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Hybrid iteration method for common fixed points of an infinite family of nonexpansive mappings in Banach spaces

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Abstract

Let E be a real uniformly convex Banach space, and let K be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of nonexpansive mappings from K to itself with $F := \{x \in K : T_i x = x, \forall i \geq 1\} \neq \emptyset$. For an arbitrary initial point $x_1 \in K$, the modified hybrid iteration scheme $\{x_n\}$ is defined as follows: $x_{n+1} = \alpha_n x_n + (1-\alpha_n)\left(T_n^* x_n - \lambda_{n+1}\mu A(T_n^* x_n)\right), n \geq 1$, where A: $K \to K$ is an L-Lipschitzian mapping, $T_n^* = T_i$ with i satisfying: $n = [(k-i+1)(i+k)/2]+[1+(i-1)(i+2)/2], k \geq i-1 (i = 1,2,\ldots), \{\lambda_n\} \subset [0,1)$, and $\{\alpha_n\}$ is a sequence in $[a, 1 - a]$ for some $a \in (0,1)$. Under some suitable conditions, the strong and weak convergence theorems of $\{x_n\}$ to a common fixed point of the nonexpansive mappings $\{T_i\}_{i=1}^{\infty}$ are obtained. The results in this article extend those of the authors whose related researches are restricted to the situation of finite families of nonexpansive mappings.
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1 Introduction

Let K be a nonempty closed convex subset of a real uniformly convex Banach space E. A self-mapping T: $K \to K$ is said to be nonexpansive if $||Tx-Ty|| \leq ||x-y||$ for all $x,y \in K$.$F : K \to K$ is said to be L-Lipschitzian if there exists a constant $L > 0$ such that $||Fx-Fy|| \leq L||x-y||$ for all $x,y \in K$.

Iterative techniques for approximating fixed points of nonexpansive mappings have been studied by various authors (see, e.g., [1-9]). In 2007, Wang [10] introduced an explicit hybrid iteration method for nonexpansive mappings in Hilbert space; and then Osilike et al. [11] extended Wang's results to arbitrary Banach spaces without the strong monotonicity assumption imposed on the hybrid operator. In 2009, Wang et al. [12] obtained the following strong and weak convergence theorems for a finite family of nonexpansive mappings in uniformly convex Banach space by using hybrid iteration method, which further extended and improved his results and partially improved those of Osilike's.

Theorem 1.1. [12]*Let E be a real uniformly convex Banach space endowed with the norm $||\cdot||$. Let $I = \{1,2,\ldots,N\}$,$\{T_i : i \in I\}$ be N nonexpansive mappings from E into itself* *with* $F = \{x \in E : T_i x = x, i \in I\} \neq \emptyset$, *and let* $A : E \to E$ *be an L-Lipschitzian mapping. For any given* $x_1 \in E$, $\{x_n\}$ *is defined by*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T^{\lambda_{n+1}} x_n, \quad n \geq 1, \tag{1.1}$$

where $T^{\lambda_{n+1}} x_n = T_n x_n - \lambda_{n+1} \mu A(T_n x_n)$, $\mu > 0$, $T_n = T_i$, $i = n(\text{mod } N)$, $1 \leq i \leq N$. *If* $\{\alpha_n\}$ *and* $\{\lambda_n\} \subset [0,1)$ *satisfy the following conditions:*

(1) $a \leq \alpha_n \leq b$ *for all* $n \geq 1$ *and some* $a,b \in (0,1)$;
(2) $\sum_{n=2}^{\infty} \lambda_n < \infty$,

then

(1) $\lim_{n \to \infty} ||x_n - q||$ *exists*, $\forall q \in F$;
(2) $\lim_{n \to \infty} ||x_n - T_i x_n|| = 0$, $\forall i \in I$;
(3) $\{x_n\}$ *converges strongly to a common fixed point of* $\{T_i : i \in I\}$ *if and only if* $\lim_{n \to \infty} d(x_n, F) = 0$.

Theorem 1.2. [12]*Let E be a real uniformly convex Banach space satisfying Opial's condition. Let* $\{T_i : i \in I\}$ *be N nonexpansive mappings from E into itself with* $F = \{x \in E : T_i x = x, i \in I\} \neq \emptyset$, *and let* $A : E \to E$ *be an L-Lipschitzian mapping. For any given* $x_1 \in E$, $\{x_n\}$ *is defined as in Theorem* 1.1, *and* $\{\alpha_n\}$ *and* $\{\lambda_n\} \subset [0,1)$ *satisfy the conditions appeared in Theorem* 1.1. *Then* $\{x_n\}$ *converges weakly to a common fixed point of the mappings* $\{T_i : i \in I\}$.

Inspired and motivated by those study mentioned above, in this article, we use a modified hybrid iteration scheme for approximating common fixed points of an infinite family of nonexpansive mappings $\{T_i\}_{i=1}^{\infty}$ and prove some strong and weak convergence theorems for such mappings in uniformly convex Banach spaces. The results extend those of Wang whose research is restricted to the situation of finite families of nonexpansive mappings.

2. Preliminaries

A Banach space E is said to satisfy Opial's condition if, for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that

$$\limsup_{n \to \infty} ||x_n - x|| < \limsup_{n \to \infty} ||x_n - y|| \tag{2.1}$$

for all $y \in E$ with $y \neq x$, where $x_n \rightharpoonup x$ denotes that $\{x_n\}$ converges weakly to x.

A mapping T with domain $D(T)$ and range $R(T)$ in E is said to be demiclosed at p if whenever $\{x_n\}$ is a sequence in $D(T)$ such that $\{x_n\}$ converges weakly to $x^* \in D(T)$ and $\{Tx_n\}$ converges strongly to p, then $Tx^* = p$.

We need the following lemmas for our main results.

Lemma 2.1. [13]*Let* $\{a_n\}$,$\{\delta_n\}$, *and* $\{b_n\}$ *be sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 + \delta_n) a_n + b_n, \ \forall n \geq 1, \tag{2.2}$$

if $\sum_{n=1}^{\infty} \delta_n < \infty$ *and* $\sum_{n=1}^{\infty} b_n < \infty$, *then* $\lim_{n \to \infty} a_n$ *exists.*

Lemma 2.2. [14]*Let E be a real uniformly convex Banach space and let a, b be two constants with* $0 < a < b < 1$. *Suppose that* $\{t_n\} \subset [a,b]$ *is a real sequence and* $\{x_n\},\{y_n\}$ *are two sequences in E. Then the conditions*

$$\lim_{n\to\infty} ||t_n x_n + (1-t_n)y_n|| = d,\ \limsup_{n\to\infty} ||x_n|| \le d,\ \limsup_{n\to\infty} ||y_n|| \le d \tag{2.3}$$

imply that $\lim_{n\to\infty}||x_n - y_n|| = 0$, *where* $d \geq 0$ *is a constant.*

Lemma 2.3. [15]*Let E be a real uniformly convex Banach space, let K be a nonempty closed convex subset of E, and let* $T : K \to K$ *be a nonexpansive mapping. Then I- T is demiclosed at zero.*

3 Main results

Lemma 3.1. *Let E be a real uniformly convex Banach space, and let K be a nonempty closed convex subset of E. Let* $\{T_i\}_{i=1}^{\infty}$ *be a sequence of nonexpansive mappings from K to itself, and let* $A : K \to K$ *be an L-Lipschitzian mapping. For an arbitrary initial point* $x_1 \in K$,$\{x_n\}$ *is defined as follows:*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T^{\lambda_{n+1}} x_n, \quad n \ge 1, \tag{3.1}$$

where $T^{\lambda_{n+1}} x_n = T_n^* x_n - \lambda_{n+1}\mu A(T_n^* x_n), \mu > 0, T_n^* = T_i$ *with i satisfying the following equation:*

$$n = [(k-i+1)(i+k)/2] + [1 + (i-1)(i+2)/2],\ k \ge i-1,\ k \in \mathbf{Z}^+, \tag{3.2}$$

that is,

$$T_1^* = T_1,\ T_2^* = T_1,\ T_3^* = T_2,\ T_4^* = T_1,\ T_5^* = T_2,\ T_6^* = T_3,\ T_7^* = T_1,\ T_8^* = T_2,\ \ldots.$$

If $F := \{x \in K : T_i x = x, \forall i \geq 1\} \neq \emptyset$, *and* $\{\alpha_n\}$*and* $\{\lambda_n\} \subset [0, 1)$ *satisfy the following conditions:*

(1) $a \le \alpha_n \le b$ *for all* $n \ge 1$ *and some* $a,b \in (0,1)$;
(2) $\sum_{n=2}^{\infty} \lambda_n < \infty$,

then

(1) $\lim_{n\to\infty}||x_n - q||$ *exists*, $\forall q \in F$;
(2) $\lim_{n\to\infty}||x_n - T_i x_n|| = 0$, $\forall i \ge 1$.

Proof. (1) For any $q \in F$, by (3.1), we have

$$\begin{aligned} ||x_{n+1} - q|| &= ||\alpha_n(x_n - q) + (1-\alpha_n)(T^{\lambda_{n+1}}x_n - q)|| \\ &\le \alpha_n||x_n - q|| + (1-\alpha_n)||x_n - q|| + \lambda_{n+1}(1-\alpha_n)\mu||A(T_n^* x_n)|| \\ &\le ||x_n - q|| + (1-\alpha_n)\lambda_{n+1}\mu||A(T_n^* x_n) - A(q)|| + (1-\alpha_n)\lambda_{n+1}\mu||A(q)|| \\ &\le \left[1 + (1-a)\mu L\lambda_{n+1}\right]||x_n - q|| + (1-a)\lambda_{n+1}\mu||A(q)||. \end{aligned} \tag{3.3}$$

Since $\sum_{n=2}^{\infty} \lambda_n < \infty$,, it follows from Lemma 2.1 that $\lim_{n\to\infty}||x_n - q||$ exists.

(2) Assume, by the conclusion of (1), $\lim_{n\to\infty}||x_n - q|| = d$. That is

$$\lim_{n\to\infty} ||\alpha_n(x_n - q) + (1-\alpha_n)(T^{\lambda_{n+1}}x_n - q)|| = d. \tag{3.4}$$

Noting that $\lim_{n\to\infty}\lambda_n = 0$ and

$$\begin{aligned} ||T^{\lambda_{n+1}}x_n - q|| &= ||T_n^* x_n - \lambda_{n+1}\mu A(T_n^* x_n) - q|| \\ &\le ||x_n - q|| + \lambda_{n+1}\mu||A(T_n^* x_n)||, \end{aligned} \tag{3.5}$$

we have

$$\limsup_{n\to\infty} ||T^{\lambda_{n+1}}x_n - q|| \le \limsup_{n\to\infty} ||x_n - q|| = d, \tag{3.6}$$

and hence, it follows from (3.4), (3.6) and Lemma 2.2 that

$$\lim_{n\to\infty} ||x_n - T^{\lambda_{n+1}}x_n|| = 0. \tag{3.7}$$

Next,

$$\begin{aligned} ||x_n - T^{\lambda_{n+1}}x_n|| &= ||x_n - T_n^* x_n + \lambda_{n+1}\mu A(T_n^* x_n)|| \\ &\ge ||x_n - T_n^* x_n|| - \lambda_{n+1}\mu||A(T_n^* x_n)||, \end{aligned} \tag{3.8}$$

thus,

$$||x_n - T_n^* x_n|| \le ||x_n - T^{\lambda_{n+1}}x_n|| + \lambda_{n+1}\mu||A(T_n^* x_n)||. \tag{3.9}$$

It follows then from (3.7) that

$$\lim_{n\to\infty} ||x_n - T_n^* x_n|| = 0. \tag{3.10}$$

On the other hand, since $x_{n+1} - T_n^* x_n = \alpha_n(x_n - T_n^* x_n) - (1-\alpha_n)\lambda_{n+1}\mu A(T_n^* x_n)$, we obtain, by (3.10),

$$\lim_{n\to\infty} ||x_{n+1} - T_n^* x_n|| = 0, \tag{3.11}$$

which, in addition to (3.10), implies that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{3.12}$$

By induction, for any nonnegative integer j, we also have

$$\lim_{n\to\infty} ||x_{n+j} - x_n|| = 0. \tag{3.13}$$

For each $i \ge 1$, since

$$\begin{aligned} ||x_n - T_{n+i}^* x_n|| &\le ||x_n - x_{n+i}|| + ||x_{n+i} - T_{n+i}^* x_n|| \\ &\le ||x_n - x_{n+i}|| + ||x_{n+i} - T_{n+i}^* x_{n+i}|| \\ &\quad + ||T_{n+i}^* x_{n+i} - T_{n+i}^* x_n|| \\ &\le 2||x_n - x_{n+i}|| + ||x_{n+i} - T_{n+i}^* x_{n+i}||, \end{aligned} \tag{3.14}$$

it follows from (3.10) and (3.13) that

$$\lim_{n\to\infty} ||x_n - T_{n+i}^* x_n|| = 0. \tag{3.15}$$

Now, for each $i \geq 1$, we claim that

$$\lim_{n\to\infty} ||x_n - T_i x_n|| = 0. \tag{3.16}$$

As a matter of fact, setting

$$n = N(k, i) + i,$$

where $N(k,i) = [(k-i+1)(i+k)/2+(1+(i-1)(i+2)/2)]-i$, we obtain that

$$\begin{aligned} ||x_n - T_i x_n|| &\leq ||x_n - x_{N(k,i)}|| + ||x_{N(k,i)} - T_i x_n|| \\ &\leq ||x_n - x_{N(k,i)}|| + ||x_{N(k,i)} - T^*_{N(k,i)+i} x_{N(k,i)}|| \\ &\quad + ||T^*_{N(k,i)+i} x_{N(k,i)} - T_i x_n|| \\ &= ||x_n - x_{N(k,i)}|| + ||x_{N(k,i)} - T^*_{N(k,i)+i} x_{N(k,i)}|| \\ &\quad + ||T_i x_{N(k,i)} - T_i x_n|| \\ &\leq 2||x_n - x_{N(k,i)}|| + ||x_{N(k,i)} - T^*_{N(k,i)+i} x_{N(k,i)}|| \\ &= 2||x_n - x_{n-i}|| + ||x_{N(k,i)} - T^*_{N(k,i)+i} x_{N(k,i)}||. \end{aligned} \tag{3.17}$$

Then, since $N(k,i) \to \infty$ as $n \to \infty$, it follows from (3.13) and (3.15) that (3.16) holds obviously. This completes the proof.

Theorem 3.2. *Let E be a real uniformly convex Banach space, and let K be a nonempty closed convex subset of E. Let* $\{T_i\}_{i=1}^{\infty}$ *be a sequence of nonexpansive mappings from K to itself. Suppose that* $\{x_n\}$ *is a sequence defined by* (3.1). *If* $F: = \{x \in K: T_i x = x, \forall i \geq 1\} \neq \emptyset$ *and there exist* $T_{i_0} \in \{T_i\}_{i=1}^{\infty}$ *and a nondecreasing function* $f: [0, \infty) \to [0, \infty)$ *with* $f(0) = 0$ *and* $f(r) > 0$ *for all* $r \in (0, \infty)$ *such that* $f\left(d(x_n, F)\right) \leq ||x_n - T_{i_0} x_n||$ *for all* $n \geq 1$, *then* $\{x_n\}$ *converges strongly to some common fixed point of* $\{T_i\}_{i=1}^{\infty}$.

Proof. Since

$$f(d(x_n, F)) \leq \|x_n - T_{i_0} x_n\|,$$

by taking limsup as $n \to \infty$ on both sides in the inequality above, we have

$$\lim_{n\to\infty} f(d(x_n, F)) = 0,$$

which implies $\lim_{n\to\infty} d(x_n,F) = 0$ by the definition of the function f.

Now we will show that $\{x_n\}$is a Cauchy sequence. Since $\{x_n\}$ is bounded, there exists a constant $M > 0$ such that $||x_n-q|| \leq M(\forall n \geq 1)$; for any $\epsilon > 0$, there exists a positive integer N_1 such that $d(x_n,F) < \epsilon/4$ for all $n \geq N_1$. On the other hand, there exists a positive integer N_2 such that $\sum_{j=n}^{\infty} \lambda_n < \epsilon/4\delta$ for all $n \geq N_2$ since $\sum_{n=2}^{\infty} \lambda_n < \infty$,

Thus, for any $q \in F$ and $n,m \geq N(= \max\{N_1,N_2\})$, it follows from (3.3) that

$$||x_n - x_m|| \leq ||x_n - q|| + ||x_m - q|| \leq 2||x_N - q|| + 2\delta \sum_{j=N}^{\infty} \lambda_{j+1}, \tag{3.18}$$

where $\delta = (1-a)\mu[LM + ||A(q)||]$. Taking the infimum in the above inequalities for all $q \in F$, we obtain

$$\|x_n - x_m\| \leq 2d(x_N, F) + 2\delta \sum_{j=N}^{\infty} \lambda_{j+1} < \epsilon. \tag{3.19}$$

This implies that $\{x_n\}$ is a Cauchy sequence.-Thus, there exists a $p \in K$ such that $x_n \to p$ as $n \to \infty$, since E is complete. Then, $\lim_{n\to\infty} d(x_n, F) = 0$ yields that $d(p, F) = 0$. Furthermore, it follows from the closedness of F that $p \in F$. This completes the proof.

Theorem 3.3. *Let E be a real uniformly convex Banach space satisfying Opial's condition, and let K be a nonempty closed convex subset of E. Let* $\{T_i\}_{i=1}^{\infty}$ *be a sequence of nonexpansive mappings from K to itself. Suppose that* $\{x_n\}$ *is a sequence defined by* (3.1). *If* $F := \{x \in K : T_i x = x, \forall i \geq 1\} \neq \emptyset$, *then* $\{x_n\}$ *converges weakly to some common fixed point of* $\{T_i\}_{i=1}^{\infty}$.

Proof. For any $q \in F$, by Lemma 3.1, we know that $\lim_{n\to\infty} ||x_n - q||$ exists. We now prove that $\{x_n\}$ has a unique weakly subsequential limit in F. First of all, Lemmas 2.3 and 3.1 guarantee that each weakly subsequential limit of $\{x_n\}$ is a common fixed point of $\{T_i\}_{i=1}^{\infty}$. Secondly, Opial's condition guarantees that the weakly subsequential limit of $\{x_n\}$ is unique. Consequently, $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^{\infty}$. This completes the proof.

Remark 3.4. The results in this article extend those of the authors whose related researches are restricted to the situation where the operators used in the iteration procedure consist of just a finite family of nonexpansive mappings.

Example 3.5. Let E be the vector space R^1 with the standard norm $||\cdot|| = |\cdot|$ and $K = [0,1]$. Consider the following iteration

$$x_{n+1} = \left(\frac{4}{5} - \frac{1}{2n}\right) x_n + \left(\frac{1}{5} + \frac{1}{2n}\right) \left[T_n^* x_n - \frac{1}{(n+1)^2} \mu A\left(T_n^* x_n\right)\right],$$

where $T_n^* = T_i,\ T_i x = x^i/i,\ \mu = 1,\ Ax = x/2,\ x \in K$, and i satisfies the equation: $n = [(k-i+1)\ (i+k)/2]+[1+(i-1)(i+2)/2]$, $k \geq i - 1$. It is clear that $\{T_i\}_{i=1}^{\infty}$ is a sequence of nonexpansive mappinps with common fixed point zero and A is a 1/2-Lipschitzian mapping from K to itself. Then it follows by Theorem 3.3 that $\{x_n\}$ converges strongly to zero, since in R^1 weak convergence is equivalent to strong convergence. The numerical experiment outcome obtained by using MATLAB 7.10.0.499 shows that as $x_1 = 1$, the computations of x_{10}, x_{30} and x_{50} are 0.249844611882226, 0.00414201993416185, and 5.55601413578197e-05, respectively. This example illustrates the efficiency of approximation of common fixed points of an infinite family of nonexpansive mappings.

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Competing interests

The authors declare that they have no competing interests.

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