $t_{k_n}\to 0$ and this contradicts the fact that $t_{k_n}\ge\epsilon$ for any $n\in\mathbb{N}$.

Therefore, $t_n\to 0$ and this proves that $\beta\in\mathcal{F}$.

A similar argument to the one used in Example 4.1 proves that the pair (A,B) has the P-property.

On the other hand, the point $(0,0)\in A$ is a best proximity point for T since

$$d\bigl((0,0),T(0,0)\bigr)=d\bigl((0,0),(1,\arctan 0)\bigr)=d\bigl((0,0),(1,0)\bigr)=1=d(A,B).$$

Moreover, $(0,0)$ is the unique best proximity point for T.

Indeed, if $(0,x)\in A$ is a best proximity point for T, then

$$1=d(A,B)=d\bigl((0,x),T(0,x)\bigr)=d\bigl((0,x),(1,\arctan x)\bigr)=\sqrt{1+(x-\arctan x)^2},$$

and this gives us

$$x=\arctan x.$$

Taking into account that the unique solution of this equation is $x=0$, we have proved that T has a unique best proximity point which is $(0,0)$.

Notice that in this case B is not closed.

Since for any nonempty subset A of X, the pair (A,A) satisfies the P-property, we have the following corollary.

Corollary 4.1 *Let (X,d) be a complete metric space and A be a nonempty closed subset of X. Let $T\colon A\to A$ be a Geraghty-contraction. Then T has a unique fixed point.*

Proof Using Theorem 3.1 when $A=B$, the desired result follows. □

Notice that when $A = X$, Corollary 4.1 is Theorem 1.1 due to Gerahty [1].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The three authors have contributed equally in this paper. They read and approval the final manuscript.

Acknowledgements
This research was partially supported by 'Universidad de Las Palmas de Gran Canaria', Project ULPGC 2010-006.

Received: 16 May 2012 Accepted: 10 December 2012 Published: 27 December 2012

References
1. Geraghty, M: On contractive mappings. Proc. Am. Math. Soc. **40**, 604-608 (1973)
2. Eldred, AA, Veeramani, P: Existence and convergence of best proximity points. J. Math. Anal. Appl. **323**, 1001-1006 (2006)
3. Anuradha, J, Veeramani, P: Proximal pointwise contraction. Topol. Appl. **156**, 2942-2948 (2009)
4. Markin, J, Shahzad, N: Best approximation theorems for nonexpansive and condensing mappings in hyperconvex spaces. Nonlinear Anal. **70**, 2435-2441 (2009)
5. Sadiq Basha, S, Veeramani, P: Best proximity pair theorems for multifunctions with open fibres. J. Approx. Theory **103**, 119-129 (2000)
6. Sankar Raj, V, Veeramani, P: Best proximity pair theorems for relatively nonexpansive mappings. Appl. Gen. Topol. **10**, 21-28 (2009)
7. Al-Thagafi, MA, Shahzad, N: Convergence and existence results for best proximity points. Nonlinear Anal. **70**, 3665-3671 (2009)
8. Kirk, WA, Reich, S, Veeramani, P: Proximinal retracts and best proximity pair theorems. Numer. Funct. Anal. Optim. **24**, 851-862 (2003)
9. Sankar Raj, V: A best proximity theorem for weakly contractive non-self mappings. Nonlinear Anal. **74**, 4804-4808 (2011)
10. Sankar Raj, V: Banach's contraction principle for non-self mappings. Preprint

doi:10.1186/1687-1812-2012-231
Cite this article as: Caballero et al.: **A best proximity point theorem for Geraghty-contractions.** *Fixed Point Theory and Applications* 2012 **2012**:231.