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Convergence theorems for mixed type asymptotically nonexpansive mappings

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Abstract

In this paper, we introduce a new two-step iterative scheme of mixed type for two asymptotically nonexpansive self-mappings and two asymptotically nonexpansive nonself-mappings and prove strong and weak convergence theorems for the new two-step iterative scheme in uniformly convex Banach spaces.

Keywords: mixed type asymptotically nonexpansive mapping; strong and weak convergence; common fixed point; uniformly convex Banach space

1 Introduction

Let K be a nonempty subset of a real normed linear space E. A mapping $T : K \to K$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\| \tag{1.1}$$

for all $x, y \in K$ and $n \ge 1$.

In 1972, Goebel and Kirk [1] introduced the class of asymptotically nonexpansive self-mappings, which is an important generalization of the class of nonexpansive self-mappings, and proved that if K is a nonempty closed convex subset of a real uniformly convex Banach space E and T is an asymptotically nonexpansive self-mapping of K, then T has a fixed point.

Since then, some authors proved weak and strong convergence theorems for asymptotically nonexpansive self-mappings in Banach spaces (see [2–16]), which extend and improve the result of Goebel and Kirk in several ways.

Recently, Chidume *et al.* [10] introduced the concept of asymptotically nonexpansive nonself-mappings, which is a generalization of an asymptotically nonexpansive self-mapping, as follows.

Definition 1.1 [10] Let K be a nonempty subset of a real normed linear space E. Let $P : E \to K$ be a nonexpansive retraction of E onto K. A nonself-mapping $T : K \to E$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \le k_n \|x - y\| \tag{1.2}$$

for all $x, y \in K$ and $n \ge 1$.

Let K be a nonempty closed convex subset of a real uniformly convex Banach space E. In 2003, also, Chidume *et al.* [10] studied the following iteration scheme:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = P((1-\alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}x_n) \end{cases} \tag{1.3}$$

for each $n \geq 1$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ and P is a nonexpansive retraction of E onto K, and proved some strong and weak convergence theorems for an asymptotically nonexpansive nonself-mapping.

In 2006, Wang [11] generalized the iteration process (1.3) as follows:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = P((1-\alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}y_n), \\ y_n = P((1-\beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n) \end{cases} \tag{1.4}$$

for each $n \geq 1$, where $T_1, T_2 : K \to E$ are two asymptotically nonexpansive nonself-mappings and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[0,1)$, and proved some strong and weak convergence theorems for two asymptotically nonexpansive nonself-mappings. Recently, Guo and Guo [12] proved some new weak convergence theorems for the iteration process (1.4).

The purpose of this paper is to construct a new iteration scheme of mixed type for two asymptotically nonexpansive self-mappings and two asymptotically nonexpansive nonself-mappings and to prove some strong and weak convergence theorems for the new iteration scheme in uniformly convex Banach spaces.

2 Preliminaries

Let E be a real Banach space, K be a nonempty closed convex subset of E and $P : E \to K$ be a nonexpansive retraction of E onto K. Let $S_1, S_2 : K \to K$ be two asymptotically nonexpansive self-mappings and $T_1, T_2 : K \to E$ be two asymptotically nonexpansive nonself-mappings. Then we define the new iteration scheme of mixed type as follows:

$$\begin{cases} x_1 \in K, \\ x_{n+1} = P((1-\alpha_n)S_1^n x_n + \alpha_n T_1(PT_1)^{n-1}y_n), \\ y_n = P((1-\beta_n)S_2^n x_n + \beta_n T_2(PT_2)^{n-1}x_n) \end{cases} \tag{2.1}$$

for each $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $[0,1)$.

If S_1 and S_2 are the identity mappings, then the iterative scheme (2.1) reduces to the sequence (1.4).

We denote the set of common fixed points of S_1, S_2, T_1 and T_2 by $F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$ and denote the distance between a point z and a set A in E by $d(z,A) = \inf_{x \in A} \|z - x\|$.

Now, we recall some well-known concepts and results.

Let E be a real Banach space, E^* be the dual space of E and $J : E \to 2^{E^*}$ be the *normalized duality mapping* defined by

$$J(x) = \{ f \in E^* : \langle x, f \rangle = \|x\| \|f\|, \|f\| = \|x\| \}$$

for all $x \in E$, where $\langle \cdot, \cdot \rangle$ denotes duality pairing between E and E^*. A single-valued normalized duality mapping is denoted by j.

A subset K of a real Banach space E is called a *retract* of E [10] if there exists a continuous mapping $P : E \to K$ such that $Px = x$ for all $x \in K$. Every closed convex subset of a uniformly convex Banach space is a retract. A mapping $P : E \to E$ is called a *retraction* if $P^2 = P$. It follows that if a mapping P is a retraction, then $Py = y$ for all y in the range of P.

A Banach space E is said to satisfy *Opial's condition* [17] if, for any sequence $\{x_n\}$ of E, $x_n \to x$ weakly as $n \to \infty$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$.

A Banach space E is said to have a *Fréchet differentiable norm* [18] if, for all $x \in U = \{x \in E : \|x\| = 1\}$,

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists and is attained uniformly in $y \in U$.

A Banach space E is said to have the *Kadec-Klee property* [19] if for every sequence $\{x_n\}$ in E, $x_n \to x$ weakly and $\|x_n\| \to \|x\|$, it follows that $x_n \to x$ strongly.

Let K be a nonempty closed subset of a real Banach space E. A nonself-mapping $T : K \to E$ is said to be *semi-compact* [11] if, for any sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to some $x^* \in K$.

Lemma 2.1 [15] *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonnegative sequences satisfying the following condition*:

$$a_{n+1} \leq (1 + b_n)a_n + c_n$$

for each $n \geq n_0$, where n_0 is some nonnegative integer, $\sum_{n=n_0}^{\infty} b_n < \infty$ and $\sum_{n=n_0}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.2 [8] *Let E be a real uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for each $n \geq 1$. Also, suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that*

$$\limsup_{n\to\infty} \|x_n\| \leq r, \qquad \limsup_{n\to\infty} \|y_n\| \leq r, \qquad \lim_{n\to\infty} \big\| t_n x_n + (1 - t_n) y_n \big\| = r$$

hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.3 [10] *Let E be a real uniformly convex Banach space, K be a nonempty closed convex subset of E and $T : K \to E$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1, \infty)$ and $k_n \to 1$ as $n \to \infty$. Then $I - T$ is demiclosed at zero, i.e., if $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then $x \in F(T)$, where $F(T)$ is the set of fixed points of T.*

Lemma 2.4 [16] *Let X be a uniformly convex Banach space and C be a convex subset of X. Then there exists a strictly increasing continuous convex function $\gamma : [0,\infty) \to [0,\infty)$ with $\gamma(0) = 0$ such that, for each mapping $S : C \to C$ with a Lipschitz constant $L > 0$,*

$$\left\|\alpha Sx + (1-\alpha)Sy - S\left[\alpha x + (1-\alpha)y\right]\right\| \le L\gamma^{-1}\left(\|x-y\| - \frac{1}{L}\|Sx - Sy\|\right)$$

for all $x, y \in C$ and $0 < \alpha < 1$.

Lemma 2.5 [16] *Let X be a uniformly convex Banach space such that its dual space X^* has the Kadec-Klee property. Suppose $\{x_n\}$ is a bounded sequence and $f_1, f_2 \in W_w(\{x_n\})$ such that*

$$\lim_{n\to\infty}\left\|\alpha x_n + (1-\alpha)f_1 - f_2\right\|$$

exists for all $\alpha \in [0,1]$, where $W_w(\{x_n\})$ denotes the set of all weak subsequential limits of $\{x_n\}$. Then $f_1 = f_2$.

3 Strong convergence theorems

In this section, we prove strong convergence theorems for the iterative scheme given in (2.1) in uniformly convex Banach spaces.

Lemma 3.1 *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $S_1, S_2 : K \to K$ be two asymptotically nonexpansive self-mappings with $\{k_n^{(1)}\}, \{k_n^{(2)}\} \subset [1,\infty)$ and $T_1, T_2 : K \to E$ be two asymptotically nonexpansive nonself-mappings with $\{l_n^{(1)}\}, \{l_n^{(2)}\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ and $\sum_{n=1}^{\infty}(l_n^{(i)} - 1) < \infty$ for $i = 1, 2$, respectively, and $F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (2.1), where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0,1)$. Then*

(1) *$\lim_{n\to\infty}\|x_n - q\|$ exists for any $q \in F$;*
(2) *$\lim_{n\to\infty} d(x_n, F)$ exists.*

Proof (1) Set $h_n = \max\{k_n^{(1)}, k_n^{(2)}, l_n^{(1)}, l_n^{(2)}\}$. For any $q \in F$, it follows from (2.1) that

$$\begin{aligned}\|y_n - q\| &\le \left\|(1-\beta_n)\left(S_2^n x_n - q\right) + \beta_n\left(T_2(PT_2)^{n-1}x_n - q\right)\right\| \\ &\le (1-\beta_n)h_n\|x_n - q\| + \beta_n h_n\|x_n - q\| \\ &= h_n\|x_n - q\| \end{aligned} \tag{3.1}$$

and so

$$\begin{aligned}\|x_{n+1} - q\| &\le \left\|(1-\alpha_n)\left(S_1^n x_n - q\right) + \alpha_n\left(T_1(PT_1)^{n-1}y_n - q\right)\right\| \\ &\le (1-\alpha_n)h_n\|x_n - q\| + \alpha_n h_n\|y_n - q\| \\ &\le (1-\alpha_n)h_n^2\|x_n - q\| + \alpha_n h_n^2\|x_n - q\| \\ &= \left[1 + \left(h_n^2 - 1\right)\right]\|x_n - q\|. \end{aligned} \tag{3.2}$$

Since $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ and $\sum_{n=1}^{\infty}(l_n^{(i)} - 1) < \infty$ for $i = 1, 2$, we have $\sum_{n=1}^{\infty}(h_n^2 - 1) < \infty$. It follows from Lemma 2.1 that $\lim_{n\to\infty}\|x_n - q\|$ exists.

(2) Taking the infimum over all $q \in F$ in (3.2), we have

$$d(x_{n+1}, F) \leq \left[1 + \left(h_n^2 - 1\right)\right] d(x_n, F)$$

for each $n \geq 1$. It follows from $\sum_{n=1}^{\infty}(h_n^2 - 1) < \infty$ and Lemma 2.1 that the conclusion (2) holds. This completes the proof. □

Lemma 3.2 *Let E be a real uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $S_1, S_2 : K \to K$ be two asymptotically nonexpansive self-mappings with $\{k_n^{(1)}\}, \{k_n^{(2)}\} \subset [1, \infty)$ and $T_1, T_2 : K \to E$ be two asymptotically nonexpansive nonself-mappings with $\{l_n^{(1)}\}, \{l_n^{(2)}\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)} - 1) < \infty$ and $\sum_{n=1}^{\infty}(l_n^{(i)} - 1) < \infty$ for $i = 1, 2$, respectively, and $F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by* (2.1) *and the following conditions hold*:

(a) *$\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[\epsilon, 1 - \epsilon]$ for some $\epsilon \in (0, 1)$;*

(b) *$\|x - T_i y\| \leq \|S_i x - T_i y\|$ for all $x, y \in K$ and $i = 1, 2$.*

Then $\lim_{n\to\infty} \|x_n - S_i x_n\| = \lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for $i = 1, 2$.

Proof Set $h_n = \max\{k_n^{(1)}, k_n^{(2)}, l_n^{(1)}, l_n^{(2)}\}$. For any given $q \in F$, $\lim_{n\to\infty} \|x_n - q\|$ exists by Lemma 3.1. Now, we assume that $\lim_{n\to\infty} \|x_n - q\| = c$. It follows from (3.2) and $\sum_{n=1}^{\infty}(h_n^2 - 1) < \infty$ that

$$\lim_{n\to\infty} \left\| (1 - \alpha_n)\left(S_1^n x_n - q\right) + \alpha_n \left(T_1 (PT_1)^{n-1} y_n - q\right) \right\| = c$$

and

$$\limsup_{n\to\infty} \left\| S_1^n x_n - q \right\| \leq \limsup_{n\to\infty} k_n^{(1)} \|x_n - q\| = c.$$

Taking lim sup on both sides in (3.1), we obtain $\limsup_{n\to\infty} \|y_n - q\| \leq c$ and so

$$\limsup_{n\to\infty} \left\| T_1 (PT_1)^{n-1} y_n - q \right\| \leq \limsup_{n\to\infty} l_n^{(1)} \|y_n - q\| \leq c.$$

Using Lemma 2.2, we have

$$\lim_{n\to\infty} \left\| S_1^n x_n - T_1 (PT_1)^{n-1} y_n \right\| = 0. \tag{3.3}$$

By the condition (b), it follows that

$$\left\| x_n - T_1 (PT_1)^{n-1} y_n \right\| \leq \left\| S_1^n x_n - T_1 (PT_1)^{n-1} y_n \right\|$$

and so, from (3.3), we have

$$\lim_{n\to\infty} \left\| x_n - T_1 (PT_1)^{n-1} y_n \right\| = 0. \tag{3.4}$$

Since

$$\begin{aligned}
\|x_n - q\| &\leq \left\| x_n - T_1 (PT_1)^{n-1} y_n \right\| + \left\| T_1 (PT_1)^{n-1} y_n - q \right\| \\
&\leq \left\| x_n - T_1 (PT_1)^{n-1} y_n \right\| + l_n^{(1)} \|y_n - q\|.
\end{aligned}$$

Taking lim inf on both sides in the inequality above, we have

$$\liminf_{n\to\infty} \|y_n - q\| \geq c$$

by (3.4) and so

$$\lim_{n\to\infty} \|y_n - q\| = c.$$

Using (3.1), we have

$$\lim_{n\to\infty} \left\|(1-\beta_n)\left(S_2^n x_n - q\right) + \beta_n\left(T_2(PT_2)^{n-1}x_n - q\right)\right\| = c.$$

In addition, we have

$$\limsup_{n\to\infty} \left\|S_2^n x_n - q\right\| \leq \limsup_{n\to\infty} k_n^{(2)} \|x_n - q\| = c$$

and

$$\limsup_{n\to\infty} \left\|T_2(PT_2)^{n-1}x_n - q\right\| \leq \limsup_{n\to\infty} l_n^{(2)} \|x_n - q\| = c.$$

It follows from Lemma 2.2 that

$$\lim_{n\to\infty} \left\|S_2^n x_n - T_2(PT_2)^{n-1}x_n\right\| = 0. \tag{3.5}$$

Now, we prove that

$$\lim_{n\to\infty} \|x_n - T_1 x_n\| = \lim_{n\to\infty} \|x_n - T_2 x_n\| = 0.$$

Indeed, since $\|x_n - T_2(PT_2)^{n-1}x_n\| \leq \|S_2^n x_n - T_2(PT_2)^{n-1}x_n\|$ by the condition (b). It follows from (3.5) that

$$\lim_{n\to\infty} \left\|x_n - T_2(PT_2)^{n-1}x_n\right\| = 0. \tag{3.6}$$

Since $S_2^n x_n = P(S_2^n x_n)$ and $P : E \to K$ is a nonexpansive retraction of E onto K, we have

$$\left\|y_n - S_2^n x_n\right\| \leq \beta_n \left\|S_2^n x_n - T_2(PT_2)^{n-1}x_n\right\|$$

and so

$$\lim_{n\to\infty} \left\|y_n - S_2^n x_n\right\| = 0. \tag{3.7}$$

Furthermore, we have

$$\|y_n - x_n\| \leq \left\|y_n - S_2^n x_n\right\| + \left\|S_2^n x_n - T_2(PT_2)^{n-1}x_n\right\| + \left\|T_2(PT_2)^{n-1}x_n - x_n\right\|.$$

Thus it follows from (3.5), (3.6) and (3.7) that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.8}$$

Since $\|x_n - T_1(PT_1)^{n-1}x_n\| \le \|S_1^n x_n - T_1(PT_1)^{n-1}x_n\|$ by the condition (b) and

$$\begin{aligned} &\left\|S_1^n x_n - T_1(PT_1)^{n-1}x_n\right\| \\ &\quad \le \left\|S_1^n x_n - T_1(PT_1)^{n-1}y_n\right\| + \left\|T_1(PT_1)^{n-1}y_n - T_1(PT_1)^{n-1}x_n\right\| \\ &\quad \le \left\|S_1^n x_n - T_1(PT_1)^{n-1}y_n\right\| + l_n^{(1)}\|y_n - x_n\|. \end{aligned}$$

Using (3.3) and (3.8), we have

$$\lim_{n\to\infty} \left\|S_1^n x_n - T_1(PT_1)^{n-1}x_n\right\| = 0 \tag{3.9}$$

and

$$\lim_{n\to\infty} \left\|x_n - T_1(PT_1)^{n-1}x_n\right\| = 0. \tag{3.10}$$

It follows from

$$\begin{aligned} \left\|x_{n+1} - S_1^n x_n\right\| &= \left\|P\left[(1-\alpha_n)S_1^n x_n + \alpha_n T_1(PT_1)^{n-1}y_n\right] - P\left(S_1^n x_n\right)\right\| \\ &\le \alpha_n \left\|S_1^n x_n - T_1(PT_1)^{n-1}y_n\right\| \end{aligned}$$

and (3.3) that

$$\lim_{n\to\infty} \left\|x_{n+1} - S_1^n x_n\right\| = 0. \tag{3.11}$$

In addition, we have

$$\left\|x_{n+1} - T_1(PT_1)^{n-1}y_n\right\| \le \left\|x_{n+1} - S_1^n x_n\right\| + \left\|S_1^n x_n - T_1(PT_1)^{n-1}y_n\right\|.$$

Using (3.3) and (3.11), we obtain that

$$\lim_{n\to\infty} \left\|x_{n+1} - T_1(PT_1)^{n-1}y_n\right\| = 0. \tag{3.12}$$

Thus, using (3.9), (3.10) and the inequality

$$\left\|S_1^n x_n - x_n\right\| \le \left\|S_1^n x_n - T_1(PT_1)^{n-1}x_n\right\| + \left\|T_1(PT_1)^{n-1}x_n - x_n\right\|,$$

we have $\lim_{n\to\infty} \|S_1^n x_n - x_n\| = 0$. It follows from (3.6) and the inequality

$$\left\|S_1^n x_n - T_2(PT_2)^{n-1}x_n\right\| \le \left\|S_1^n x_n - x_n\right\| + \left\|x_n - T_2(PT_2)^{n-1}x_n\right\|$$

that

$$\lim_{n\to\infty} \left\|S_1^n x_n - T_2(PT_2)^{n-1}x_n\right\| = 0. \tag{3.13}$$

Since

$$\left\|x_{n+1} - T_2(PT_2)^{n-1}y_n\right\| \leq \left\|x_{n+1} - S_1^n x_n\right\| + \left\|S_1^n x_n - T_2(PT_2)^{n-1}x_n\right\| + l_n^{(2)}\|x_n - y_n\|,$$

from (3.8), (3.11) and (3.13), it follows that

$$\lim_{n\to\infty}\left\|x_{n+1} - T_2(PT_2)^{n-1}y_n\right\| = 0. \tag{3.14}$$

Again, since $(PT_i)(PT_i)^{n-2}y_{n-1}, x_n \in K$ for $i = 1, 2$ and T_1, T_2 are two asymptotically nonexpansive nonself-mappings, we have

$$\begin{aligned}
&\left\|T_i(PT_i)^{n-1}y_{n-1} - T_i x_n\right\| \\
&\quad = \left\|T_i\left[(PT_i)(PT_i)^{n-2}y_{n-1}\right] - T_i(Px_n)\right\| \\
&\quad \leq \max\left\{l_1^{(1)}, l_1^{(2)}\right\}\left\|(PT_i)(PT_i)^{n-2}y_{n-1} - Px_n\right\| \\
&\quad \leq \max\left\{l_1^{(1)}, l_1^{(2)}\right\}\left\|T_i(PT_i)^{n-2}y_{n-1} - x_n\right\|
\end{aligned} \tag{3.15}$$

for $i = 1, 2$. It follows from (3.12), (3.14) and (3.15) that

$$\lim_{n\to\infty}\left\|T_i(PT_i)^{n-1}y_{n-1} - T_i x_n\right\| = 0 \tag{3.16}$$

for $i = 1, 2$. Moreover, we have

$$\|x_{n+1} - y_n\| \leq \left\|x_{n+1} - T_1(PT_1)^{n-1}y_n\right\| + \left\|T_1(PT_1)^{n-1}y_n - x_n\right\| + \|x_n - y_n\|.$$

Using (3.4), (3.8) and (3.12), we have

$$\lim_{n\to\infty}\|x_{n+1} - y_n\| = 0. \tag{3.17}$$

In addition, we have

$$\begin{aligned}
\|x_n - T_i x_n\| &\leq \left\|x_n - T_i(PT_i)^{n-1}x_n\right\| + \left\|T_i(PT_i)^{n-1}x_n - T_i(PT_i)^{n-1}y_{n-1}\right\| \\
&\quad + \left\|T_i(PT_i)^{n-1}y_{n-1} - T_i x_n\right\| \\
&\leq \left\|x_n - T_i(PT_i)^{n-1}x_n\right\| + \max\left\{\sup_{n\geq 1} l_n^{(1)}, \sup_{n\geq 1} l_n^{(2)}\right\}\|x_n - y_{n-1}\| \\
&\quad + \left\|T_i(PT_i)^{n-1}y_{n-1} - T_i x_n\right\|
\end{aligned}$$

for $i = 1, 2$. Thus it follows from (3.6), (3.10), (3.16) and (3.17) that

$$\lim_{n\to\infty}\|x_n - T_1 x_n\| = \lim_{n\to\infty}\|x_n - T_2 x_n\| = 0.$$

Finally, we prove that

$$\lim_{n\to\infty}\|x_n - S_1 x_n\| = \lim_{n\to\infty}\|x_n - S_2 x_n\| = 0.$$

In fact, by the condition (b), we have

$$\begin{aligned}\|x_n - S_i x_n\| &\leq \|x_n - T_i(PT_i)^{n-1}x_n\| + \|S_i x_n - T_i(PT_i)^{n-1}x_n\| \\ &\leq \|x_n - T_i(PT_i)^{n-1}x_n\| + \|S_i^n x_n - T_i(PT_i)^{n-1}x_n\|\end{aligned}$$

for $i = 1, 2$. Thus it follows from (3.5), (3.6), (3.9) and (3.10) that

$$\lim_{n\to\infty} \|x_n - S_1 x_n\| = \lim_{n\to\infty} \|x_n - S_2 x_n\| = 0.$$

This completes the proof. □

Now, we find two mappings, $S_1 = S_2 = S$ and $T_1 = T_2 = T$, satisfying the condition (b) in Lemma 3.2 as follows.

Example 3.1 [20] Let $\mathbb{R}$ be the real line with the usual norm $|\cdot|$ and let $K = [-1, 1]$. Define two mappings $S, T : K \to K$ by

$$Tx = \begin{cases} -2\sin\frac{x}{2}, & \text{if } x \in [0,1], \\ 2\sin\frac{x}{2}, & \text{if } x \in [-1,0), \end{cases}$$

and

$$Sx = \begin{cases} x, & \text{if } x \in [0,1], \\ -x, & \text{if } x \in [-1,0). \end{cases}$$

Now, we show that T is nonexpansive. In fact, if $x, y \in [0,1]$ or $x, y \in [-1, 0)$, then we have

$$|Tx - Ty| = 2\left|\sin\frac{x}{2} - \sin\frac{y}{2}\right| \leq |x - y|.$$

If $x \in [0,1]$ and $y \in [-1,0)$ or $x \in [-1,0)$ and $y \in [0,1]$, then we have

$$\begin{aligned}|Tx - Ty| &= 2\left|\sin\frac{x}{2} + \sin\frac{y}{2}\right| \\ &= 4\left|\sin\frac{x+y}{4}\cos\frac{x-y}{4}\right| \\ &\leq |x+y| \\ &\leq |x-y|.\end{aligned}$$

This implies that T is nonexpansive and so T is an asymptotically nonexpansive mapping with $k_n = 1$ for each $n \geq 1$. Similarly, we can show that S is an asymptotically nonexpansive mapping with $l_n = 1$ for each $n \geq 1$.

Next, we show that two mappings S, T satisfy the condition (b) in Lemma 3.2. For this, we consider the following cases:

Case 1. Let $x, y \in [0,1]$. Then we have

$$|x - Ty| = \left|x + 2\sin\frac{y}{2}\right| = |Sx - Ty|.$$

Case 2. Let $x, y \in [-1,0)$. Then we have

$$|x - Ty| = \left|x - 2\sin\frac{y}{2}\right| \le \left|-x - 2\sin\frac{y}{2}\right| = |Sx - Ty|.$$

Case 3. Let $x \in [-1,0)$ and $y \in [0,1]$. Then we have

$$|x - Ty| = \left|x + 2\sin\frac{y}{2}\right| \le \left|-x + 2\sin\frac{y}{2}\right| = |Sx - Ty|.$$

Case 4. Let $x \in [0,1]$ and $y \in [-1,0)$. Then we have

$$|x - Ty| = \left|x - 2\sin\frac{y}{2}\right| = |Sx - Ty|.$$

Therefore, the condition (b) in Lemma 3.2 is satisfied.

Theorem 3.1 *Under the assumptions of Lemma* 3.2, *if one of* S_1, S_2, T_1 *and* T_2 *is completely continuous, then the sequence* $\{x_n\}$ *defined by* (2.1) *converges strongly to a common fixed point of* S_1, S_2, T_1 *and* T_2.

Proof Without loss of generality, we can assume that S_1 is completely continuous. Since $\{x_n\}$ is bounded by Lemma 3.1, there exists a subsequence $\{S_1x_{n_j}\}$ of $\{S_1x_n\}$ such that $\{S_1x_{n_j}\}$ converges strongly to some q^*. Moreover, we know that

$$\lim_{j\to\infty}\|x_{n_j} - S_1x_{n_j}\| = \lim_{j\to\infty}\|x_{n_j} - S_2x_{n_j}\| = 0$$

and

$$\lim_{j\to\infty}\|x_{n_j} - T_1x_{n_j}\| = \lim_{j\to\infty}\|x_{n_j} - T_2x_{n_j}\| = 0$$

by Lemma 3.2, which imply that

$$\left\|x_{n_j} - q^*\right\| \le \|x_{n_j} - S_1x_{n_j}\| + \left\|S_1x_{n_j} - q^*\right\| \to 0$$

as $j \to \infty$ and so $x_{n_j} \to q^* \in K$. Thus, by the continuity of S_1, S_2, T_1 and T_2, we have

$$\left\|q^* - S_iq^*\right\| = \lim_{j\to\infty}\|x_{n_j} - S_ix_{n_j}\| = 0$$

and

$$\left\|q^* - T_iq^*\right\| = \lim_{j\to\infty}\|x_{n_j} - T_ix_{n_j}\| = 0$$

for $i = 1,2$. Thus it follows that $q^* \in F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$. Furthermore, since

$\lim_{n\to\infty} \|x_n - q^*\|$ exists by Lemma 3.1, we have $\lim_{n\to\infty} \|x_n - q^*\| = 0$. This completes the proof. □

Theorem 3.2 *Under the assumptions of Lemma* 3.2, *if one of S_1, S_2, T_1 and T_2 is semi-compact, then the sequence $\{x_n\}$ defined by* (2.1) *converges strongly to a common fixed point of S_1, S_2, T_1 and T_2.*

Proof Since $\lim_{n\to\infty} \|x_n - S_i x_n\| = \lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for $i = 1, 2$ by Lemma 3.2 and one of S_1, S_2, T_1 and T_2 is semi-compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to some $q^* \in K$. Moreover, by the continuity of S_1, S_2, T_1 and T_2, we have $\|q^* - S_i q^*\| = \lim_{j\to\infty} \|x_{n_j} - S_i x_{n_j}\| = 0$ and $\|q^* - T_i q^*\| = \lim_{j\to\infty} \|x_{n_j} - T_i x_{n_j}\| = 0$ for $i = 1, 2$. Thus it follows that $q^* \in F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$. Since $\lim_{n\to\infty} \|x_n - q^*\|$ exists by Lemma 3.1, we have $\lim_{n\to\infty} \|x_n - q^*\| = 0$. This completes the proof. □

Theorem 3.3 *Under the assumptions of Lemma* 3.2, *if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that*

$$f\big(d(x,F)\big) \le \|x - S_1 x\| + \|x - S_2 x\| + \|x - T_1 x\| + \|x - T_2 x\|$$

for all $x \in K$, where $F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$, then the sequence $\{x_n\}$ defined by (2.1) *converges strongly to a common fixed point of S_1, S_2, T_1 and T_2.*

Proof Since $\lim_{n\to\infty} \|x_n - S_i x_n\| = \lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for $i = 1, 2$ by Lemma 3.2, we have $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since $f : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ and $\lim_{n\to\infty} d(x_n, F)$ exists by Lemma 3.1, we have $\lim_{n\to\infty} d(x_n, F) = 0$.

Now, we show that $\{x_n\}$ is a Cauchy sequence in K. In fact, from (3.2), we have

$$\|x_{n+1} - q\| \le \big[1 + \big(h_n^2 - 1\big)\big]\|x_n - q\|$$

for each $n \ge 1$, where $h_n = \max\{k_n^{(1)}, k_n^{(2)}, l_n^{(1)}, l_n^{(2)}\}$ and $q \in F$. For any m, n, $m > n \ge 1$, we have

$$\begin{aligned} \|x_m - q\| &\le \big[1 + \big(h_{m-1}^2 - 1\big)\big]\|x_{m-1} - q\| \\ &\le e^{h_{m-1}^2 - 1}\|x_{m-1} - q\| \\ &\le e^{h_{m-1}^2 - 1} e^{h_{m-2}^2 - 1}\|x_{m-2} - q\| \\ &\le \cdots \\ &\le e^{\sum_{i=n}^{m-1}(h_i^2 - 1)}\|x_n - q\| \\ &\le M\|x_n - q\|, \end{aligned}$$

where $M = e^{\sum_{i=1}^{\infty}(h_i^2 - 1)}$. Thus, for any $q \in F$, we have

$$\begin{aligned} \|x_n - x_m\| &\le \|x_n - q\| + \|x_m - q\| \\ &\le (1 + M)\|x_n - q\|. \end{aligned}$$

Taking the infimum over all $q \in F$, we obtain

$$\|x_n - x_m\| \leq (1 + M)d(x_n, F).$$

Thus it follows from $\lim_{n\to\infty} d(x_n, F) = 0$ that $\{x_n\}$ is a Cauchy sequence. Since K is a closed subset of E, the sequence $\{x_n\}$ converges strongly to some $q^* \in K$. It is easy to prove that $F(S_1)$, $F(S_2)$, $F(T_1)$ and $F(T_2)$ are all closed and so F is a closed subset of K. Since $\lim_{n\to\infty} d(x_n, F) = 0$, $q^* \in F$, the sequence $\{x_n\}$ converges strongly to a common fixed point of S_1, S_2, T_1 and T_2. This completes the proof. □

4 Weak convergence theorems

In this section, we prove weak convergence theorems for the iterative scheme defined by (2.1) in uniformly convex Banach spaces.

Lemma 4.1 *Under the assumptions of Lemma* 3.1, *for all* $q_1, q_2 \in F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$, *the limit*

$$\lim_{n\to\infty} \left\| tx_n + (1-t)q_1 - q_2 \right\|$$

exists for all $t \in [0,1]$, *where* $\{x_n\}$ *is the sequence defined by* (2.1).

Proof Set $a_n(t) = \|tx_n + (1-t)q_1 - q_2\|$. Then $\lim_{n\to\infty} a_n(0) = \|q_1 - q_2\|$ and, from Lemma 3.1, $\lim_{n\to\infty} a_n(1) = \lim_{n\to\infty} \|x_n - q_2\|$ exists. Thus it remains to prove Lemma 4.1 for any $t \in (0,1)$.

Define the mapping $G_n : K \to K$ by

$$G_n x = P\big[(1-\alpha_n)S_1^n x + \alpha_n T_1(PT_1)^{n-1} P\big((1-\beta_n)S_2^n x + \beta_n T_2(PT_2)^{n-1} x\big)\big]$$

for all $x \in K$. It is easy to prove that

$$\|G_n x - G_n y\| \leq h_n^4 \|x - y\| \tag{4.1}$$

for all $x, y \in K$, where $h_n = \max\{k_n^{(1)}, k_n^{(2)}, l_n^{(1)}, l_n^{(2)}\}$. Letting $h_n = 1 + \nu_n$, it follows from $1 \leq \prod_{j=n}^{\infty} h_j^4 \leq e^{4\sum_{j=n}^{\infty} \nu_j}$ and $\sum_{n=1}^{\infty} \nu_n < \infty$ that $\lim_{n\to\infty} \prod_{j=n}^{\infty} h_j^4 = 1$. Setting

$$S_{n,m} = G_{n+m-1} G_{n+m-2} \cdots G_n \tag{4.2}$$

for each $m \geq 1$, from (4.1) and (4.2), it follows that

$$\|S_{n,m}x - S_{n,m}y\| \leq \left(\prod_{j=n}^{n+m-1} h_j^4 \right) \|x - y\|$$

for all $x, y \in K$ and $S_{n,m}x_n = x_{n+m}$, $S_{n,m}q = q$ for any $q \in F$. Let

$$b_{n,m} = \left\| tS_{n,m}x_n + (1-t)S_{n,m}q_1 - S_{n,m}\big(tx_n + (1-t)q_1\big) \right\|. \tag{4.3}$$

Then, using (4.3) and Lemma 2.4, we have

$$\begin{aligned}
b_{n,m} &\le \left(\prod_{j=n}^{n+m-1} h_j^4\right)\gamma^{-1}\left(\|x_n - q_1\| - \left(\prod_{j=n}^{n+m-1} h_j^4\right)^{-1}\|S_{n,m}x_n - S_{n,m}q_1\|\right) \\
&\le \left(\prod_{j=n}^{\infty} h_j^4\right)\gamma^{-1}\left(\|x_n - q_1\| - \left(\prod_{j=n}^{\infty} h_j^4\right)^{-1}\|x_{n+m} - q_1\|\right).
\end{aligned}$$

It follows from Lemma 3.1 and $\lim_{n\to\infty}\prod_{j=n}^{\infty} h_j^4 = 1$ that $\lim_{n\to\infty} b_{n,m} = 0$ uniformly for all m. Observe that

$$\begin{aligned}
a_{n+m}(t) &\le \left\|S_{n,m}\big(tx_n + (1-t)q_1\big) - q_2\right\| + b_{n,m} \\
&= \left\|S_{n,m}\big(tx_n + (1-t)q_1\big) - S_{n,m}q_2\right\| + b_{n,m} \\
&\le \left(\prod_{j=n}^{n+m-1} h_j^4\right)\left\|tx_n + (1-t)q_1 - q_2\right\| + b_{n,m} \\
&\le \left(\prod_{j=n}^{\infty} h_j^4\right)a_n(t) + b_{n,m}.
\end{aligned}$$

Thus we have $\limsup_{n\to\infty} a_n(t) \le \liminf_{n\to\infty} a_n(t)$, that is, $\lim_{n\to\infty}\|tx_n + (1-t)q_1 - q_2\|$ exists for all $t \in (0,1)$. This completes the proof. □

Lemma 4.2 *Under the assumptions of Lemma* 3.1, *if E has a Fréchet differentiable norm, then, for all $q_1, q_2 \in F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$, the limit*

$$\lim_{n\to\infty}\big\langle x_n, j(q_1 - q_2)\big\rangle$$

exists, where $\{x_n\}$ is the sequence defined by (2.1). *Furthermore, if $W_w(\{x_n\})$ denotes the set of all weak subsequential limits of $\{x_n\}$, then $\langle x^* - y^*, j(q_1 - q_2)\rangle = 0$ for all $q_1, q_2 \in F$ and $x^*, y^* \in W_w(\{x_n\})$.*

Proof This follows basically as in the proof of Lemma 3.2 of [12] using Lemma 4.1 instead of Lemma 3.1 of [12]. □

Theorem 4.1 *Under the assumptions of Lemma* 3.2, *if E has a Fréchet differentiable norm, then the sequence $\{x_n\}$ defined by* (2.1) *converges weakly to a common fixed point of S_1, S_2, T_1 and T_2.*

Proof Since E is a uniformly convex Banach space and the sequence $\{x_n\}$ is bounded by Lemma 3.1, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to some $q \in K$. By Lemma 3.2, we have

$$\lim_{k\to\infty}\|x_{n_k} - S_i x_{n_k}\| = \lim_{k\to\infty}\|x_{n_k} - T_i x_{n_k}\| = 0$$

for $i = 1, 2$. It follows from Lemma 2.3 that $q \in F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$.

Now, we prove that the sequence $\{x_n\}$ converges weakly to q. Suppose that there exists a subsequence $\{x_{m_j}\}$ of $\{x_n\}$ such that $\{x_{m_j}\}$ converges weakly to some $q_1 \in K$. Then, by the same method given above, we can also prove that $q_1 \in F$. So, $q, q_1 \in F \cap W_w(\{x_n\})$. It follows from Lemma 4.2 that

$$\|q - q_1\|^2 = \langle q - q_1, j(q - q_1)\rangle = 0.$$

Therefore, $q_1 = q$, which shows that the sequence $\{x_n\}$ converges weakly to q. This completes the proof. □

Theorem 4.2 *Under the assumptions of Lemma* 3.2, *if the dual space* E^* *of* E *has the Kadec-Klee property, then the sequence* $\{x_n\}$ *defined by* (2.1) *converges weakly to a common fixed point of* S_1, S_2, T_1 *and* T_2.

Proof Using the same method given in Theorem 4.1, we can prove that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to some $q \in F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$.

Now, we prove that the sequence $\{x_n\}$ converges weakly to q. Suppose that there exists a subsequence $\{x_{m_j}\}$ of $\{x_n\}$ such that $\{x_{m_j}\}$ converges weakly to some $q^* \in K$. Then, as for q, we have $q^* \in F$. It follows from Lemma 4.1 that the limit

$$\lim_{n\to\infty} \left\| t x_n + (1-t)q - q^* \right\|$$

exists for all $t \in [0,1]$. Again, since $q, q^* \in W_w(\{x_n\})$, $q^* = q$ by Lemma 2.5. This shows that the sequence $\{x_n\}$ converges weakly to q. This completes the proof. □

Theorem 4.3 *Under the assumptions of Lemma* 3.2, *if* E *satisfies Opial's condition, then the sequence* $\{x_n\}$ *defined by* (2.1) *converges weakly to a common fixed point of* S_1, S_2, T_1 *and* T_2.

Proof Using the same method as given in Theorem 4.1, we can prove that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to some $q \in F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$.

Now, we prove that the sequence $\{x_n\}$ converges weakly to q. Suppose that there exists a subsequence $\{x_{m_j}\}$ of $\{x_n\}$ such that $\{x_{m_j}\}$ converges weakly to some $\bar{q} \in K$ and $\bar{q} \neq q$. Then, as for q, we have $\bar{q} \in F$. Using Lemma 3.1, we have the following two limits exist:

$$\lim_{n\to\infty} \|x_n - q\| = c, \qquad \lim_{n\to\infty} \|x_n - \bar{q}\| = c_1.$$

Thus, by Opial's condition, we have

$$c = \limsup_{k\to\infty} \|x_{n_k} - q\| < \limsup_{k\to\infty} \|x_{n_k} - \bar{q}\| = \limsup_{j\to\infty} \|x_{m_j} - \bar{q}\| < \limsup_{j\to\infty} \|x_{m_j} - q\| = c,$$

which is a contradiction and so $q = \bar{q}$. This shows that the sequence $\{x_n\}$ converges weakly to q. This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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